Diamond-alpha Grüss type inequalities on time scales

Martin Bohner* and Thomas Matthews

Department of Mathematics and Statistics, Missouri University of Science and Technology, Rolla, MO 65409-0020, USA
E-mail: bohner@mst.edu
E-mail: tmnqb@mst.edu
*Corresponding author

Adnan Tuna

Faculty of Science and Arts, Department of Mathematics, University of Niğde, Merkez, 51240, Niğde, Turkey
E-mail: atuna@nigde.edu.tr

Abstract: In this paper, we study a more general version of Grüss type inequalities on time scales by using the recent theory of combined dynamic derivatives on time scales. In the case $\alpha = 1$, we obtain delta-integral Grüss type inequalities on time scales. For $\alpha = 0$, we obtain nabla-integral Grüss type inequalities. We supply numerous examples throughout.

Keywords: Grüss type inequality; diamond-$\alpha$ derivative; time scales.

Reference to this paper should be made as follows: Bohner, M., Matthews, T. and Tuna, A. (2011) 'Diamond-alpha Grüss type inequalities on time scales', Int. J. Dynamical Systems and Differential Equations, Vol. 3, Nos. 1/2, pp. 234–247.

Biographical notes: Martin Bohner is a Professor of Mathematics at Missouri S&T. His research interests centre around differential, difference, and dynamic equations as well as their applications to economics, finance, biology, physics, and engineering.

Thomas Matthews is a Graduate Student of Mathematics at Missouri S&T. His research interest lies in the application of differential, difference, and dynamic equations to finance, economics, and statistics.

Adnan Tuna is an Assistant Professor of Mathematics at Nigde University. His research interest lies in dynamic equations, nonlinear evolution equations, and their solutions.

Copyright © 2011 Inderscience Enterprises Ltd.
1 Introduction

In 1988, Hilger (1988) introduced the time scales theory to unify continuous and discrete analysis. Since then, many authors have studied certain integral inequalities on time scales in Agarwal et al. (2002), Bohner and Duman (2010), Bohner and Matthews (2007, 2008), Hilscher (2002), Liu and Ngô (2009), Ngô and Liu (2009) and Wong et al. (2006).

Sidi Ammi and Torres (2010b) have established the diamond-α Grüss inequality on time scales as follows.

Theorem 1.1 (see Sidi Ammi and Torres (2010b, Theorem 3.4)): Let \( T \) be a time scale and \( a, b \in T \) with \( a < b \). If \( f, g \in C(T, \mathbb{R}) \) satisfy \( \varphi \leq f(x) \leq \Phi \) and \( \gamma \leq g(x) \leq \Gamma \) for all \( x \in [a, b] \cap T \), then

\[
\left| \frac{1}{b-a} \int_a^b f(x)g(x)\Diamond_\alpha x - \frac{1}{(b-a)^2} \int_a^b f(x)\Diamond_\alpha x \int_a^b g(x)\Diamond_\alpha x \right| \leq \frac{1}{4} (\Phi - \varphi)(\Gamma - \gamma).
\]

Dragomir (2000) gave some classical and new integral inequalities of Grüss type, for example, the following two results.

Theorem 1.2 (see Dragomir (2000, Theorem 2.1)): Let \( f, g : [a, b] \to \mathbb{R} \) be two Lipschitzian mappings with Lipschitz constants \( L_1 > 0 \) and \( L_2 > 0 \), respectively, i.e.,

\[
|f(x) - f(y)| \leq L_1 |x - y| \quad \text{and} \quad |g(x) - g(y)| \leq L_2 |x - y|
\]

for all \( x, y \in [a, b] \). If \( p : [a, b] \to [0, \infty) \) is integrable, then

\[
\left| \int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx - \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx \right| \\
\leq L_1 L_2 \left[ \int_a^b p(x)dx \int_a^b p(x)x^2dx - \left( \int_a^b p(x)dx \right)^2 \right],
\]

and the inequality is sharp.

Theorem 1.3 (see Dragomir (2000, Theorem 4.1)): Let \( f, g : [a, b] \to \mathbb{R} \) be two integrable mappings on \([a, b]\) such that

\[
|f(x) - f(y)| \leq M |g(x) - g(y)|
\]

for all \( x, y \in [a, b] \). If \( p : [a, b] \to [0, \infty) \) is integrable, then

\[
\left| \int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx - \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx \right| \\
\leq M \left[ \int_a^b p(x)dx \int_a^b p(x)g^2(x)dx - \left( \int_a^b p(x)g(x)dx \right)^2 \right],
\]

and the inequality is sharp.
In 2006, Sheng et al. (2006) studied a combined dynamic ‘diamond-alpha’ derivative as a linear combination of $\Delta$ and $\nabla$ dynamic derivatives on time scales. The diamond-$\alpha$ derivative reduces to the standard $\Delta$ derivative for $\alpha = 1$ and to the standard $\nabla$ derivative for $\alpha = 0$. Since then, many authors have established diamond-$\alpha$ inequalities on time scales (Sidi Ammi and Torres, 2008, 2010b; Bohner and Duman, 2010; Ferreira et al., 2009; Özkan and Kaymakçalan, 2009). We refer the reader to Malinowska and Torres (2009), Mozyrska and Torres (2009), Rogers and Sheng (2007), Sheng et al. (2006) and Sheng (2008) for an account of the calculus with diamond-$\alpha$ dynamic derivatives.

This work is organised as follows: In Section 2, we briefly present some general definitions and theorems connected to the time scales calculus. Next, in Sections 3–5, we generalise Theorems 1.1–1.3, respectively, for general time scales by using the recent theory of combined dynamic derivatives on time scales. In the case $\alpha = 1$, we obtain delta-integral Grüss type inequalities on time scales, while for $\alpha = 0$, we obtain nabla-integral Grüss type inequalities. In order to illustrate the theoretical results, we supply numerous examples throughout.

2 General definitions

For the general theory of calculus on time scales we refer to Agarwal et al. (2001), Bohner and Peterson (2001, 2003) and Hilger (1988). We now introduce the diamond-$\alpha$ integral, referring the reader to Sidi Ammi and Torres (2010a, 2010b), Ferreira et al. (2009), Özkan and Kaymakçalan (2009) for more on the associated calculus.

Definition 2.1: Let $0 \leq \alpha \leq 1$ and $f \in C(\mathbb{T}, \mathbb{R})$. Then the diamond-alpha integral of $f$ is defined by

$$\int_{a}^{b} f(x) \diamond \alpha x = \alpha \int_{a}^{b} f(x) \Delta x + (1 - \alpha) \int_{a}^{b} f(x) \nabla x,$$

where $a, b \in \mathbb{T}$.

Theorem 2.2: Let $0 \leq \alpha \leq 1$ and $f, g \in C(\mathbb{T}, \mathbb{R})$. If $a, b, c \in \mathbb{T}$ and $\beta \in \mathbb{R}$, then

(i) $\int_{a}^{b} [f(x) + g(x)] \diamond \alpha x = \int_{a}^{b} f(x) \diamond \alpha x + \int_{a}^{b} g(x) \diamond \alpha x$

(ii) $\int_{a}^{b} (\beta f(x)) \diamond \alpha x = \beta \int_{a}^{b} f(x) \diamond \alpha x$

(iii) $\int_{a}^{b} f(x) \diamond \alpha x = - \int_{b}^{a} f(x) \diamond \alpha x$

(iv) $\int_{a}^{b} f(x) \diamond \alpha x = \int_{a}^{c} f(x) \diamond \alpha x + \int_{c}^{b} f(x) \diamond \alpha x$

(v) $\int_{a}^{b} f(x) \diamond \alpha x = 0$
Example 2.3: If we let $\mathbb{T} = \mathbb{R}$ in Definition 2.1, then we obtain
\[ \int_{a}^{b} f(x)\diamond_{\alpha}x = \int_{a}^{b} f(x)dx, \text{ where } a, b \in \mathbb{R}. \]

Example 2.4: If we let $\mathbb{T} = \mathbb{Z}$ in Definition 2.1 and $m < n$, then we obtain
\[ \int_{m}^{n} f(x)\diamond_{\alpha}x = \sum_{i=m}^{n-1} [\alpha f_i + (1 - \alpha)f_{i+1}], \text{ where } m, n \in \mathbb{N}_0, \]
and where we put for convenience $f_i = f(i)$.

Example 2.5: If we let $\mathbb{T} = q^{\mathbb{N}_0}$ in Definition 2.1 and $m < n$, then we obtain
\[ \int_{q^m}^{q^n} f(x)\diamond_{\alpha}x = (q - 1) \sum_{i=m}^{n-1} q^i \left[ \alpha f(q^i) + (1 - \alpha)f(q^{i+1}) \right], \text{ where } m, n \in \mathbb{N}_0. \]

Example 2.6: Let $t_i < t_{i+1}$ for all $i \in \mathbb{N}_0$. If we let $\mathbb{T} = \{t_i : i \in \mathbb{N}_0\}$ in Definition 2.1 and $m < n$, then we obtain
\[ \int_{t_m}^{t_n} f(x)\diamond_{\alpha}x = \sum_{i=m}^{n-1} (t_{i+1} - t_i) \left[ \alpha f(t_i) + (1 - \alpha)f(t_{i+1}) \right], \text{ where } m, n \in \mathbb{N}_0, \]
and from here we may obtain Example 2.4 by letting $t_i = i$ for all $i \in \mathbb{N}_0$ and Example 2.5 by letting $t_i = q^i$ for all $i \in \mathbb{N}_0$.

3 The weighted diamond-alpha Grüss inequality

We first extend Theorem 1.1 to the weighted case.

**Theorem 3.1:** Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$ with $a < b$. If $f, g \in C(\mathbb{T}, \mathbb{R})$ and $p \in C(\mathbb{T}, [0, \infty))$ satisfy $\varphi \leq f(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$ for all $x \in [a, b] \cap \mathbb{T}$ and $\int_{a}^{b} p(x)\diamond_{\alpha}x > 0$, then
\[
\left| \int_{a}^{b} p(x)\diamond_{\alpha}x \int_{a}^{b} p(x)f(x)\diamond_{\alpha}x - \int_{a}^{b} p(x)f(x)\diamond_{\alpha}x \int_{a}^{b} p(x)g(x)\diamond_{\alpha}x \right| \leq \frac{1}{4} (\Phi - \varphi)(\Gamma - \gamma) \left( \int_{a}^{b} p(x)\diamond_{\alpha}x \right)^2. \tag{3.1}
\]
Proof: We have
\[
\frac{1}{\int_a^b p(x)\diamond x} \int_a^b p(x)f(x)g(x)\diamond x
- \frac{1}{\int_a^b p(x)\diamond x} \int_a^b p(x)f(x)\diamond x \frac{1}{\int_a^b p(x)\diamond x} \int_a^b p(x)g(x)\diamond x
= \frac{1}{2 \left(\int_a^b p(x)\diamond x\right)^2} \int_a^b \int_a^b p(x)p(y)(f(x) - f(y))(g(x) - g(y))\diamond x\diamond y.
\]
(3.2)

Applying the two-dimensional diamond-$\alpha$ Cauchy–Schwarz inequality from (Sidi Ammi and Torres, 2010a, Theorem 3.5), we get
\[
\left[\frac{1}{2 \left(\int_a^b p(x)\diamond x\right)^2} \int_a^b \int_a^b p(x)p(y)(f(x) - f(y))(g(x) - g(y))\diamond x\diamond y\right]^2
\leq \frac{1}{2 \left(\int_a^b p(x)\diamond x\right)^2} \int_a^b \int_a^b p(x)p(y)(f(x) - f(y))\diamond x\diamond y
\times \frac{1}{2 \left(\int_a^b p(x)\diamond x\right)^2} \int_a^b \int_a^b p(x)p(y)(g(x) - g(y))\diamond x\diamond y
= \left\{\frac{1}{\int_a^b p(x)\diamond x} \int_a^b p(x)f^2(x)\diamond x - \left(\frac{1}{\int_a^b p(x)\diamond x} \int_a^b p(x)f(x)\diamond x\right)^2\right\}
\times \left\{\frac{1}{\int_a^b p(x)\diamond x} \int_a^b p(x)g^2(x)\diamond x - \left(\frac{1}{\int_a^b p(x)\diamond x} \int_a^b p(x)g(x)\diamond x\right)^2\right\}.
\]
(3.3)

We also have
\[
\frac{1}{\int_a^b p(x)\diamond x} \int_a^b p(x)f^2(x)\diamond x - \left(\frac{1}{\int_a^b p(x)\diamond x} \int_a^b p(x)f(x)\diamond x\right)^2
= \left(\Phi - \frac{1}{\int_a^b p(x)\diamond x} \int_a^b p(x)f(x)\diamond x\right) \left(\frac{1}{\int_a^b p(x)\diamond x} \int_a^b p(x)f(x)\diamond x - \varphi\right)
- \frac{1}{\int_a^b p(x)\diamond x} \int_a^b p(x)(\Phi - f(x))(f(x) - \varphi)\diamond x
\leq \left(\Phi - \frac{1}{\int_a^b p(x)\diamond x} \int_a^b p(x)f(x)\diamond x\right) \left(\frac{1}{\int_a^b p(x)\diamond x} \int_a^b p(x)f(x)\diamond x - \varphi\right).
\]
(3.4)
Similarly, we have
\[
\frac{1}{\int_a^b p(x) \diamond_{\alpha} x} \int_a^b p(x) g^2(x) \diamond_{\alpha} x - \left( \frac{1}{\int_a^b p(x) \diamond_{\alpha} x} \int_a^b p(x) g(x) \diamond_{\alpha} x \right)^2 \\
\leq \left( \frac{1}{\int_a^b p(x) \diamond_{\alpha} x} \int_a^b p(x) g(x) \diamond_{\alpha} x \right) \left( \frac{1}{\int_a^b p(x) \diamond_{\alpha} x} \int_a^b p(x) g(x) \diamond_{\alpha} x - \gamma \right).
\]

Using (3.4) and (3.5) in (3.3), (3.2) implies
\[
\left| \int_a^b p(x) f(x) g(x) \diamond_{\alpha} x - \int_a^b p(x) f(x) \diamond_{\alpha} x \int_a^b p(x) g(x) \diamond_{\alpha} x \right| \\
\leq \left( \frac{1}{\int_a^b p(x) \diamond_{\alpha} x} \int_a^b p(x) f(x) \diamond_{\alpha} x \right)^{\frac{1}{2}} \left( \frac{1}{\int_a^b p(x) \diamond_{\alpha} x} \int_a^b p(x) f(x) \diamond_{\alpha} x - \varphi \right)^{\frac{1}{2}} \\
\times \left( \frac{1}{\int_a^b p(x) \diamond_{\alpha} x} \int_a^b p(x) g(x) \diamond_{\alpha} x \right)^{\frac{1}{2}} \left( \frac{1}{\int_a^b p(x) \diamond_{\alpha} x} \int_a^b p(x) g(x) \diamond_{\alpha} x - \gamma \right)^{\frac{1}{2}}.
\]

Applying the elementary inequality
\[4\beta\gamma \leq (\beta + \gamma)^2 \quad \text{for all } \beta, \gamma \in \mathbb{R},\]
we can state
\[
4 \left( \frac{1}{\int_a^b p(x) \diamond_{\alpha} x} \int_a^b p(x) f(x) \diamond_{\alpha} x \right) \left( \frac{1}{\int_a^b p(x) \diamond_{\alpha} x} \int_a^b p(x) f(x) \diamond_{\alpha} x - \varphi \right) \leq (\Phi - \varphi)^2
\]
(3.7)
and
\[
4 \left( \frac{1}{\int_a^b p(x) \diamond_{\alpha} x} \int_a^b p(x) g(x) \diamond_{\alpha} x \right) \left( \frac{1}{\int_a^b p(x) \diamond_{\alpha} x} \int_a^b p(x) g(x) \diamond_{\alpha} x - \gamma \right) \leq (\Gamma - \gamma)^2.
\]
(3.8)
Combining (3.6) with (3.7) and (3.8), we obtain (3.1). \qed

Example 3.2: If we let \( p(x) \equiv 1 \) on \( T \) in Theorem 3.1, then we obtain Theorem 1.1.

Example 3.3: If we let \( T = \mathbb{R} \) in Theorem 3.1, then we obtain the inequality
\[
\left| \int_a^b p(x) dx \int_a^b p(x) f(x) g(x) dx - \int_a^b p(x) f(x) dx \int_a^b p(x) g(x) dx \right| \\
\leq \frac{1}{4} (\Phi - \varphi) (\Gamma - \gamma) \left( \int_a^b p(x) dx \right)^2.
\]
This result can be found in Dragomir (2000, Theorem 1.1), where the constant \( \frac{1}{4} \) is also shown to be the best possible.
Example 3.4: If we let $\mathbb{T} = \mathbb{R}$ in Example 3.2, then we obtain the inequality
\[
\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx \right| \leq \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma).
\]

Example 3.5: If we let $\mathbb{T} = \mathbb{Z}$ and $\alpha = 1$ in Theorem 3.1, then we obtain the inequality
\[
\left| \frac{1}{n-m} \sum_{i=m}^{n-1} \left[ \alpha f_i g_i + (1 - \alpha) f_{i+1} g_{i+1} \right] - \frac{1}{(n-m)^2} \sum_{i=m}^{n-1} \left[ \alpha f_i + (1 - \alpha) f_{i+1} \right] \sum_{i=m}^{n-1} \left[ \alpha g_i + (1 - \alpha) g_{i+1} \right] \right| \leq \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma).
\]

Example 3.6: If we let $\mathbb{T} = \mathbb{Z}$ in Example 3.2, then we obtain the inequality
\[
\left| \frac{1}{n-m} \sum_{i=m}^{n-1} f_i g_i - \frac{1}{(n-m)^2} \sum_{i=m}^{n-1} f_i \sum_{i=m}^{n-1} g_i \right| \leq \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma).
\]

If, additionally, $\alpha = 1$, then we obtain the inequality
\[
\left| \frac{1}{n-m} \sum_{i=m}^{n-1} f_i g_i - \frac{1}{(n-m)^2} \sum_{i=m}^{n-1} f_i \sum_{i=m}^{n-1} g_i \right| \leq \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma).
\]

Example 3.7: If we let $\mathbb{T} = q^{\mathbb{N}_0}$ and $\alpha = 1$ in Theorem 3.1, then we obtain the inequality
\[
\left| \sum_{i=m}^{n-1} q^i p(q^i) \sum_{i=m}^{n-1} q^i p(q^i) f(q^i) g(q^i) - \sum_{i=m}^{n-1} q^i p(q^i) \sum_{i=m}^{n-1} q^i p(q^i) g(q^i) \right| \leq \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma) \left( \sum_{i=m}^{n-1} q^i p(q^i) \right)^2.
\]

Example 3.8: If we let $\mathbb{T} = q^{\mathbb{N}_0}$ in Example 3.2, then we obtain the inequality
\[
\left| \frac{q-1}{q^n-q^m} \sum_{i=m}^{n-1} q^i \left[ \alpha f(q^i) g(q^i) + (1 - \alpha) f(q^{i+1}) g(q^{i+1}) \right] - \left( \frac{q-1}{q^n-q^m} \right)^2 \sum_{i=m}^{n-1} q^i \left[ \alpha f(q^i) + (1 - \alpha) f(q^{i+1}) \right] \times \sum_{i=m}^{n-1} q^i \left[ \alpha g(q^i) + (1 - \alpha) g(q^{i+1}) \right] \right| \leq \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma).
\]
Proof

Let two Lipschitzian mappings with Lipschitz constants $L_1 > 0$ and $L_2 > 0$, respectively, i.e.,

$$|f(x) - y| \leq L_1 |x - y| \quad \text{and} \quad |g(x) - y| \leq L_2 |x - y|$$

for all $x, y \in [a, b] \cap \mathbb{T}$. Multiplying this inequality by $p(x)p(y) \geq 0$ and integrating over $[a, b] \times [a, b]$, we have

$$\left| \int_a^b \int_a^b p(x)p(y)(f(x) - f(y))(g(x) - g(y))\Diamond_{\alpha}x\Diamond_{\alpha}y \right|$$

$$\leq \int_a^b \int_a^b p(x)p(y) \|(f(x) - f(y))(g(x) - g(y))\| \Diamond_{\alpha}x\Diamond_{\alpha}y$$

$$\leq L_1 L_2 \int_a^b \int_a^b p(x)p(y) |(x - y)|^2 \Diamond_{\alpha}x\Diamond_{\alpha}y.$$ 

We also have

$$\frac{1}{2} \int_a^b \int_a^b p(x)p(y)(f(x) - f(y))(g(x) - g(y))\Diamond_{\alpha}x\Diamond_{\alpha}y$$

$$= \int_a^b p(x)\Diamond_{\alpha}x \int_a^b p(x)f(x)g(x)\Diamond_{\alpha}x - \int_a^b p(x)f(x)\Diamond_{\alpha}x \int_a^b p(x)g(x)\Diamond_{\alpha}x.$$

4 The case when both mappings are Lipschitzian

We now extend Theorem 1.2 to time scales.

**Theorem 4.1:** Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$ with $a < b$. Let $f, g \in C(\mathbb{T}, \mathbb{R})$ be two Lipschitzian mappings with Lipschitz constants $L_1 > 0$ and $L_2 > 0$, respectively, i.e.,

$$|f(x) - f(y)| \leq L_1 |x - y| \quad \text{and} \quad |g(x) - y| \leq L_2 |x - y|$$

for all $x, y \in [a, b] \cap \mathbb{T}$. If $p \in C(\mathbb{T}, [0, \infty))$, then

$$\int_a^b p(x)\Diamond_{\alpha}x \int_a^b p(x)f(x)g(x)\Diamond_{\alpha}x - \int_a^b p(x)f(x)\Diamond_{\alpha}x \int_a^b p(x)g(x)\Diamond_{\alpha}x$$

$$\leq L_1 L_2 \left[ \int_a^b p(x)\Diamond_{\alpha}x \int_a^b p(x)x^2\Diamond_{\alpha}x - \left( \int_a^b p(x)\Diamond_{\alpha}x \right)^2 \right],$$

and the inequality is sharp.

**Proof:** Using condition (4.1), we get

$$\left| (f(x) - f(y))(g(x) - y) \right| \leq L_1 L_2 |x - y|^2$$

for all $x, y \in [a, b] \cap \mathbb{T}$. Multiplying this inequality by $p(x)p(y) \geq 0$ and integrating over $[a, b] \times [a, b]$, we have

$$\left| \int_a^b \int_a^b p(x)p(y)(f(x) - f(y))(g(x) - g(y))\Diamond_{\alpha}x\Diamond_{\alpha}y \right|$$

$$\leq \int_a^b \int_a^b p(x)p(y) |(f(x) - f(y))(g(x) - g(y))| \Diamond_{\alpha}x\Diamond_{\alpha}y$$

$$\leq L_1 L_2 \int_a^b \int_a^b p(x)p(y) |(x - y)|^2 \Diamond_{\alpha}x\Diamond_{\alpha}y.$$
and
\[
\frac{1}{2} \int_a^b \int_a^b p(x)p(y)(x - y)^2 \, \alpha x \, \alpha y
= \int_a^b p(x) \alpha x \left( \int_a^b p(x) x^2 \, \alpha x \right) - \left( \int_a^b p(x) x \, \alpha x \right)^2,
\]

which completes the proof of inequality (4.2). Moreover, if we choose \( L_1, L_2 > 0 \), \( f(x) = L_1 x \) and \( g(x) = L_2 x \) for \( x \in \mathbb{T} \), then \( f \) and \( g \) are Lipschitzian with Lipschitz constants \( L_1 > 0 \) and \( L_2 > 0 \), respectively, and equality holds in (4.2) for any \( p \in C(\mathbb{T}, [0, \infty)) \). \( \Box \)

**Example 4.2:** If we let \( p(x) \equiv 1 \) on \( \mathbb{T} \) in Theorem 4.1, then we obtain the inequality
\[
\left| \frac{1}{b-a} \int_a^b f(x)g(x) \, \alpha x - \frac{1}{b-a} \int_a^b f(x) \, \alpha x \int_a^b g(x) \, \alpha x \right|
\leq L_1 L_2 \left[ \frac{1}{b-a} \int_a^b x^2 \, \alpha x - \left( \frac{1}{b-a} \int_a^b x \, \alpha x \right)^2 \right].
\]

**Example 4.3:** If we let \( \mathbb{T} = \mathbb{R} \) in Theorem 4.1, then we obtain Theorem 1.2.

**Example 4.4:** If we let \( \mathbb{T} = \mathbb{R} \) in Example 4.2, then we obtain the inequality
\[
\left| \frac{1}{b-a} \int_a^b f(x)g(x) \, dx - \frac{1}{b-a} \int_a^b f(x) \, dx \int_a^b g(x) \, dx \right|
\leq L_1 L_2 \frac{(b-a)^2}{12},
\]
which can be found in Dragomir (2000, Corollary 2.2).

**Example 4.5:** If we let \( \mathbb{T} = \mathbb{Z} \) and \( \alpha = 1 \) in Theorem 4.1, then we obtain the inequality
\[
\sum_{i=m}^{n-1} p_i \sum_{i=m}^{n-1} p_i f_i g_i - \sum_{i=m}^{n-1} p_i f_i \sum_{i=m}^{n-1} p_i g_i
\leq L_1 L_2 \left[ \sum_{i=m}^{n-1} p_i \sum_{i=m}^{n-1} p_i i^2 - \left( \sum_{i=m}^{n-1} p_i i \right)^2 \right].
\]

**Example 4.6:** If we let \( \mathbb{T} = \mathbb{Z} \) in Example 4.2, then we obtain the inequality
\[
\left| \frac{1}{n-m} \sum_{i=m}^{n-1} [\alpha f_i g_i + (1 - \alpha) f_{i+1} g_{i+1}] \right.
\leq L_1 L_2 \left[ \frac{(n-m)^2 - 1}{12} + \alpha (1 - \alpha) \right].
\]
If, additionally, $\alpha = 1$, then we obtain the inequality
\[
\left| \frac{1}{n-m} \sum_{i=m}^{n-1} f_i g_i - \frac{1}{(n-m)^2} \sum_{i=m}^{n-1} f_i \sum_{i=m}^{n-1} g_i \right| \leq L_1 L_2 \left( \frac{(n-m)^2 - 1}{12} \right).
\]

Note also that we have in the discrete case the same bound on the right-hand side than in the continuous case if and only if $\alpha = \frac{1}{2} - \frac{1}{\sqrt{6}}$ or $\alpha = \frac{1}{2} + \frac{1}{\sqrt{6}}$.

**Example 4.7:** If we let $\mathbb{T} = q^{\mathbb{N}_0}$ and $\alpha = 1$ in Theorem 4.1, then we obtain the inequality
\[
\left| \sum_{i=m}^{n-1} q^i p(q^i) \sum_{i=m}^{n-1} q^i p(q^i) f(q^i) g(q^i) - \sum_{i=m}^{n-1} q^i p(q^i) f(q^i) \sum_{i=m}^{n-1} q^i p(q^i) g(q^i) \right| \leq L_1 L_2 \left[ \sum_{i=m}^{n-1} q^i p(q^i) \sum_{i=m}^{n-1} q^i p(q^i) - \left( \sum_{i=m}^{n-1} q^i p(q^i) \right)^2 \right].
\]

**Example 4.8:** If we let $\mathbb{T} = q^{\mathbb{N}_0}$ in Example 4.2, then we obtain the inequality
\[
\left| \frac{q - 1}{q^n - q^m} \sum_{i=m}^{n-1} q^i \left[ \alpha f(q^i) g(q^i) + (1 - \alpha) f(q^{i+1}) g(q^{i+1}) \right] \right. - \left( \frac{q - 1}{q^n - q^m} \right)^2 \sum_{i=m}^{n-1} q^i \left[ \alpha f(q^i) + (1 - \alpha) f(q^{i+1}) \right] \sum_{i=m}^{n-1} q^i \left[ \alpha g(q^i) + (1 - \alpha) g(q^{i+1}) \right] \left| \leq L_1 L_2 \left[ \frac{q^{2m} + q^m + q^{2m}}{q^2 + q + 1} \left( \alpha + (1 - \alpha) q^2 \right) - \left( \frac{q^n + q^m}{q + 1} \right)^2 \left( \alpha + (1 - \alpha) q^2 \right) \right]. \right.
\]

If, additionally, $\alpha = 1$, then we obtain the inequality
\[
\left| \frac{q - 1}{q^n - q^m} \sum_{i=m}^{n-1} q^i f(q^i) g(q^i) - \left( \frac{q - 1}{q^n - q^m} \right)^2 \sum_{i=m}^{n-1} q^i f(q^i) \sum_{i=m}^{n-1} q^i g(q^i) \right| \leq L_1 L_2 \frac{(q^n - q^m)(q^{n+1} - q^m)}{(q^2 + q + 1)(q + 1)^2}.
\]

**5 The case when $f$ is $M$-$g$-Lipschitzian**

**Theorem 5.1:** Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$ with $a < b$. Let $f, g \in C(\mathbb{T}, \mathbb{R})$ be such that $f$ is $M$-$g$-Lipschitzian with $M > 0$, i.e.,
\[
|f(x) - f(y)| \leq M |g(x) - g(y)| \tag{5.1}
\]
for all \( x, y \in [a, b] \cap \mathbb{T} \). If \( p \in C(\mathbb{T}, [0, \infty)) \), then

\[
\left| \int_a^b p(x)\Diamond_\alpha x \int_a^b p(x)f(x)g(x)\Diamond_\alpha x - \int_a^b p(x)f(x)\Diamond_\alpha x \int_a^b p(x)g(x)\Diamond_\alpha x \right| \\
\leq M \left[ \int_a^b p(x)\Diamond_\alpha x \int_a^b p(x)g^2(x)\Diamond_\alpha x - \left( \int_a^b p(x)g(x)\Diamond_\alpha x \right)^2 \right], \tag{5.2}
\]

and the inequality is sharp.

**Proof:** Using condition (5.1), we get

\[
|p(x)f(x)g(x) - p(y)f(y)| \leq M|g(x) - g(y)|
\]

for all \( x, y \in [a, b] \cap \mathbb{T} \). Multiplying this inequality by \( p(x)p(y) \geq 0 \) and integrating over \([a, b] \times [a, b]\), we have

\[
\frac{1}{2} \left| \int_a^b \int_a^b p(x)p(y)(f(x) - f(y))(g(x) - g(y))\Diamond_\alpha x \Diamond_\alpha y \right| \\
\leq \frac{1}{2} \int_a^b \int_a^b p(x)p(y) |(f(x) - f(y))(g(x) - g(y))| \Diamond_\alpha x \Diamond_\alpha y \\
\leq \frac{M}{2} \int_a^b \int_a^b p(x)p(y)((g(x) - g(y))^2 \Diamond_\alpha x \Diamond_\alpha y \\
= M \left[ \int_a^b p(x)\Diamond_\alpha x \int_a^b p(x)g^2(x)\Diamond_\alpha x - \left( \int_a^b p(x)g(x)\Diamond_\alpha x \right)^2 \right],
\]

which completes the proof of inequality (5.2). Moreover, if we choose \( f(x) = Mx \) with \( M > 0 \) and \( g(x) = x \), then \( f \) is \( M \)-Lipschitzian and equality holds in (5.2) for any \( p \in C(\mathbb{T}, [0, \infty)) \).

**Example 5.2:** If we let \( p(x) \equiv 1 \) on \( \mathbb{T} \) in Theorem 5.1, then we obtain the inequality

\[
\frac{1}{b-a} \int_a^b f(x)g(x)\Diamond_\alpha x - \frac{1}{b-a} \int_a^b f(x)\Diamond_\alpha x \frac{1}{b-a} \int_a^b g(x)\Diamond_\alpha x \\
\leq M \left[ \frac{1}{b-a} \int_a^b g^2(x)\Diamond_\alpha x - \left( \frac{1}{b-a} \int_a^b g(x)\Diamond_\alpha x \right)^2 \right].
\]

**Example 5.3:** If we let \( \mathbb{T} = \mathbb{R} \) in Theorem 5.1, then we obtain Theorem 1.3.

**Example 5.4:** If we let \( \mathbb{T} = \mathbb{R} \) in Example 5.2, then we obtain the inequality

\[
\frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \frac{1}{b-a} \int_a^b g(x)dx \\
\leq M \left[ \frac{1}{b-a} \int_a^b g^2(x)dx - \left( \frac{1}{b-a} \int_a^b g(x)dx \right)^2 \right],
\]

which can be found in Dragomir (2000, Remark 4.2).
Example 5.5: If we let $\mathbb{T} = \mathbb{Z}$ and $\alpha = 1$ in Theorem 5.1, then we obtain the inequality

$$\left| \sum_{i=m}^{n-1} p_i \sum_{i=m}^{n-1} p_i f_i g_i - \sum_{i=m}^{n-1} p_i f_i \sum_{i=m}^{n-1} p_i g_i \right| \leq M \left[ \left| \sum_{i=m}^{n-1} p_i \sum_{i=m}^{n-1} p_i g_i^2 - \left( \sum_{i=m}^{n-1} p_i g_i \right)^2 \right| \right].$$

Example 5.6: If we let $\mathbb{T} = \mathbb{Z}$ in Example 5.2, then we obtain the inequality

$$\left| \sum_{i=m}^{n-1} f_i g_i \right| \leq \left( \frac{1}{n-m} \sum_{i=m}^{n-1} \left[ \alpha f_i g_i + (1 - \alpha) f_{i+1} g_{i+1} \right] \right) \leq M \left[ \frac{1}{n-m} \sum_{i=m}^{n-1} \left[ \alpha g_i^2 + (1 - \alpha) g_{i+1}^2 \right] \right].$$

If, additionally, $\alpha = 1$, then we obtain the inequality

$$\left| \sum_{i=m}^{n-1} f_i g_i - \frac{1}{(n-m)^2} \sum_{i=m}^{n-1} f_i \sum_{i=m}^{n-1} g_i \right| \leq M \left[ \frac{1}{n-m} \sum_{i=m}^{n-1} \left[ \alpha g_i^2 + (1 - \alpha) g_{i+1}^2 \right] \right].$$

Example 5.7: If we let $\mathbb{T} = q^\mathbb{N}_0$ and $\alpha = 1$ in Theorem 5.1, then we obtain the inequality

$$\sum_{i=m}^{n-1} q^i p(q^i) \sum_{i=m}^{n-1} q^i p(q^i) f(q^i) g(q^i) - \sum_{i=m}^{n-1} q^i p(q^i) f(q^i) \sum_{i=m}^{n-1} q^i p(q^i) g(q^i) \leq M \left[ \sum_{i=m}^{n-1} q^i p(q^i) \sum_{i=m}^{n-1} q^i p(q^i) g^2(q^i) - \left( \sum_{i=m}^{n-1} q^i p(q^i) g(q^i) \right)^2 \right].$$

Example 5.8: If we let $\mathbb{T} = q^\mathbb{N}_0$ in Example 5.2, then we obtain the inequality

$$\left| \sum_{i=m}^{n-1} q^i \left[ \alpha f(q^i) g(q^i) + (1 - \alpha) f(q^{i+1}) g(q^{i+1}) \right] \right| \leq M \left[ \sum_{i=m}^{n-1} q^i \left[ \alpha g(q^i)^2 + (1 - \alpha) g(q^{i+1})^2 \right] \right].$$
If, additionally, $\alpha = 1$, then we obtain the inequality

$$\left| \frac{q - 1}{q^n - q^m} \sum_{i=1}^{n-1} q^i f(q^i)g(q^i) - \left( \frac{q - 1}{q^n - q^m} \sum_{i=1}^{n-1} q^i f(q^i) \right) \left( \frac{q - 1}{q^n - q^m} \sum_{i=1}^{n-1} q^i g(q^i) \right) \right| \leq M \left[ \frac{q - 1}{q^n - q^m} \sum_{i=1}^{n-1} q^i g^2(q^i) - \left( \frac{q - 1}{q^n - q^m} \sum_{i=1}^{n-1} q^i g(q^i) \right)^2 \right].$$

**References**


