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DISCRETE REID ROUNDABOUT THEOREMS

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ABSTRACT. We present a Reid Roundabout Theorem that relates positive definiteness of a discrete quadratic functional to disconjugacy of a linear Hamiltonian difference system. This main result applies to the so-called singular case, and it does not require a controllability assumption on the system. First, we consider the case of general self-adjoint boundary conditions. Furthermore, a simpler result is derived for the case of separated boundary conditions.

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1. INTRODUCTION AND PRELIMINARIES

Throughout we assume that $N, n \in \mathbb{N}$, that A_k, B_k, C_k are given real $n \times n$-matrices for $k \in J = [0, N] \cap \mathbb{Z}$ with

(A) $\qquad \tilde{A}_k = (I - A_k)^{-1}$ exists and B_k, C_k are symmetric for $k \in J$,

and that R and S are real $2n \times 2n$-matrices such that

(A$_1$) $\qquad S$ is symmetric.

We consider discrete quadratic functionals $\mathcal{F}$ of the form

$$\mathcal{F}(x) = \mathcal{F}_0(x) + \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T S \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}$$

where

$$\mathcal{F}_0(x) = \sum_{k=0}^{N} \left\{ x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k \right\}.$$

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We say that $\mathcal{F}$ is *positive definite* if $\mathcal{F}(x) > 0$ holds for all $x \neq 0$ (i.e., $x_k \neq 0$ for some $k \in J^* = [0, N+1] \cap \mathbb{Z}$) such that (x, u) is $\mathcal{F}$-admissible, i.e.,

$$\Delta x_k = A_k x_{k+1} + B_k u_k \text{ for } k \in J, \quad \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} \in \operatorname{Im} R^T \tag{1.1}$$

holds, where $\Delta x_k = x_{k+1} - x_k$ and where Im denotes the image of a matrix. Reid Roundabout Theorems, i.e., characterizations of positive definiteness of $\mathcal{F}$, have been given e.g. in [1, 6] (for invertible B_k) and in [3, 5] (for arbitrary symmetric B_k). An up-to-date monograph on the subject is the recently appeared book by Ahlbrandt and Peterson [2]. Note that the case of invertible B_k does not cover such important functionals related to Sturm-Liouville difference equations of higher order. In the present work, however, we require only symmetry of B_k (see our assumption (A)). The **purpose** of this paper is to present a Reid Roundabout Theorem that **does not** require an additional **controllability** assumption as is needed in [3, Theorem 3]. This we will do for the general case described above as well as for the case of separated boundary conditions, i.e., if there exist real $n \times n$-matrices R_0, R_{N+1}, S_0, and S_{N+1} such that

$$R = \begin{pmatrix} R_0 & 0 \\ 0 & R_{N+1} \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} -S_0 & 0 \\ 0 & S_{N+1} \end{pmatrix} \text{ with } S_0, S_{N+1} \text{ symmetric.} \tag{A$_2$}$$

For this case our corresponding Reid Roundabout Theorem will be in some sense "better" than the one for the general case.

Let us introduce some terminology that is needed below. We deal with the linear Hamiltonian difference system

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A_k^T u_k, \tag{H}$$

for $k \in J$. The *principal solution* (X, U) of (H) (at 0) is the matrix solution (X, U) of (H) (i.e., X_k and U_k are $n \times n$-matrices solving (H)) that satisfies the initial conditions $X_0 = 0$ and $U_0 = I$. Note that it exists uniquely due to our assumption (A). A *conjoined basis* (X, U) of (H) (i.e., a matrix solution (X, U) of (H) satisfying $X_0^T U_0 = U_0^T X_0$ and $\operatorname{rank}(X_0^T, U_0^T) = n$) is said to have no *focal point* in $(k, k+1]$ provided

$$\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k \quad \text{and} \quad D_k = X_k X_{k+1}^\dagger \tilde{A}_k B_k \geq 0$$

holds, where Ker denotes the kernel of a matrix, "≥ 0" means positive semidefinitess, and where we use $\dagger$ to denote the Moore-Penrose inverse of a matrix. Finally, the system (H) is called *disconjugate* on J^* if

$$\text{the principal solution of (H) has no focal points in } (0, N+1].$$

Note that this definition of disconjugacy is not formulated in terms of so-called generalized zeros of vector solutions (x, u) of (H) as is done in [3, Definition 4]. But, by

[3, Theorem 2], our definition here is the same as [3, Definition 4]. We will use the following result (see [3, Theorem 2]).

Lemma 1.1 (Reid Roundabout Theorem when $R = S = 0$). *Suppose* (A) *and* $R = S = 0$. *Then the following statements are equivalent:*

(i) $\mathcal{F} = \mathcal{F}_0$ *is positive definite;*
(ii) (H) *is disconjugate on* J^*.

The next section contains an analogue of Lemma 1.1 for general R and S without any further assumptions (e.g., controllability of (H) as is required in [3, Theorem 3]), and we will present the corresponding result for the case of separated boundary conditions (i.e., when R and S are of the form (A_2)) in the concluding Section 3.

2. THE GENERAL CASE

We need some additional notation. Let

$$(2.1)\qquad \begin{cases} Q_k = U_k X_k^\dagger + (U_k X_k^\dagger \tilde{X}_k - \tilde{U}_k)(I - X_k^\dagger X_k) U_k^T, \\ \tilde{Q}_k = X_k^\dagger + X_k^\dagger \tilde{X}_k (I - X_k^\dagger X_k) U_k^T, \\ Q_k^* = \begin{pmatrix} -X_k^\dagger \tilde{X}_k X_k^\dagger X_k & \tilde{Q}_k \\ \tilde{Q}_k^T & Q_k \end{pmatrix} \end{cases}$$

and

$$(2.2)\qquad X_k^* = \begin{pmatrix} 0 & I \\ X_k & \tilde{X}_k \end{pmatrix} \quad \text{and} \quad U_k^* = \begin{pmatrix} I & 0 \\ U_k & \tilde{U}_k \end{pmatrix},$$

where (X, U) and $(\tilde{X}, \tilde{U})$ denote the special normalized conjoined bases of (H) (at 0), i.e., the matrix solutions of (H) satisfying the initial conditions

$$(2.3)\qquad X_0 = 0, \qquad U_0 = I, \qquad \tilde{X}_0 = -I, \qquad \tilde{U}_0 = 0.$$

By a direct computation (see also [3, Remark 8 (iii)]) the matrices $X_k^{*T} U_k^*$ are symmetric and

$$(2.4)\qquad Q_k^* X_k^* = U_k^* X_k^{*\dagger} X_k^* \quad \text{for all} \quad k \in J^*.$$

The following results are crucial for the proof of our Theorem 2.3 below.

Lemma 2.1. *Assume* (A), *suppose that* $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$ *and that* (x, u) *satisfies the equation of motion* $\Delta x_k = A_k x_{k+1} + B_k u_k$ *for all* $k \in J$. *Then*

$$\mathcal{F}_0(x) = \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T Q_{N+1}^* \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} + \sum_{k=0}^{N} z_k^T D_k z_k,$$

where $z_k = u_k - (U_k X_k^\dagger)^T x_k + (X_k^\dagger)^T x_0$, $D_k = X_k X_{k+1}^\dagger \tilde{A}_k B_k$ *as in Section 1, for* $k \in J$, *and where we use the notation* (2.1), (2.2), *and* (2.3).

Proof. This is a special case of Picone's identity [3, Proposition 4] with $\alpha = -x_0$ (observe that $Q_0^* = 0$). $\square$

Lemma 2.2. *Under the assumptions and with the notation of Lemma 2.1 we have that*

$$\begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} \in \operatorname{Im} X_{N+1}^*.$$

Proof. This is a special case of [3, Remark 3 (iii)]. Indeed, by [3, Lemma 3], (X^*, U^*) is a conjoined basis of the "large" Hamiltonian system (H^*) (as introduced in [3]), and, by our assumptions, $\operatorname{Ker} X_{k+1}^* \subset \operatorname{Ker} X_k^*$ for all $k \in J$, (x^*, u^*) with $x_k^* = \binom{-x_0}{x_k}$, $u_k^* = \binom{0}{u_k}$ satisfies the corresponding equation of motion, and $\binom{-x_0}{x_0} = -X_0^* x_0 \in \operatorname{Im} X_0^*$. Hence, [3, Remark 3 (iii)] implies recursively that $x_{k+1}^* \in \operatorname{Im} X_{k+1}^*$ for $k \in J$. $\square$

Our main result now reads as follows.

Theorem 2.3 (General Reid Roundabout Theorem). *Assume* (A) *and* (A_1). *Then the following statements are equivalent:*

(i) $\mathcal{F}$ *is positive definite;*

(ii) (H) *is disconjugate on* J^* *and*

$$\begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T S \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} + \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T \begin{pmatrix} u_0 \\ u_{N+1} \end{pmatrix} > 0$$

holds for all solutions (x, u) *of* (H) *with* $\binom{-x_0}{x_{N+1}} \in \operatorname{Im} R^T \setminus \{0\}$;

(iii) (H) *is disconjugate on* J^* *and*

$$d^T (S + Q_{N+1}^*) d > 0 \quad \text{for all} \quad d \in \operatorname{Im} R^T \cap \operatorname{Im} X_{N+1}^* \setminus \{0\},$$

where we use the notation (2.1), (2.2), *and* (2.3).

Proof. First, (ii) follows from (i) by the implication "(i) $\Longrightarrow$ (ii)" of Lemma 1.1 and by the formula $\mathcal{F}_0(x) = \binom{-x_0}{x_{N+1}}^T \binom{u_0}{u_{N+1}}$ which holds for solutions (x, u) of (H) (see e.g. [3, Lemma 1]).

Next, suppose (ii) and let $d \in \operatorname{Im} R^T \cap \operatorname{Im} X_{N+1}^* \setminus \{0\}$. Then there exists $c \in \mathbb{R}^n$ such that $d = X_{N+1}^* c \in \operatorname{Im} R^T$. We put

$$\begin{pmatrix} x_k \\ u_k \end{pmatrix} = \begin{pmatrix} X_k & \tilde{X}_k \\ U_k & \tilde{U}_k \end{pmatrix} c \quad \text{for} \quad k \in J^*.$$

Then (x, u) is a solution of (H) satisfying

$$\begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} = \begin{pmatrix} -X_0 & -\tilde{X}_0 \\ X_{N+1} & \tilde{X}_{N+1} \end{pmatrix} c = X_{N+1}^* c = d \in \operatorname{Im} R^T \setminus \{0\}$$

so that by (ii),

$$\begin{aligned}
-d^T S d &< d^T \begin{pmatrix} U_0 & \tilde{U}_0 \\ U_{N+1} & \tilde{U}_{N+1} \end{pmatrix} c = d^T U^*_{N+1} c = c^T X^{*^T}_{N+1} U^*_{N+1} c \\
&= c^T U^{*^T}_{N+1} X^*_{N+1} c = c^T U^{*^T}_{N+1} X^*_{N+1} X^{*^\dagger}_{N+1} X^*_{N+1} c = c^T X^{*^T}_{N+1} U^*_{N+1} X^{*^\dagger}_{N+1} X^*_{N+1} c \\
&= c^T X^{*^T}_{N+1} Q^*_{N+1} X^*_{N+1} c = d^T Q^*_{N+1} d,
\end{aligned}$$

where we also used (2.4). This yields (iii).

Finally, assume (iii), and let (x, u) be $\mathcal{F}$-admissible. Without loss of generality (observe the implication "(ii) $\Longrightarrow$ (i)" from Lemma 1.1) we may assume $d := \binom{-x_0}{x_{N+1}} \neq 0$. Now Lemma 2.2 yields $d \in \operatorname{Im} X^*_{N+1}$ so that we have by (1.1)

$$d \in \operatorname{Im} R^T \cap \operatorname{Im} X^*_{N+1} \setminus \{0\},$$

and hence, by (iii) and Lemma 2.1,

$$\begin{aligned}
0 &< d^T (S + Q^*_{N+1}) d \leq d^T (Q^*_{N+1} + S) d + \sum_{k=0}^{N} z_k^T D_k z_k \\
&= \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T Q^*_{N+1} \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} + \sum_{k=0}^{N} z_k^T D_k z_k + d^T S d = \mathcal{F}(x).
\end{aligned}$$

Hence $\mathcal{F}$ is positive definite and the proof of our result is complete. □

Remark 2.4. Observe that, if $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$ for all $k \in J$, Lemma 2.2 together with the proof of the implication "(ii) $\Longrightarrow$ (iii)" from Theorem 2.3 above describes exactly the set of all "reachable boundary states", more precisely: There exists an (x, u) satisfying the equation of motion and $\binom{-x_0}{x_{N+1}} = c$ if and only if $c \in \operatorname{Im} X^*_{N+1}$. Moreover, all these boundary states can be reached with trajectories (x, u) that solve the full Hamiltonian system (H).

3. THE CASE OF SEPARATED BOUNDARY CONDITIONS

Besides of (A) we also assume (A$_2$) in this section, and then (x, u) is $\mathcal{F}$-admissible provided

(3.1) $$\Delta x_k = A_k x_{k+1} + B_k u_k \text{ for } k \in J, \quad x_0 \in \operatorname{Im} R_0^T, \quad x_{N+1} \in \operatorname{Im} R_{N+1}^T.$$

According to [7, Corollary 3.1.3] there exists an $n \times n$-matrix S_0^* with

(3.2) $$\operatorname{rank}(S_0^*, R_0) = n \quad \text{and} \quad \operatorname{Im} S_0^{*^T} = \operatorname{Ker} R_0.$$

We put

(3.3) $$R_0^* = R_0 S_0 + S_0^*$$

so that

(3.4) $$\operatorname{rank}(R_0^*, R_0) = n \quad \text{and} \quad R_0^* R_0^T = R_0 S_0 R_0^T \quad \text{is symmetric.}$$

Therefore [7, Theorem 3.1.2] implies the existence of some $\varepsilon > 0$ such that

$$\tilde{A}_{-1} = (R_0^* + \varepsilon R_0)^{-1} \quad \text{exists.}$$

Moreover, let us put

$$A_{-1} = I - \tilde{A}_{-1}^{-1},\ \tilde{A}_{N+1} = I,\ A_{N+1} = 0,\ B_{-1} = -\tilde{A}_{-1}^{-1} R_0^T,\ B_{N+1} = R_{N+1}^T R_{N+1}.$$

The following result is needed for the proof of our Theorem 3.2 below.

Lemma 3.1. *Let* $\underset{\sim}{C}_k = C_k$ *for* $0 \le k \le N-1$, $\underset{\sim}{C}_{-1} = \varepsilon I$, $\underset{\sim}{C}_N = C_N + S_{N+1} - B_{N+1}^\dagger$, *and* $\underset{\sim}{C}_{N+1} = 0$. *Then* $\underset{\sim}{\mathcal{F}}_0$, *defined by*

$$\underset{\sim}{\mathcal{F}}_0(x) = \sum_{k=-1}^{N+1} \left\{ x_{k+1}^T \underset{\sim}{C}_k x_{k+1} + u_k^T B_k u_k \right\}$$

is positive definite (i.e., $\underset{\sim}{\mathcal{F}}_0(x) > 0$ *for all* $x \neq 0$ *such that* (x,u) *is* $\underset{\sim}{\mathcal{F}}_0$*-admissible (i.e., with* $\Delta x_k = A_k x_{k+1} + B_k u_k$ *for all* $-1 \le k \le N+1$, $x_{-1} = x_{N+2} = 0$*)) if and only if* $\mathcal{F}$ *is positive definite.*

Proof. First, we assume that $\underset{\sim}{\mathcal{F}}_0$ is positive definite in the above described sense. Let (x,u) be $\mathcal{F}$-admissible with $x \neq 0$. According to (3.1) there exist $c_0, c_{N+1} \in \mathbb{R}^n$ with $x_0 = R_0^T c_0$ and $x_{N+1} = R_{N+1}^T c_{N+1}$. We define

$$x_{-1} = x_{N+2} = 0, \quad u_{-1} = -c_0, \quad \text{and} \quad u_{N+1} = -R_{N+1}^\dagger c_{N+1}.$$

Then

$$A_{-1} x_0 + B_{-1} u_{-1} = (I - \tilde{A}_{-1}^{-1}) x_0 + \tilde{A}_{-1}^{-1} R_0^T c_0 = x_0 = \Delta x_{-1}$$

and

$$A_{N+1} x_{N+1} + B_{N+1} u_{N+1} = -R_{N+1}^T R_{N+1} R_{N+1}^\dagger c_{N+1} = -R_{N+1}^T c_{N+1} = -x_{N+1} = \Delta x_{N+1}$$

so that (x,u) is $\underset{\sim}{\mathcal{F}}_0$-admissible. Hence by (3.4)

$$\begin{aligned} 0 < \underset{\sim}{\mathcal{F}}_0(x) &= x_0^T \underset{\sim}{C}_{-1} x_0 + u_{-1}^T B_{-1} u_{-1} + \mathcal{F}_0(x) + x_{N+1}^T (S_{N+1} - B_{N+1}^\dagger) x_{N+1} \\ &\qquad + u_{N+1}^T B_{N+1} u_{N+1} \\ &= \varepsilon c_0^T R_0 R_0^T c_0 - c_0^T (R_0^* + \varepsilon R_0) R_0^T c_0 + \mathcal{F}(x) + x_0^T S_0 x_0 \\ &\qquad - x_{N+1}^T B_{N+1}^\dagger x_{N+1} + u_{N+1}^T B_{N+1} u_{N+1} \\ &= \mathcal{F}(x) - x_{N+1}^T B_{N+1}^\dagger x_{N+1} + u_{N+1}^T B_{N+1} u_{N+1} = \mathcal{F}(x), \end{aligned}$$

where the last equality sign follows from

$$u_{N+1}^T B_{N+1} u_{N+1} = u_{N+1}^T B_{N+1} B_{N+1}^\dagger B_{N+1} u_{N+1} = x_{N+1}^T B_{N+1}^\dagger x_{N+1}.$$

Therefore $\mathcal{F}$ is positive definite.

Conversely, assume that $\mathcal{F}$ is positive definite and let (x,u) be $\underset{\sim}{\mathcal{F}}_0$-admissible such that $x \neq 0$. Then

$$x_0 = \tilde{A}_{-1}x_{-1} + \tilde{A}_{-1}B_{-1}u_{-1} = -R_0^T u_{-1} \in \operatorname{Im} R_0^T$$

and

$$x_{N+1} = -B_{N+1}u_{N+1} = -R_{N+1}^T R_{N+1}u_{N+1} \in \operatorname{Im} R_{N+1}^T$$

imply that (x,u) is also $\mathcal{F}$-admissible. Hence by (3.4) as above

$$\begin{aligned} 0 \;&<\; \mathcal{F}(x) = \underset{\sim}{\mathcal{F}}_0(x) - x_0^T \underset{\sim}{C}_{-1}x_0 - u_{-1}^T B_{-1}u_{-1} - x_{N+1}^T(S_{N+1} - B_{N+1}^\dagger)x_{N+1} \\ &\quad -u_{N+1}^T B_{N+1}u_{N+1} + x_{N+1}^T S_{N+1}x_{N+1} - x_0^T S_0 x_0 \\ &= \underset{\sim}{\mathcal{F}}_0(x) - \varepsilon u_{-1}^T R_0 R_0^T u_{-1} + u_{-1}^T(R_0^* + \varepsilon R_0)R_0^T u_{-1} - u_{-1}^T R_0 S_0 R_0^T u_{-1} = \underset{\sim}{\mathcal{F}}_0(x) \end{aligned}$$

so that positive definiteness of $\underset{\sim}{\mathcal{F}}_0$ follows. □

By the aid of Lemma 3.1 and Lemma 1.1 we can now prove our final result which has the advantage (compared to Theorem 2.3) that we need only one solution of (H) (instead of two of them in Theorem 2.3). As before we do not require any additional assumption in contrast to the corresponding result given in [4, Theorem 3].

Theorem 3.2 (Reid Roundabout Theorem for separated boundary conditions)**.** *Assume* (A) *and* (A_2)*. Let* (X,U) *be the conjoined basis of* (H) *satisfying*

$$X_0 = -R_0^T \quad \textit{and} \quad U_0 = R_0^{*T},$$

where R_0^* *is given by* (3.2) *and* (3.3)*. Then* $\mathcal{F}$ *is positive definite if and only if* (X,U) *has no focal points in* $(0, N+2]$*, where we put*

$$X_{N+2} = \left(I - B_{N+1}B_{N+1}^\dagger + B_{N+1}S_{N+1}\right) X_{N+1} + B_{N+1}U_{N+1} \textit{ and } B_{N+1} = R_{N+1}^T R_{N+1}.$$

Proof. From the equivalence "(i) $\Longleftrightarrow$ (ii)" of Lemma 1.1 it follows that $\underset{\sim}{\mathcal{F}}_0$ is positive definite (in the sense of Lemma 3.1) if and only if the solution $(\underset{\sim}{X}, \underset{\sim}{U})$ of

$$\Delta X_k = A_k X_{k+1} + B_k U_k, \quad \Delta U_k = \underset{\sim}{C}_k X_{k+1} - A_k^T U_k \quad \text{for } -1 \le k \le N+1$$

with $\underset{\sim}{X}_{-1} = 0$ and $\underset{\sim}{U}_{-1} = I$ satisfies

$$\operatorname{Ker} \underset{\sim}{X}_{k+1} \subset \operatorname{Ker} \underset{\sim}{X}_k, \quad \underset{\sim}{D}_k := \underset{\sim}{X}_k \underset{\sim}{X}_{k+1}^\dagger \tilde{A}_k B_k \ge 0 \quad \text{for } -1 \le k \le N+1.$$

We have

$$\underset{\sim}{X}_0 = \tilde{A}_{-1}B_{-1} = -R_0^T = X_0 \quad \text{and} \quad \underset{\sim}{U}_0 = \underset{\sim}{C}_{-1}X_0 + (R_0^* + \varepsilon R_0)^T = R_0^{*T} = U_0$$

so that $\underset{\sim}{X}_k = X_k$, $\underset{\sim}{U}_k = U_k$ follows for $k \in J$ (again note that the corresponding initial value problems have unique solutions according to our assumption (A)). Moreover, $\underset{\sim}{X}_{N+1} = X_{N+1}$, $\underset{\sim}{U}_{N+1} = U_{N+1} + (S_{N+1} - B_{N+1}^\dagger)X_{N+1}$, and

$$\begin{aligned} \underset{\sim}{X}_{N+2} &= X_{N+1} + B_{N+1}\underset{\sim}{U}_{N+1} = \left(I + B_{N+1}S_{N+1} - B_{N+1}B_{N+1}^\dagger\right) X_{N+1} + B_{N+1}U_{N+1} \\ &= X_{N+2}. \end{aligned}$$

Observe that $\operatorname{Ker} \underset{\sim}{X}_0 \subset \operatorname{Ker} \underset{\sim}{X}_{-1} = \mathbb{R}^n$ and $\underset{\sim}{D}_{-1} = 0 \geq 0$ hold trivially, and then our assertion follows from Lemma 3.1. $\square$

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