Dynamic cobweb models with conformable fractional derivatives

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\begin{abstract}
In this paper, basic and more realistic dynamic cobweb models are developed in terms of conformable fractional derivatives. The general solutions and stability criteria for the proposed models are given. Moreover, the developed models are illustrated with examples on several time scales.
\end{abstract}

\section{1. Introduction}
The cobweb model, one of the basic models in economic dynamics, has been widely used to illustrate supply and demand. The model describes the equilibrium price in a market with a time lag between supply and demand decisions. Mostly, it is used in agricultural markets with a time lag between planting and harvesting. We refer the reader to \cite{1–3} for detailed information on both discrete and continuous time cobweb models. Dynamic cobweb models, i.e., unification of both discrete and continuous time cobweb models, have been developed in \cite{4–6}. In 2014, the authors in \cite{7} introduced a new fractional derivative definition which has a local kernel, called conformable fractional derivative. Due to advantages of enabling the simple analytical calculation, conformable fractional derivatives have become popular rapidly. Relative simplicity in calculation and the possibility of obtaining more realistic results are the most important reasons to prefer conformable fractional derivatives for modeling economical phenomena as well as real-world phenomena. For readers interested in various definitions of fractional derivatives and recent history of fractional calculus, we refer to the paper \cite{8}. In \cite{9}, the authors studied a conformable fractional-order cobweb model with continuous supply and demand flow. They constructed a conformable fractional model and gave stability conditions. Recently, several articles on conformable fractional calculus on time scales \cite{10–12} have been published. Moreover, some recent papers concern applications of conformable fractional calculus on time scales. For example, in \cite{13}, the authors studied conformable fractional Sturm–Liouville equations, and in \cite{14}, conformable fractional Dirac systems on time scales were investigated. In this paper, motivated by \cite{9} and under the inspiration of previous studies, we develop a generalized dynamic cobweb model with conformable fractional derivatives on time scales.

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The rest of the paper is organized as follows. Some basic definitions and theorems on conformable fractional calculus and cobweb theory on time scales are presented in the next section. In the third section, the price equilibria of both the basic dynamic cobweb model and the more realistic dynamic cobweb model containing an additional parameter $c$ in the supply equation are investigated and illustrated with examples on different time scales. In this section, all graphs are plotted by using Mathematica 11 software. In the last section, some conclusions are offered.

2. Preliminaries

In this section, we give a brief introduction to the time scales theory and refer the interested reader to the monograph [15]. A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. For $a, b \in \mathbb{T}$, we define the closed interval $[a, b]_\mathbb{T} = [a, b] \cap \mathbb{T}$. For $t \in \mathbb{T}$, the forward jump operator $\sigma$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, where $\inf \emptyset = \sup \mathbb{T}$. If $\sigma(t) > t$, then we say that $t$ is right-scattered. Additionally, if $\sigma(t) = t$, then $t$ is called right-dense. The forward graininess function $\mu$ is defined by $\mu(t) = \sigma(t) - t$ for all $t \in \mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum $m$, then $\mathbb{T}^* = \mathbb{T} - m$. Otherwise, $\mathbb{T}^* = \mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is called delta differentiable at $t \in \mathbb{T}^*$ if there exists a number $f^\Delta(t)$ such that for each $\varepsilon > 0$, there exists a neighborhood $U$ of $t$ such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U$. $f^\Delta(t)$ is called the delta derivative of $f$ at $t$, and we say that $f$ is delta differentiable if $f$ is delta differentiable at all $t \in \mathbb{T}^*$. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted as $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$. A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive given that

$$1 + \mu(t)p(t) \neq 0 \text{ for all } t \in \mathbb{T}^*.$$ 

The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$. The time scales exponential function is defined for $p \in \mathcal{R}$ as

$$e_p(t, s) = \exp \left( \int_s^t \xi_h(\tau)p(\tau)\Delta \tau \right) \text{ for } s, t \in \mathbb{T},$$

where $\xi_h$ is the cylinder transformation defined by

$$\xi_h(z) = \frac{1}{h} \log(1 + zh) \text{ for } h > 0, h \neq -\frac{1}{z} \text{ and } \xi_0(z) = z.$$ 

Note that $e_p(\cdot, t_0)$ is the unique solution of the initial value problem

$$y^\Delta(t) = p(t)y, \quad y(t_0) = 1,$$

where $p \in \mathcal{R}$ and $t_0 \in \mathbb{T}$. Also note that

$$\int_{t_0}^t f(\tau)\Delta \tau = \mu(t)f(t).$$

**Theorem 1** (See [15, Page 64]). If $p \in \mathcal{R}$ and $a, b, c \in \mathbb{T}$, then

$$\int_a^b p(\tau)e_p(c, \sigma(\tau))\Delta \tau = e_p(c, a) - e_p(c, b).$$

**Definition 2** (See [12]). Let $f : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^*$, and $\alpha \in (0, 1]$. For $t > 0$, we define $T_\alpha(f)(t)$ to be the number (provided it exists) with the property that, given any $\varepsilon > 0$, there is a neighborhood $U$ of $t$ such that

$$|f(\sigma(t)) - f(s) - T_\alpha(f)(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U$. $T_\alpha(f)(t)$ is called the conformable fractional derivative of $f$ of order $\alpha$ at $t$, and we define the conformable fractional derivative at $0$ as $T_\alpha(f)(0) = \lim_{t \to 0^+} T_\alpha(f)(t)$.

The following theorem provides important properties of the conformable fractional derivative on time scales.

**Theorem 3** (See [12]). Let $\alpha \in (0, 1]$ and $\mathbb{T}$ be a time scale. Assume $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^*$. The following properties hold.

1. If $f$ is conformable fractional differentiable of order $\alpha$ at $t > 0$, then $f$ is continuous at $t$.
2. If $f$ is continuous at $t$ and $t$ is right-scattered, then $f$ is conformable fractional differentiable of order $\alpha$ at $t$ with

$$T_\alpha(f)(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}t^{1-\alpha}.$$
(3) If $t$ is right-dense, then $f$ is conformable fractional differentiable of order $\alpha$ at $t$ if and only if the limit $\lim_{s \to t} \frac{f(t) - f(s)}{t - s} t^{1-\alpha}$ exists as a finite number. In this case,

$$T_\alpha(f)(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} t^{1-\alpha}.$$ (4)

If $f$ is conformable fractional differentiable of order $\alpha$ at $t$, then

$$f(\sigma(t)) = f(t) + \mu(t) t^{\alpha-1} T_\alpha(f)(t).$$

The conformable fractional integral on time scales has been introduced in [12] as follows.

Definition 4. Let $f \in C_\alpha(T, \mathbb{R})$. Then the conformable fractional integral of $f$, $\alpha \in (0, 1]$, is defined by

$$F_\alpha(t) = \int_a^t f(t) \Delta^\alpha t := \int_a^t f(t) t^{\alpha-1} \Delta t. \tag{1}$$

Note that the Cauchy conformable integral on time scales is defined in [12] by

$$\int_a^b f(t) \Delta^\alpha t := F_\alpha(b) - F_\alpha(a). \tag{2}$$

3. Conformable fractional cobweb model on time scales

A dynamic cobweb model has been given in [4] as

$$\begin{align*}
D(\sigma(t)) &= a(t) + b(t) (p(t) + p^4(t)), \\
S(\sigma(t)) &= a_1(t) + b_1(t) p(t), \\
D(t) &= S(t),
\end{align*} \tag{3}$$

where $a, a_1, b, b_1 : T \to \mathbb{R}, b(t) \neq b_1(t)$ and $b(t) \neq 0$. We introduce the basic fractional-order dynamic cobweb model as

$$\begin{align*}
D(\sigma(t)) &= a(t) + b(t) (p(t) + T_\alpha(p)(t)), \\
S(\sigma(t)) &= a_1(t) + b_1(t) p(t), \\
D(t) &= S(t),
\end{align*} \tag{4}$$

where the expected price $p_e = \frac{a_1(t) - a(t)}{b(t) - b_1(t)}$ is assumed to be constant on $T$, $0 < t < T^\alpha$, and $T_\alpha(f)$ is the conformable derivative with $0 < \alpha \leq 1$.

Theorem 5. Assume $a, b, a_1, b_1 : T \to \mathbb{R}, p_0 \in \mathbb{R}, b(t) \neq 0$ and $b(t) \neq b_1(t)$ for all $t \in T$. Define $h_\alpha(t) := \frac{b(t) - b_1(t)}{b(t) - b(t)} t^{\alpha-1}$ for all $t \in T$ and assume $h_\alpha \in C(T)$. Then the unique solution $p$ of (4) satisfying $p(t_0) = p_0$ is

$$p(t) = (p_0 - p_e)e_{h_\alpha}(t, t_0) + p_e. \tag{5}$$

Proof. If $p$ solves (4), then we have

$$a(t) + b(t) (p(t) + T_\alpha(p)(t)) = a_1(t) + b_1(t) p(t). \tag{6}$$

Rewriting (6), we get

$$T_\alpha(p)(t) = \frac{b_1(t) - b(t)}{b(t)} p(t) + \frac{a_1(t) - a(t)}{b(t)}. \tag{7}$$

Multiplying both sides of (7) by the integrating factor $e_{\int h_\alpha(s, t_0)}$ and using the conformable integral definitions (1) and (2), we find the general solution of (7) as

$$p(t) = e_{h_\alpha}(t, t_0) p_0 + \int_{t_0}^t e_{h_\alpha}(\sigma(s), t) \frac{a_1(s) - a(s)}{b(s)} \Delta^\alpha s = e_{h_\alpha}(t, t_0) p_0 - p_e \int_{t_0}^t e_{h_\alpha}(t, \sigma(s)) h_\alpha(s) \Delta s = e_{h_\alpha}(t, t_0) p_0 - p_e \left( e_{h_\alpha}(t, t_0) - e_{h_\alpha}(t, t) \right) = (p_0 - p_e) e_{h_\alpha}(t, t_0) + p_e,$$

where we have used Theorem 1. Conversely, $p$ given by (5) can easily be seen to be a solution of (4). \qed
Theorem 6. If $h_a \in \mathcal{R}$ and
\[
\lim_{t \to \infty} \int_{t_0}^{t} \psi_{\mu}(h_a(\tau)) \Delta \tau = -\infty,
\]
where
\[
\psi_h(z) := \begin{cases} \log \left| \frac{1 + h z}{h z} \right| & \text{if } h > 0 \quad \text{for } z \in \mathbb{C} \setminus \left\{ -\frac{1}{h} \right\}, \\ \text{Re}(z) & \text{if } h = 0 \end{cases}
\]
then the price of (4) converges to its equilibrium value $p_e$.

Proof. From [16, Proof of Theorem 3.4] it is known that $h_a \in \mathcal{R}$ implies
\[
|e_{h_a}(t, t_0)| = \exp \left( \int_{t_0}^{t} \psi_{\mu}(h_a(\tau)) \Delta \tau \right).
\]
Using the assumptions, we get
\[
0 \leq |e_{h_a}(t, t_0)| = \exp \left( \int_{t_0}^{t} \psi_{\mu}(h_a(\tau)) \Delta \tau \right) \to 0
\]
and thus $e_{h_a}(t, t_0) \to 0$ as $t \to \infty$. By (5), $p(t) \to p_e$ as $t \to \infty$. □

Next, we will investigate the price function (5) on different time scales.

Example 7. For $\mathbb{T} = \mathbb{R}$, in [9, Theorem 2] the authors showed that the conformable fractional cobweb model
\[
\begin{cases}
D(t) = a + b \left( p(t) + T_a(p)(t) \right), \\
S(t) = a_1 + b_1 p(t), \\
D(t) = S(t),
\end{cases}
\]
(9)
where $T_a(f)$ is the conformable fractional derivative with $0 < \alpha \leq 1$, has a solution in the form
\[
p(t) = (p_0 - p_e) e^{\left(\frac{b_1}{b-a}\right)(t^\alpha - 0^\alpha)} + p_e,
\]
(10)
where $p_e = \frac{a_1 - a}{b-a}$. Note that (9) and (10) are the continuous cases of (4) and (5), respectively, when $a, a_1, b, b_1$ are constant.

Example 8. If $\mathbb{T} = [0.001, 1] \cup [2, 3] \cup [4, 5] \cup [6, 7]$, then (4) is in the form
\[
\begin{cases}
D(\sigma(t)) = a(t) + b(t) \left( p(t) + T_a(p)(t) \right), \\
S(\sigma(t)) = a_1(t) + b_1(t) p(t), \\
D(t) = S(t),
\end{cases}
\]
(11)
Suppose $a(t) \equiv 80, a_1(t) \equiv -10, b(t) \equiv -4, b_1(t) \equiv 2$ are constant functions and $p_0 = 18$. By Theorem 5, the general solution of (11) for $\alpha = 1$ is
\[
p(t) = \begin{cases} 3e^{\frac{3}{2}(t-0.001)} + 15, & 0.001 \leq t \leq 1, \\
-0.3452e^{\frac{3}{2}(t-2)} + 15, & 2 \leq t \leq 3, \\
0.0374e^{\frac{3}{2}(t-4)} + 15, & 4 \leq t \leq 5, \\
-0.0042e^{\frac{3}{2}(t-6)} + 15, & 6 \leq t \leq 7.
\end{cases}
\]
(12)
The illustration of the price function of (11) for different values of $\alpha \in (0, 1]$ is presented in Fig. 1 on the time scale $\mathbb{T} = [0.001, 1] \cup [2, 3] \cup [4, 5] \cup [6, 7]$.

Example 9. If $\mathbb{T} = \mathbb{N}$, then (4) can be written in the form
\[
\begin{cases}
D(t + 1) = a(t) + b(t) \left( p(t) + T_a(p)(t) \right), \\
S(t + 1) = a_1(t) + b_1(t) p(t), \\
D(t) = S(t).
\end{cases}
\]
(13)
By Theorem 5, the general solution of (13) is
\[
p(t) = (p_0 - p_e) \prod_{\tau = t_0}^{t-1} \left( 1 + \frac{b_1(\tau) - b(\tau)}{b(\tau)} \right)^{\alpha - 1} + p_e,
\]
(14)
where $p_e = \frac{a_1(t) - a(t)}{b(\tau) - b(\tau)}$. 

(a) If $0 < \alpha < 1$, then the fractional-order solutions are obtained for \((13)\). By Theorem 5, the price function \((14)\) converges to its equilibrium price, if it satisfies 
\[ \frac{b_1(t) - b(t)}{b(t)} t^{\alpha - 1} < 0 \] for all $t \in T$. If $\alpha = 1$, $a(t) \equiv a$, $b(t) \equiv b$, $a_1(t) \equiv a_1$, $b_1(t) \equiv b_1$, and $t_0 = 0$, then \((14)\) turns into the solution of the difference equation presented in \([3]\),
\[
p(t) = (p_0 - p_e) \left( \frac{b_1}{b} \right)^t + p_e,
\]
where $p_e = \frac{a_1 - a}{b - b_1}$.

(b) The price function \((14)\) is illustrated in Fig. 2 with the particular functions $a(t) \equiv 80$, $a_1(t) \equiv -10$, $b(t) \equiv -4$, $b_1(t) \equiv 2$, with $p_0 = 18$ for $t_0 > 0$ and different orders of $0 < \alpha \leq 1$.

**Example 10.** If $T = q^{t_0}$, then \((4)\) turns into
\[
\begin{align*}
D(qt) &= a(t) + b(t) \left( p(t) + \frac{p(qt) - p(t)}{(q-1)q^t} t^{1-\alpha} \right), \\
S(qt) &= a_1(t) + b_1(t)p(t), \\
D(t) &= S(t).
\end{align*}
\]
By Theorem 5, the general solution of \((15)\) is
\[
p(t) = (p_0 - p_e) \prod_{\tau \in [t_0, t]_T} \left( 1 + (q - 1) \frac{b_1(\tau) - b(\tau)}{b(\tau)} \tau^\alpha \right) + p_e.
\]

(a) For $\alpha = 1$, \((16)\) coincides with the solution of the $q$-difference equation presented in \([4]\),
\[
p(t) = (p_0 - p_e) \prod_{\tau \in [t_0, t]_T} \left( 1 + (q - 1) \frac{b_1(\tau)}{b(\tau)} \tau \right) + p_e.
\]
where \( p_e = \frac{a_1(t) - a(t)}{b(t) - b_1(t)} \). If \( 0 < \alpha < 1 \), then the fractional-order solutions are obtained for (15). By Theorem 6, the price function (16) converges to its equilibrium price, since \( 1 - \frac{2}{(q-1)^\alpha} < \frac{b(t)}{b(t)} < 1 \) for all \( t \in \mathbb{T} \).

(b) By considering \( q = 1.5 \), (16) is illustrated in Fig. 3 with the same functions and initial value as in the previous examples, \( a(t) \equiv 80, a_1(t) \equiv -10, b(t) \equiv -4, b_1(t) \equiv 2 \), with \( p_0 = 18 \) for \( t_0 > 0 \) and \( 0 < \alpha \leq 1 \).

In [9], the authors construct a more realistic model on continuous time by considering the expectations in the form of \( p(t) + cT_a(p(t)) \) for \( c \in \mathbb{R} \). We define the fractional-order dynamic cobweb model as

\[
\begin{align*}
D(\sigma(t)) &= a(t) + b(t)p(t), \\
S(\sigma(t)) &= a_1(t) + b_1(t)(p(t) + cT_a(p(t))), \\
D(t) &= S(t),
\end{align*}
\]

where \( p_e = \frac{a_1(t) - a(t)}{b(t) - b_1(t)} \) is assumed to be a constant on \( \mathbb{T} \), \( 0 < t \in \mathbb{T}^\alpha \), and \( T_a(f) \) is the conformable fractional derivative with \( 0 < \alpha \leq 1 \). In (17), for \( c > 0 \), expectations follow the same pattern, i.e., if the price increases, then producers expect a further increase, and if the price decreases, then producers expect a further decrease. Contrary, when \( c < 0 \), expectations follow the opposite direction.

**Theorem 11.** Assume \( a, b, a_1, b_1 : \mathbb{T} \rightarrow \mathbb{R}, p_0 \in \mathbb{R}, b(t) \neq 0 \) and \( b(t) \neq b_1(t) \) for all \( t \in \mathbb{T} \). Define \( g_a(t) = \frac{b(t) - b_1(t)}{b(t)}t^{\alpha - 1} \) for all \( t \in \mathbb{T} \) and assume \( g_a \in \mathbb{R} \). Then the unique solution \( p \) of (17) satisfying \( p(t_0) = p_0 \) is

\[
p(t) = (p_0 - p_e)g_a(t, t_0) + p_e. \tag{18}
\]

**Proof.** The proof is similar to the one of Theorem 5 by using the integrating factor \( e^{\int_{\sigma(t)} g_a(t)} \). □

**Theorem 12.** If \( g_a \in \mathbb{R} \) and

\[
\lim_{t \to -\infty} \int_{t_0}^t \Psi_{\mu(\tau)}(g_a(\tau))\Delta \tau = -\infty,
\]

where \( \Psi_{\mu}(z) \) is as in (8), then the price of (17) converges to its equilibrium value \( p_e \).

**Proof.** The proof is similar to the one of Theorem 6. □

**Example 13.** For \( \mathbb{T} = \mathbb{R} \), in [9, Theorem 4], the authors proved that the model

\[
\begin{align*}
D(t) &= a + bp(t), \\
S(t) &= a_1 + b_1(p(t) + cT_a(p(t))), \\
D(t) &= S(t),
\end{align*}
\]

where \( T_a(f) \) is the conformable fractional derivative with \( 0 < \alpha \leq 1 \), has a solution in the form

\[
p(t) = (p_0 - p_e)e^{\frac{b_1 - b}{bT_a}(t^{\alpha} - t_0^{\alpha})} + p_e, \tag{20}
\]

where \( p_e = \frac{a_1(t) - a(t)}{b(t) - b_1(t)} \). Note that (19) and (20) are the continuous cases of (17) and (18), respectively, when \( a, a_1, b, b_1 \) are constant. In particular, the behavior of the model for some values of \( c > 0 \) and \( c < 0 \) is illustrated in [9, Example 4].
Example 14. If \( T = [0.001, 1] \cup [2, 3] \cup [4, 5] \cup [6, 7] \), then (17) is in the form
\[
\begin{align*}
D(\sigma(t)) &= a(t) + b(t)p(t), \\
S(\sigma(t)) &= a_1(t) + b_1(t)(p(t) + cT_\alpha(p(t))), \\
D(t) &= S(t).
\end{align*}
\]  

(21)

Suppose \( p_0 = 18 \) and \( a(t) \equiv 80, a_1(t) \equiv -10, b(t) \equiv -4, b_1(t) \equiv 2 \) are constant functions. In particular, the general solution of (21) for \( \alpha = 1 \) and \( c = 5 \) is
\[
p(t) = \begin{cases} 
3e^{\frac{2}{\alpha}(t-0.001)} + 15, & 0.001 \leq t \leq 1, \\
0.6590e^{\frac{2}{\alpha}(t-2)} + 15, & 2 \leq t \leq 3, \\
0.1447e^{\frac{2}{\alpha}(t-4)} + 15, & 4 \leq t \leq 5, \\
0.0318e^{\frac{2}{\alpha}(t-6)} + 15, & 6 \leq t \leq 7.
\end{cases}
\]  

(22)

The illustration of the price function of (21) for different values of \( \alpha \in (0, 1) \) is presented in Figs. 4, 5, and 6, where \( c = 1, c = 5, \) and \( c = -10 \) respectively, on \( T = [0.001, 1] \cup [2, 3] \cup [4, 5] \cup [6, 7] \).

Example 15. If \( T = \mathbb{N} \), then (17) can be written in the form
\[
\begin{align*}
D(t + 1) &= a(t) + b(t)p(t), \\
S(t + 1) &= a_1(t) + b_1(t)(p(t) + cT_\alpha(p(t))), \\
D(t) &= S(t).
\end{align*}
\]  

(23)
By Theorem 11, the general solution of (23) is

\[ p(t) = (p_0 - p_e) \prod_{\tau=t_0}^{t-1} \left( 1 + \frac{b(\tau) - b_1(\tau)}{b_1(\tau)c} \tau^{\alpha-1} \right) + p_e, \tag{24} \]

where \( p_e = \frac{a_1(t) - a(t)}{b(t) - b_1(t)} \). By Theorem 12, the price function (24) converges to its equilibrium price, if it satisfies \(-2 < \frac{b(t) - b_1(t)}{b(t)c} t^{\alpha-1} < 0\) for all \( t \in T \).

For \( c = 1 \) and \( c = 5 \), the price function (24) is illustrated in Figs. 7 and 8 respectively, with the particular functions \( a(t) \equiv 80, a_1(t) \equiv -10, b(t) \equiv -4, b_1(t) \equiv 2, \) with \( p_0 = 18 \) for \( t_0 > 0 \) and different orders of \( 0 < \alpha \leq 1 \). The behavior of (24) is visualized in Fig. 9 for \( c = -10 \).

**Example 16.** If \( T = q^{n_0} \), then (17) is in the form

\[
\begin{align*}
D(qt) &= a(t) + b(t)p(t), \\
S(qt) &= a_1(t) + b_1(t) \left( p(t) + c \frac{b(qt) - p(t)}{b_1(t) c} t^{1-\alpha} \right), \\
D(t) &= S(t).
\end{align*}
\tag{25}
\]

By Theorem 11, the general solution of (25) is

\[ p(t) = (p_0 - p_e) \prod_{\tau \in \{q^n, n \geq 0, t\}} \left( 1 + (q - 1) \frac{b(\tau) - b_1(\tau)}{b_1(\tau)c} \tau^{\alpha} \right) + p_e, \tag{26} \]

where \( p_e = \frac{a_1(t) - a(t)}{b(t) - b_1(t)} \). By Theorem 12, the price function (26) converges to its equilibrium price, since \(-\frac{2}{(q-1)^{\alpha}} < \frac{b(t) - b_1(t)}{b(t)c} < 0\) for all \( t \in T \).
Fig. 8. Graph of (24) for different values of $\alpha$ with $c = 5$.

Fig. 9. Graph of (24) for different values of $\alpha$ with $c = -10$.

By considering $q = 1.2$, the graph of (26) is presented for $c = 0.5$ and $c = 1$ in Figs. 10 and 11, respectively, with the same functions as in the previous examples, $a(t) \equiv 80$, $a_1(t) \equiv -10$, $b(t) \equiv -4$, $b_1(t) \equiv 2$, with $p_0 = 18$ for $t_0 > 0$ and $0 < \alpha \leq 1$. In Fig. 12, the graph of (26) is given to illustrate the behavior of the equation for $c = -0.5$.

4. Conclusion

In this study, we investigate a fractional analogue of the well-known supply demand model on time scales. Also, stability conditions for the price function around the equilibrium price are given. It is seen that the solution of the dynamic cobweb model involving local fractional-order derivatives can be calculated analytically. For $\alpha = 1$, the results coincide with well-known results from the literature, and when $0 < \alpha < 1$, local fractional-order derivatives of the price function are obtained. In the more realistic model, it is seen that when the parameter $c$ is positive, expectations are of extrapolative kind. On the other hand, when $c$ is negative, producers expect the opposite tendency of the current price trend. Our results also cover the results of [9] when $\mathbb{T} = \mathbb{R}$. Therefore, this study is a generalization of the dynamic cobweb model to noninteger-order derivatives for arbitrary time scales. For further studies, dynamic cobweb model with conformable fractional derivatives can be improved to accommodate more realistic cases.

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Fig. 10. Graph of (26) for different values of $\alpha$ with $c = 0.5$.

Fig. 11. Graph of (26) for different values of $\alpha$ with $c = 1$.

Fig. 12. Graph of (26) for different values of $\alpha$ with $c = -0.5$.

References


