CONTROLLABILITY AND DISCONJUGACY FOR LINEAR HAMILTONIAN DIFFERENCE SYSTEMS.

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Abstract. For matrices A_k, B_k, C_k, and S we consider the discrete quadratic functional

$$F(x,u)=\sum_{k=0}^{N}\{x_{k+1}^T C_k x_{k+1}+u_k^T B_k u_k\}+\begin{pmatrix}-x_0\\ x_{N+1}\end{pmatrix}^T S\begin{pmatrix}-x_0\\ x_{N+1}\end{pmatrix}$$

where (x,u) are pairs that satisfy certain boundary conditions as well as a difference equation of the form $x_{k+1}-x_k=:\Delta x_k=A_k x_{k+1}+B_k u_k$.
The concept of controllability and disconjugacy of a certain related linear Hamiltonian Difference System is introduced and a necessary and sufficient condition for F to be positive for all non-trivial (x,u) under consideration is stated. This condition is in a sense a discrete version of the well-known Jacobi's Condition from the calculus of variations.

1. INTRODUCTION.

In this chapter we are concerned with a discrete quadratic functional of the form

$$F(x,u)=\sum_{k=0}^{N}\{x_{k+1}^T C_k x_{k+1}+u_k^T B_k u_k\}+\begin{pmatrix}-x_0\\ x_{N+1}\end{pmatrix}^T S\begin{pmatrix}-x_0\\ x_{N+1}\end{pmatrix},$$

where S, B_k, C_k are symmetric matrices, and where the pairs (x,u) have to satisfy the difference equation $x_{k+1}-x_k=:\Delta x_k=A_k x_{k+1}+B_k u_k$ and boundary conditions of the form $\binom{-x_0}{x_{N+1}}\in \mathrm{Im}R^T$ (A_k and R being matrices; Im denoting the image of a matrix). In various problems (for example when trying to solve variational problems or control problems, where F occurs as the second variation of the problem; see e.g. [11, Chapter 8]), the following question arises: What conditions must we impose on the matrices to ensure that the functional will be positive for all non-trivial pairs (x, u) under consideration? The converse question is also important: If F is positive definite in the above sense, what do the matrices look like? In the "continuous" case with $R = S = 0$ (i.e., the sum is an integral, the difference equation is a differential equation, and the considered interval is $[a, b] \subset \mathbb{R}$), the answer to these questions is the well-known Jacobi's Condition: Assuming the controllability of a certain related Hamiltonian Differential System, the functional is positive definite if and only if Jacobi's Equation is disconjugate on $(a, b]$. Of course, the concepts of controllability and disconjugacy in the present "discrete" case will look quite different from the "continuous" case and it is the purpose of this chapter to introduce these concepts and to state (without giving a proof) the discrete version of Jacobi's Condition, which, using our definition of controllability and disconjugacy, then reads exactly as the classical Jacobi's Condition.

It should be mentioned that the related linear Hamiltonian Difference System, which will be introduced in Section 2, can trivially seen to be controllable in our sense for the case that all the matrices B_k are invertible. Erbe and Yan recently introduced such Hamiltonian Systems (see [8]) where they assume the B_k to be non-singular. They gave a definition of disconjugacy involving the inverse of B_k and for this case they characterized the positive definiteness of F (where they have $R = S = 0$) in terms of disconjugacy ([8, 9, 10]). Ahlbrandt ([3]) and Peterson ([13]) also dealt with these problems and they also impose the non-singularity assumption on the B_k. Special cases of Hamiltonian Systems with invertible B_k are for example self-adjoint vector difference equations which were studied intensively by many authors, among them Ahlbrandt ([2]), Peterson and Ridenhour ([14, 15, 16]), Peil and Peterson ([12]), and Chen and Erbe ([7]). Even more specific are the second order linear difference equations, see for instance Chen ([6]).

But there are also cases where the matrices B_k are singular, and those cases cannot be treated within the framework of the previous work. It turns out that Hamiltonian Systems with certain singular B_k are in some sense (see Lemma 2 in Section 2) equivalent to a scalar self-adjoint difference equation of even order, a Sturm-Liouville Equation. It has been an open question to find a Jacobi's Condition (respectively a Reid Roundabout Theorem, using Ahlbrandt's terminology; see [3]) for these problems, but recently Ahlbrandt and Peterson answered this question in one direction (Theorem 1 in [1]). The corresponding Hamiltonian Systems turn out to be controllable in our sense, so that our theorem below (Section 4) applies to all of these problems.

In fact, it applies to many more systems and solves a problem posed by Ahlbrandt in [3]. He writes on p.515: "An open question is that of existence of a Reid Roundabout Theorem for systems which allows B_n to be singular." Crucial for giving such a Reid Roundabout Theorem are our concepts of controllability and disconjugacy which are introduced in Section 3.

2. TERMINOLOGY AND EXAMPLES.

Let us introduce more precisely the expressions and terms which already have been used in Section 1. For a matrix- or vector-valued function f on a set $J^* := [0, N+1] \cap \mathbb{Z}$ with $N \in \mathbb{Z}$ we write $f_k := f(k)$, $k \in J^*$. The forward difference operator Δ is given by $\Delta f_k := f_{k+1} - f_k$ and the shift operator E (see also [11, Chapter 2]) by $E f_k := f_{k+1}$ for $k \in J := [0, N] \cap \mathbb{Z}$, so that we have $\Delta = E - I$, where I denotes the identity operator. For $n \in \mathbb{N}$ we are given $n \times n$-matrix-valued functions A, B, and C on J and constant $2n \times 2n$-matrices R and S which satisfy the following general assumption:

$$(V) \qquad \begin{cases} R \ \text{arbitrary}, S \ \text{symmetric}; \\ B_k, C_k \ \text{symmetric} \quad \forall\, k \in J; \\ \tilde{A}_k := (I - A_k)^{-1} \ \text{exists} \quad \forall\, k \in J. \end{cases}$$

On J we consider the *linear Hamiltonian Difference System*

$$(H) \qquad \Delta \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \begin{pmatrix} Ex \\ u \end{pmatrix}$$

with solutions $(x, u) : J^* \to \mathbb{R}^{2n}$ and we introduce the *discrete quadratic functional*

$$F(x,u) = \sum_{k=0}^{N} \{ x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k \} + \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T S \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}.$$

Definition 1.

(i) $(x,u): J^* \to \mathbb{R}^{2n}$ is called *admissible*, if it satisfies (the equation of motion) $\Delta x = AEx + Bu$ on J.

(ii) $(x,u): J^* \to \mathbb{R}^{2n}$ is said to satisfy the *boundary conditions* (notation: $(x,u) \in \mathcal{R}$), if $\binom{-x_0}{x_{N+1}} \in \operatorname{Im} R^T$.

(iii) F is called *positive definite* (notation: $F > 0$), if $F(x,u) > 0$ for all admissible $(x,u) \in \mathcal{R}$ with $x \neq 0$. ■

Remark 1.

(i) The choice $R = 0$ yields the boundary conditions $x_0 = x_{N+1} = 0$ which have been treated in the literature, and in this case, S can be chosen to be the zero-matrix also.

(ii) (H) actually consists of two equations: While $\Delta x = AEx + Bu$ is the equation of motion, $\Delta u = CEx - A^T u$ is called the Euler Equation.

(iii) Note that the invertibility assumption on $I - A_k$ for $k \in J$ ensures the unique solvability of any initial value problem of the form $\{(H),\ x_m = x^*,\ u_m = u^*\}$ where $x^*, u^* \in \mathbb{R}^n$ and $m \in J^*$. In fact, as was pointed out by Ahlbrandt in [3], the system (H) can be written as $E\binom{x}{u} = T\binom{x}{u}$ on J, where the $2n \times 2n$-matrix-valued function T on J is given by

$$T_k = \begin{pmatrix} \tilde{A}_k & \tilde{A}_k B_k \\ C_k \tilde{A}_k & C_k \tilde{A}_k B_k + I - A_k^T \end{pmatrix}, \qquad k \in J.$$

Furthermore, each matrix T_k is invertible with

$$T_k^{-1} = \begin{pmatrix} B_k \tilde{A}_k^T C_k + I - A_k & -B_k \tilde{A}_k^T \\ -\tilde{A}_k^T C_k & \tilde{A}_k^T \end{pmatrix}.$$

■

Because of (iii) in Remark 1, the following is well-defined, and these objects will play an important role in our characterization of positive definiteness.

Definition 2. By (X,U) and $(\tilde{X},\tilde{U})$ we denote the $n \times n$-matrix-valued solutions of (H) (i.e., $\Delta X = AEX + BU,\ \Delta U = CEX - A^T U$

and $\Delta\tilde{X} = AE\tilde{X} + B\tilde{U},\ \Delta\tilde{U} = CE\tilde{X} - A^T\tilde{U}$ hold on J) which satisfy the initial conditions

$$X_0 = \tilde{U}_0 = 0, \qquad U_0 = -\tilde{X}_0 = I.$$

(X, U) is called the *principal solution* of (H) at 0 and $(\tilde{X}, \tilde{U})$ the *associate solution* of (H) at 0. ■

Lemma 1. Let (X, U) and $(\tilde{X}, \tilde{U})$ denote the principal solution of (H) at 0 and the associate solution of (H) at 0, respectively. With $N := \begin{pmatrix} X & \tilde{X} \\ U & \tilde{U} \end{pmatrix}$ on J^*, we have

$$N \begin{pmatrix} \tilde{U}^T & -\tilde{X}^T \\ -U^T & X^T \end{pmatrix} = I \text{ on } J^* \qquad \text{and} \qquad EN = TN \text{ on } J.$$

Proof. $N_{k+1} = T_k N_k$ for all $k \in J$ follows from Remark 1 (iii). ¿From

$$N_0^{-1} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and the assumption $N_k^{-1} = \begin{pmatrix} \tilde{U}_k^T & -\tilde{X}_k^T \\ -U_k^T & X_k^T \end{pmatrix}$ for $k \in J$, we have

$$\begin{aligned} N_{k+1}^{-1} &= N_k^{-1} T_k^{-1} = \begin{pmatrix} \tilde{U}_k^T & -\tilde{X}_k^T \\ -U_k^T & X_k^T \end{pmatrix} \begin{pmatrix} B_k \tilde{A}_k^T C_k + I - A_k & -B_k \tilde{A}_k^T \\ -\tilde{A}_k^T C_k & \tilde{A}_k^T \end{pmatrix} \\ &= \begin{pmatrix} \tilde{U}_{k+1}^T & -\tilde{X}_{k+1}^T \\ -U_{k+1}^T & X_{k+1}^T \end{pmatrix}, \end{aligned}$$

so that the assertion follows. ■

We now give some examples of Hamiltonian Difference Systems. While the case of a second order linear self-adjoint difference equation on J:

$$\Delta(r\Delta x) + pEx = 0, \qquad \text{(each } r_k \text{ is real and positive)}$$

and the case of a self-adjoint vector difference equation on J:

$$\Delta(P\Delta x) + QEx = 0, \qquad \text{(each } P_k \text{ is a positive definite } n \times n\text{-matrix)}$$

deal with invertible matrices B_k ($A_k \equiv 0, B_k = \frac{1}{r_k}, C_k = -p_k$ in the first case and $A_k \equiv 0, B_k = P_k^{-1}, C_k = -Q_k$ in the second case), there exist other important examples where the B_k are singular. To be more specific, we will impose the following condition on the $n \times n$-matrices A_k, B_k, and C_k ($k \in J$):

$$(V_S) \qquad \begin{cases} A_k \equiv (a_{ij})_{1 \le i,j \le n} \text{ with } a_{ij} = \begin{cases} 1 & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}; \\ B_k = \operatorname{diag}(0, \ldots, 0, \frac{1}{r_k^{(n)}}), \text{ where } r_k^{(n)} \in \mathbb{R} \setminus \{0\}; \\ C_k = \operatorname{diag}(r_k^{(0)}, \ldots, r_k^{(n-1)}), \text{ where each } r_k^{(\nu)} \in \mathbb{R}. \end{cases}$$

Here, $\operatorname{diag}(a_1, \ldots, a_n)$ denotes the diagonal $n \times n$-matrix with diagonal entries $a_1, \ldots, a_n$. Assuming (V_S), we have that $(x, u) : J^* \to \mathbb{R}^{2n}$ solves (H) iff the following holds on J:

$$\Delta x^{(\nu)} = E x^{(\nu+1)} \ (0 \le \nu \le n-2), \qquad \Delta x^{(n-1)} = \frac{1}{r^{(n)}} u^{(n)};$$

$$\Delta u^{(\nu)} = r^{(\nu)} E x^{(\nu)} - u^{(\nu-1)} \ (1 \le \nu \le n-1), \qquad \Delta u^{(0)} = r^{(0)} E x^{(0)}.$$

Under (V_S) the Hamiltonian System (H) is "equivalent" to a *Sturm-Liouville Equation*. More precisely, we have the following result:

Lemma 2. Assume (V_S). Then (x, u) where $x = (x^{(\nu)})_{0 \le \nu \le n-1}$ and $u = (u^{(\nu)})_{0 \le \nu \le n-1}$ solves (H) if and only if

$$(*) \qquad \begin{cases} x^{(\nu)} = (E^{-1}\Delta)^{\nu} y, \\ u^{(\nu)} = \sum\limits_{\mu=\nu+1}^{n} (-\Delta)^{\mu-\nu-1} [r^{(\mu)} (E^{-1}\Delta)^{\mu} E y] \end{cases}$$

holds on J with some $y : [1-n, N+1+n] \cap \mathbb{Z} \to \mathbb{R}$ such that

$$L(y) := \sum_{\mu=0}^{n} (-\Delta)^{\mu} [r^{(\mu)} (E^{-1}\Delta)^{\mu} E y] = 0.$$

Proof. Let $(*)$ hold on J with some $y : [1-n, N+1+n] \cap \mathbb{Z} \to \mathbb{R}$. Then we have on J for $0 \le \nu \le n-2$ (since $E^{-1}\Delta = E^{-1}(E - I) = \Delta E^{-1}$):

$$\Delta x^{(\nu)} = \Delta(\Delta E^{-1})^{\nu} y = (\Delta E^{-1})^{\nu+1} E y = E x^{(\nu+1)},$$

$$\Delta x^{(n-1)} = \Delta(\Delta E^{-1})^{n-1} y = E(\Delta E^{-1})^{n} y = \frac{1}{r^{(n)}} u^{(n)},$$

$$\Delta u^{(\nu+1)} = -\sum_{\mu=\nu+2}^{n} (-\Delta)^{\mu-\nu-1}[r^{(\mu)}(E^{-1}\Delta)^{\mu}Ey] = r^{(\nu+1)}Ex^{(\nu+1)} - u^{(\nu)},$$

$$\Delta u^{(0)} = -\sum_{\mu=1}^{n}(-\Delta)^{\mu}[r^{(\mu)}(E^{-1}\Delta)^{\mu}Ey] = r^{(0)}Ex^{(0)} - L(y),$$

so that $\Delta x = AEx + Bu$ and $\Delta u = CEx - A^T u - \mathrm{diag}(L(y), 0, \ldots, 0)$ hold on J, which yields (since any solution (x, u) of (H) may be written in the form $(*)$) directly the assertion. ■

Note that $L(y) = 0$ is a self-adjoint scalar difference equation of order $2n$. When studying these Sturm-Liouville Equations (or also when studying Sturm-Liouville Eigenvalue Problems), it is important in view of Lemma 2 to have a characterization of the positive definiteness of the functional F when the matrices B_k are singular. We will now provide the main tools for achieving this goal.

3. CONTROLLABILITY AND DISCONJUGACY.

In the "continuous" case, a Hamiltonian Differential System is called controllable (or identically normal) on an interval $[a, b] \subset \mathbb{R}$, if for any solution (x, u) of (H) we have that $x = 0$ on any non-degenerate subinterval $I \subset [a, b]$ implies that $x = u = 0$ holds on $[a, b]$. Such non-degenerate subintervals correspond in a sense to certain integer intervals in the present "discrete" case. To be more precise, we will use the following definition:

Definition 3. For $\kappa \in J^*$ we define a set of intervals of length $\kappa + 1$ by

$$\mathcal{J}(\kappa) := \{[m, m+\kappa] \cap \mathbb{Z} |\ m, m+\kappa \in J^*\}.$$

The Hamiltonian Difference System (H) is called *controllable* on J^*, if there exists $\kappa \in J^* \setminus \{0\}$ such that for any solution $(x, u) : J^* \to \mathbb{R}^{2n}$ of (H) we have that $x = 0$ on any $\tilde{J} \in \mathcal{J}(\kappa)$ implies $x = u = 0$ on J^*. The minimal $\kappa \in \mathbb{N}$ with this property is then called the *controllability index* of (H). ■

We will now give a characterization of controllability which allows us to determine easily whether a Hamiltonian System (H) is controllable or not. As an application of this characterization we will show in a corollary that systems under (V) with invertible B_k as well as systems under the assumption (V_S) are controllable.

Lemma 3. Assume (V) and let $\Phi_{km} := \tilde{A}_{k-1} \cdot \tilde{A}_{k-2} \cdot \ldots \cdot \tilde{A}_m$ for $k > m$. Let $\kappa \in J^* \setminus \{0\}$. For $m \in J$ with $m + \kappa \in J^*$ put

$$G_m(\kappa) := (\Phi_{m+\kappa,m}B_m, \quad \Phi_{m+\kappa,m+1}B_{m+1}, \ldots, \Phi_{m+\kappa,m+\kappa-1}B_{m+\kappa-1}).$$

Then the following statements are equivalent.

(i) (x,u) solves (H), $x = 0$ on $\tilde{J} \in \mathcal{J}(\kappa) \Longrightarrow x = u = 0$ on J^*.

(ii) $a, b \in \mathbb{R}^n$, $m \in J$ with $m + \kappa \in J^*$
$\Longrightarrow$ there exists an admissible (x,u) with $x_m = a$ and $x_{m+\kappa} = b$.

(iii) $m \in J$ with $m + \kappa \in J^* \Longrightarrow \text{rank} G_m(\kappa) = n$.

Proof. Observe that, by induction and $x_{k+1} = \tilde{A}_k x_k + B_k u_k$, for admissible (x,u) and for $m \in J$ with $m + \kappa \in J^*$ we have that

$$x_{m+\kappa} = \Phi_{m+\kappa,m}x_m + G_m(\kappa) \begin{pmatrix} u_m \\ \vdots \\ u_{m+\kappa-1} \end{pmatrix}.$$

This shows (ii) $\Longleftrightarrow$ (iii).

Now, whenever (x,u) solves (H) with $x_m = \ldots = x_{m+\kappa} = 0$ we have $\Delta u = -A^T u$ and $Bu = 0$ on $[m, m+\kappa] \cap \mathbb{Z}$, i.e.,

$$u_{m+k} = \Phi^T_{m+\kappa,m+k} u_{m+\kappa} \quad \text{and} \quad B_{m+k}\Phi^T_{m+\kappa,m+k} u_{m+\kappa} = 0, \quad 0 \le k \le \kappa - 1.$$

Since $\bigcap_{k=0}^{\kappa-1} \text{Ker}(B_{m+k}\Phi^T_{m+\kappa,m+k}) = \text{Ker} G^T_m(\kappa)$ (where Ker denotes the kernel of a matrix), (i) $\Longleftrightarrow$ (iii) follows and the proof is complete. ■

Remark 2.

(i) Assume (V) and that B_k is invertible. Then $G_m(1) = \tilde{A}_m B_m$ with $\text{rank} G_m(1) = n$ for all $m \in J$, and (H) is controllable with controllability index 1 because of Lemma 3.
Now, assume (V_S). Then, whenever $m + n \in J^*$, $\Phi_{m+n,m+k} = \tilde{A}^{n-k}$ for $0 \le k \le n-1$ with $\tilde{A} = (\tilde{a}_{ij})_{1 \le i,j \le n}$, $\tilde{a}_{ij} = \begin{cases} 1 & \text{if } i \le j \\ 0 & \text{otherwise} \end{cases}$;

$$G_m(n) = (\tilde{A}^n B_m, \quad \tilde{A}^{n-1}B_{m+1}, \ldots, \tilde{A}B_{m+n-1}),$$

and $\text{rank} G_m(n) = n$ can be verified for this case. Thus, again by Lemma 3, (H) is controllable with controllability index n.

(ii) If we are given a *time-invariant* (or autonomous) system (H) satisfying (V), i.e., $A_k \equiv A$ and $B_k \equiv B$ on J, then (H) is controllable if and only if the controllability matrix $(B,\ \tilde{A}B, \ldots, \tilde{A}^{n-1}B)$ has full rank (where we put $\tilde{A} = (I - A)^{-1}$). ■

We now introduce our concept of disconjugacy. For doing so, we will need the following well-know fact (see e.g. [5]): Given any matrix V, there exists a unique matrix $V^\dagger$ satisfying $VV^\dagger V = V$, $V^\dagger V V^\dagger = V^\dagger$, and such that both $V^\dagger V$ and $VV^\dagger$ are symmetric. This matrix $V^\dagger$ is called the *Moore-Penrose (Pseudo-)Inverse* of V, and with this notation our definition of disconjugacy of a Hamiltonian Difference Systems (H) reads as follows:

Definition 4. Let (X, U) be the principal solution of (H) at 0. (H) is called *disconjugate* on $(0, N+1]$, if the following two conditions hold:

(i) $\mathrm{Ker} X_{k+1} \subset \mathrm{Ker} X_k \quad \forall\, k \in J$,

(ii) $D_k := X_k X_{k+1}^\dagger \tilde{A}_k B_k \geq 0 \quad \forall\, k \in J$. ■

Of course, "≥ 0" in Definition 4 (ii) means positive semidefinite, and this can only be satisfied if the matrices D_k are symmetric. We have the following result:

Lemma 4. Let (X, U) be the principal solution of (H) at 0 and suppose that $\mathrm{Ker} X_{k+1} \subset \mathrm{Ker} X_k$ holds for $k \in J$. Then $D_k = X_k X_{k+1}^\dagger \tilde{A}_k B_k$ is symmetric.

Proof. We first show that we have for arbitrary matrices V and W:

$$\mathrm{Ker} V \subset \mathrm{Ker} W \Longleftrightarrow W = W V^\dagger V.$$

One direction of this claim is trivial. Now, suppose $\mathrm{Ker} V \subset$ KerW, i.e., $\mathrm{Im} W^T \subset \mathrm{Im} V^T$, and let c be such that $W^T c$ is defined. Hence, there exists $d = V d_1 + d_2$ with $d_2 \in \mathrm{Ker} V^T$ such that

$$W^T c = V^T d = V^T (V d_1 + d_2) = V^T V d_1$$

and

$$\begin{aligned} (W V^\dagger)^T c &= (V^\dagger)^T W^T c = (V^\dagger)^T V^T V d_1 \\ &= (V V^\dagger)^T V d_1 = V V^\dagger V d_1 = V d_1. \end{aligned}$$

Thus we have $W^Tc = V^TVd_1 = V^T(WV^\dagger)^Tc = (WV^\dagger V)^Tc$, and therefore $W^T = (WV^\dagger V)^T$ and $W = WV^\dagger V$ follows.

To show the assertion of the lemma, suppose that $c \in \operatorname{Ker} X_{k+1}$ for $k \in J$. Then (see Lemma 1) $0 = \tilde{X}_{k+1}X_{k+1}^Tc = X_{k+1}\tilde{X}_{k+1}^Tc$, and, assuming $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$,

$$0 = X_k\tilde{X}_{k+1}^Tc = (\tilde{X}_kX_{k+1}^T + B_k\tilde{A}_k^T)c = B_k\tilde{A}_k^Tc.$$

So we have $\operatorname{Ker} X_{k+1}^T \subset \operatorname{Ker} B_k\tilde{A}_k^T$ and hence part (i) yields the formula $X_{k+1}X_{k+1}^\dagger\tilde{A}_kB_k = \tilde{A}_kB_k$. Therefore

$$\begin{aligned} D_k &= X_kX_{k+1}^\dagger\tilde{A}_kB_k \\ &= [(I - A_k + B_k\tilde{A}_k^TC_k)X_{k+1} - B_k\tilde{A}_k^TU_{k+1}]X_{k+1}^\dagger\tilde{A}_kB_k \\ &= B_k + B_k\tilde{A}_k^TC_k\tilde{A}_kB_k - B_k\tilde{A}_k^T(X_{k+1}^\dagger)^TX_{k+1}^TU_{k+1}X_{k+1}^\dagger\tilde{A}_kB_k, \end{aligned}$$

which is symmetric (using Lemma 1 again). ■

Note that, if X_{k+1} is invertible for $k \in J$, it is easy to see that

$$D_k = X_kX_{k+1}^{-1}\tilde{A}_kB_k = X_k(X_k + B_kU_k)^{-1}B_k = (I + B_kQ_k)^{-1}B_k$$

is symmetric, since $Q_k := U_kX_k^{-1}$ is symmetric (see Lemma 1). In this case, Q solves on J the discrete Riccati Matrix Equation

$$EQ = C + (I - A^T)Q(I + BQ)^{-1}(I - A),$$

as is well-known (see e.g. [13]).

Now, if B_k is invertible for $k \in J$, which is the case that has been treated in the literature, we have the following result (compare also the definitions of disconjugacy in [8, 3, 16]):

Lemma 5. Assume that (V) holds and that B_k is non-singular for all $k \in J$. Then (H) is disconjugate on $(0, N+1]$ iff

$$(*) \qquad \begin{cases} x_k^TB_k^{-1}(I - A_k)x_{k+1} > 0 \quad \forall\, k \in J \setminus \{0\} \\ \text{for all solutions } (x, u) \text{ of } (H) \text{ with } x_0 = 0 \text{ and } x \neq 0. \end{cases}$$

Proof. Let (X, U) be the principal solution of (H) at 0. First suppose that (H) is disconjugate on $(0, N+1]$, i.e., $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$ and $D_k = X_kX_{k+1}^\dagger\tilde{A}_kB_k \geq 0$ for all $k \in J$. It follows, since $X_1 = \tilde{A}_0B_0$ is invertible, that X_{k+1} is an invertible matrix for all $k \in J$ and hence that $D_k = X_k X_{k+1}^{-1} \tilde{A}_k B_k$ is invertible for all $k \in J \setminus \{0\}$. Let (x, u) solve (H) with $x_0 = 0$ and $x \neq 0$. Therefore, $x = Xu_0$ and $u = Uu_0$ hold on J^* (see Remark 1 (iii)) with $u_0 \neq 0$, and we have for $k \in J \setminus \{0\}$

$$\begin{aligned} x_k^T B_k^{-1}(I - A_k)x_{k+1} &= x_k^T B_k^{-1}(I - A_k)X_{k+1}X_k^{-1}x_k \\ &= x_k^T D_k^{-1} x_k > 0. \end{aligned}$$

Now, suppose $(*)$ holds. Let $\alpha \in \mathbb{R}^n \setminus \{0\}$ and $x = X\alpha$, $u = U\alpha$ on J^*. Then (x, u) solves (H) with $x_0 = 0$ and $(*)$ implies that

$$0 < x_k^T B_k^{-1}(I - A_k)x_{k+1} = \alpha^T X_k B_k^{-1}(I - A_k)X_{k+1}\alpha \quad (k \in J \setminus \{0\}).$$

This, in turn, shows that X_k is invertible for all $k \in J \setminus \{0\}$ and, since $\alpha^T X_k D_k^{-1} X_k \alpha = \alpha^T X_k B_k^{-1}(I - A_k)X_{k+1}\alpha > 0$, that $D_k > 0$ holds for all $k \in J \setminus \{0\}$. ■

4. JACOBI'S CONDITION.

To state our main theorem we need one last auxiliary result.

Lemma 6. Suppose that (H) is controllable on J^* and let (X, U) be the principal solution of (H) at 0. Then disconjugacy of (H) on $(0, N+1]$ implies invertibility of X_{N+1}.

Proof. Let $c \in \operatorname{Ker} X_{N+1}$. Of course, (x, u) with $x = Xc$ and $u = Uc$ solves (H) and $x = 0$ on J^* because $\operatorname{Ker} X_{N+1} \subset \operatorname{Ker} X_k$ for all $k \in J$. Since (H) is controllable on J^*, $x = u = 0$ on J^* follows, which implies $0 = u_0 = U_0 c = c$. Thus X_{N+1} is invertible. ■

Now, Jacobi's Condition, i.e., the characterization of positive definiteness of the discrete quadratic functional

$$F(x, u) = \sum_{k=0}^{N} \{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\} + \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T S \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}$$

(see the notation introduced at the beginning of Section 2 and in Definition 1) reads as follows:

Theorem. Let (X, U) and $(\tilde{X}, \tilde{U})$ be the principal solution of (H) at 0 and the associate solution of (H) at 0, respectively (see Def. 2). Suppose that (V) holds and that (H) is controllable on J^* (see Def. 3). Then F is positive definite (see Def. 1) if and only if the following two conditions hold:

(1) (H) is disconjugate on $(0, N+1]$ (see Def. 4);

(2) $M := R(M_{N+1} + S)R^T$ is positive definite on $\mathrm{Im}R$, where

$$M_{N+1} := \begin{pmatrix} -X_{N+1}^{-1}\tilde{X}_{N+1} & X_{N+1}^{-1} \\ (X_{N+1}^{-1})^T & U_{N+1}X_{N+1}^{-1} \end{pmatrix}.$$

■

As already announced we shall not give a proof of this result here. The proof requires a discrete version of a "Picone's Identity" (see e.g. [4]) and it will appear elsewhere. Several concluding remarks are in order.

Remark 3.

(i) Of course, condition (2) only makes sense when X_{N+1} is invertible. But, if (1) holds, we have shown in Lemma 6 (assuming controllability), that X_{N+1} is indeed non-singular. Furthermore, it can easily be verified with the aid of Lemma 1 that in this case M_{N+1} is symmetric, so that positive definiteness makes sense, also.

(ii) If $R = 0$, i.e., $x_0 = x_{N+1} = 0$ are the boundary conditions (see Remark 1 (i)), then condition (2) is empty and in this case positive definiteness of F is equivalent to disconjugacy of (H).

(iii) Observe that the case of linear self-adjoint difference equations of even order has been included in this result, as was shown in Lemma 2 and Remark 2 (i). Of course, the case where B_k is invertible for $k \in J$ is also included, and the result for this special case (where additionally $R = S = 0$ is assumed) has already been shown before (see e.g. [8]). It does not require the concept of controllability (see Remark 2 (i)) and disconjugacy has been defined involving the inverse of B_k (see Lemma 5), which is not possible in the general case of our theorem.

(iv) Note that the controllability assumption on (H) implies that $N + 1 \geq \kappa$, where κ is the controllability index of (H) (compare Definition 3). Furthermore, in the time-invariant case, the controllability assumption may for convenience be replaced by the rank condition given in Remark 2 (ii).

(v) Finally observe that $F > 0$ is hard to check (one has to show $F(x,u) > 0$ for any admissible $(x,u) \in \mathcal{R}$ with $x \neq 0$), while the conditions (1) und (2) are very easy to check, since Lemma 1 gives an easy formula on how to compute the important objects (X,U) and $(\tilde{X},\tilde{U})$. Of course, if $R=0$, one does not even need to compute the associate solution of (H) at 0. ■

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