Discrete Hamiltonian Systems: Difference Equations, Continued Fractions, and Riccati Equations
by Calvin D. Ahlbrandt and Allan C. Peterson

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> On becoming familiar with difference equations and their close relation to differential equations, I was in hopes that the theory of difference equations could be brought completely abreast with that for ordinary differential equations.
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> [Hugh L. Turrittin, "My Mathematical Expectations", Springer Lecture Notes 312 (page 10), 1973]

To find out about the truth of such hopes, we first must extensively study the discrete calculus. Only recently this has become a research object in its own right, and some excellent introductions to the subject include [2, 6, 9]. Advanced textbooks on special topics include [3, 10, 12]. Other major activities in this field have been initiated five years ago by Gerry Ladas and Saber Elaydi, e.g., the founding of the "Journal of Difference Equations and Applications", the introduction of an annual conference on difference equations, together with the publication of valuable proceedings that reflect very nicely the numerous research efforts in this fast growing area.

The book under review offers an introduction into the exciting subject of Hamiltonian difference systems as well as most of the recent results in this area. It hence can be used as a textbook for an introductory course at the senior undergraduate level, but it also contains material appropriate for a graduate level course as well as suggestions for research projects of graduate students. Use of the book as a textbook for a lecture is greatly facilitated due to the numerous exercises (I counted 147 of them), distributed throughout the book, which help the student to absorb the material extensively.

The major topic of the book are so-called discrete symplectic systems $\Delta z_k = S_k z_k$, where Δ is the usual forward difference operator and S_k are $2n \times 2n$-symplectic matrices (for each $k \in \mathbb{Z}$). Special cases of such symplectic systems are linear Hamiltonian difference systems

$$\text{(H)} \qquad \Delta x_k = A_k x_{k+1} + B_k u_k, \qquad \Delta u_k = C_k x_{k+1} - A_k^T u_k,$$

where A_k, B_k, and C_k are $n \times n$-matrices (such that B_k and C_k are symmetric and $I - A_k$ are invertible). Systems (H) have been introduced by Lynn Erbe and Pengxiang Yan in [7]. Special cases of systems (H), in turn, are Sturm-Liouville difference equations of second order

$$\text{(SL)} \qquad \Delta[r_k \Delta x_k] + p_k x_{k+1} = 0,$$

where $r_k, p_k \in \mathbb{R}$ with $r_k \neq 0$, and also of higher order as well as other self-adjoint vector difference equations. The notion of a (generalized) zero of solutions of such equations (which are sequences rather than continuous functions) is somewhat more complicated than in the continuous case as sign changes have to be taken into account. In his landmark paper on the subject [8], Philip Hartman defined generalized zeros for solutions of (SL) only when $r_k > 0$ (which is suggested by the continuous case, where $r(t) > 0$ is the so-called strengthened Legendre condition), but it is now the first time in a textbook that this notion for the discrete case is also defined for arbitrary $r_k \neq 0$. This then allows the construction of a discrete Sturmian theory that applies to a wider class of equations, including e.g. the simple Fibonacci recurrence relation (which is not covered by Hartman's definition). Disconjugacy of symplectic systems is studied at various places in the book, and so-called Reid roundabout theorems, that give conditions equivalent to disconjugacy, among them positivity of a certain discrete quadratic functional and solvability of a matrix Riccati difference equation (which gives rise to the study of matrix continued fractions) are established. Closely connected to disconjugacy are topics in discrete variational analysis (i.e., the optimization of functionals of the form $\sum_{t=a}^{b-1} f\left(t, x(t+1), \Delta x(t)\right)$ subject to certain boundary conditions) which is very nicely described in the book. (Watch out for the discrete Legendre condition which is on the first view not too much related to its continuous counterpart!) A novel approach is presented by investigating the effect when variable step size is introduced in the discrete variational problem. The book ends at the edge of current research with a chapter on disconjugacy for systems (H) with non-invertible matrices B_k (see [4, 5]). Only those systems include the important case of Sturm-Liouville difference equations of higher order. As work in this area is continuing, this book is of course not the final word on the subject. For even further results connected to discrete Hamiltonian systems we refer the interested reader to the special issue [1], which will appear soon.

The authors, Calvin D. Ahlbrandt and Allan C. Peterson, are Professors of Mathematics at the University of Missouri–Columbia and at the University of Nebraska–Lincoln, respectively. During the last almost twenty years they both have contributed immensely to the theory of difference equations. Both of them are on the Editorial Board of the "Journal of Difference Equations and Applications". It is of great value that they are also both profound experts in the "continuous counterparts" of the presented theory. Readers who know the authors' work in differential equations or the work of William T. Reid [13] or Werner Kratz [11] will especially appreciate studying the discrete results and observing the common features as well as the numerous discrepancies between both theories. Much work needs to be done before Turrittin's hopes become truth one day.

References

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