Discrete Sturmian Theory

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Abstract

Using the concept of focal points for so-called conjoined bases of linear Hamiltonian difference systems we present discrete analoga of Sturm's separation and comparison results.

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1. Introduction

The books [8] by Werner Kratz and [1] by Calvin Ahlbrandt and Allan Peterson contain surveys on linear Hamiltonian systems

$$\dot{X} = A(t)X + B(t)U, \qquad \dot{U} = C(t)X - A^T(t)U \tag{$\mathrm{H_C}$}$$

and on their discrete counterparts, so-called *linear Hamiltonian difference systems*

$$\Delta X_k = A_k X_{k+1} + B_k U_k, \qquad \Delta U_k = C_k X_{k+1} - A_k^T U_k, \tag{H}$$

respectively. (All occuring objects are real $n \times n$-matrices and $t \in \mathbb{R}$ while $k \in \mathbb{Z}$.) *Conjoined bases* (X, U) of ($\mathrm{H_C}$) and (H), respectively, are solutions of ($\mathrm{H_C}$) with

$$X^T(t_0)U(t_0) \text{ symmetric}, \quad \operatorname{rank}\begin{pmatrix} X(t_0) \\ U(t_0) \end{pmatrix} = n \qquad \text{for some } t_0 \in \mathbb{R}$$

and of (H) with

$$X_{k_0}^T U_{k_0} \text{ symmetric}, \quad \operatorname{rank}\begin{pmatrix} X_{k_0} \\ U_{k_0} \end{pmatrix} = n \qquad \text{for some } k_0 \in \mathbb{Z},$$

respectively. It is common to call $t_0 \in \mathbb{R}$ a *focal point* for a conjoined basis (X, U) of ($\mathrm{H_C}$) provided

$$X(t_0) \text{ is singular} \tag{$\mathrm{FP_C}$}$$

holds (see [8, Definition 1.1.1 (ii)]). Chapter 7.3 of [8] deals with Sturmian theory for conjoined bases of ($\mathrm{H_C}$), i.e., gives separation results of focal points for two conjoined bases (X, U) and $(\tilde{X}, \tilde{U})$ of ($\mathrm{H_C}$) and gives comparison results of focal points for a conjoined basis (X, U) of ($\mathrm{H_C}$) and another conjoined basis $(\tilde{X}, \tilde{U})$ of some other system

$$\dot{X} = \underset{\sim}{A}(t)X + \underset{\sim}{B}(t)U, \qquad \dot{U} = \underset{\sim}{C}(t)X - \underset{\sim}{A}^T(t)U \tag{$\underset{\sim}{H}_{\mathrm{C}}$}$$

in the following manner: Subject to certain assumptions, if (X, U) has a focal point in $(a, b] \subset \mathbb{R}$, then so does $(\tilde{X}, \tilde{U})$. It is the purpose of this survey to present discrete analoga of these kind of results for system (H), and for the comparison results we will compare conjoined bases of (H) with conjoined bases of some other linear Hamiltonian difference system

$$\Delta X_k = \underset{\sim}{A}_k X_{k+1} + \underset{\sim}{B}_k U_k, \qquad \Delta U_k = \underset{\sim}{C}_k X_{k+1} - \underset{\sim}{A}_k^T U_k. \tag{$\underset{\sim}{H}$}$$

In order to do this it is necessary to define focal points for conjoined bases of discrete systems, and this has been done by the author in [3, Definition 3]. According to this definition, we say that the interval $(k, k+1]$ contains a focal point of a conjoined basis (X, U) of (H) provided

$$\text{(FP)} \quad \text{Ker}X_{k+1} \not\subset \text{Ker}X_k \quad \text{or} \quad \text{Ker}X_{k+1} \subset \text{Ker}X_k,\ X_k X_{k+1}^\dagger (I - A_k)^{-1} B_k \not\geq 0$$

holds. Here, Ker denotes the kernel, the dagger denotes the Moore-Penrose inverse (i.e., $M^\dagger$ is the unique matrix that satisfies $MM^\dagger M = M$ and $M^\dagger M M^\dagger = M^\dagger$ such that both $MM^\dagger$ and $M^\dagger M$ are symmetric), and $\geq$ means positive semidefiniteness. Also, throughout, we impose the assumption

$$\text{(A)} \qquad I - A_k \text{ is invertible and } B_k,\ C_k \text{ are symmetric} \qquad \text{for all } k \in \mathbb{Z}$$

on systems (H). Justifications for this kind of strange looking definition (FP) are contained in [3] as well as in the next section. However, it is clear on the first view that working with condition (FP) is much more complicated compared to condition (FP_C). Discrete versions of Sturm-type separation and comparison results are given in Section 3 and 4, respectively (see also [5]). We finally wish to remark that those results also apply to so-called Sturm-Liouville difference equations (see also [4]) for they are special cases of systems (H).

2. Focal Points

The starting objects for the theory are discrete quadratic functionals of the form (with throughout the paper fixed integers M and N such that $M \leq N$)

$$\mathcal{F}(x, u) = \sum_{k=M}^{N} \left\{ x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k \right\},$$

and they are called *positive definite* whenever

$$\mathcal{F}(x, u) > 0 \text{ for all admissible } (x, u) \text{ with } x_M = x_{N+1} = 0 \text{ and } x \neq 0$$

holds. (Here, x and u are sequences of $\mathbb{R}^n$-vectors, and they are called admissible provided $\Delta x_k = A_k x_{k+1} + B_k u_k$ holds for $k \in \mathbb{Z}$.) This notion of positive definiteness is motivated from the study of discrete variational problems where functionals of the form $\mathcal{F}$ arise as so-called second variations (see [2]). Now, the main tool for the theory is the following discrete version of Picone's identity.

Lemma 1 *(Picone's Identity).* Let be given a conjoined basis (X, U) of (H) with $\mathrm{Ker}X_{k+1} \subset \mathrm{Ker}X_k$ for all $M \le k \le N$, and let (x, u) be admissible with $x_M \in \mathrm{Im}X_M$. We then for all $M \le k \le N$ have $D_k z_k = X_k X_{k+1}^{\dagger} x_{k+1} - x_k$ and

$$x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k = \Delta \left\{ x_k^T Q_k x_k \right\} + z_k^T D_k z_k,$$

where $D_k = X_k X_{k+1}^{\dagger}(I - A_k)^{-1} B_k$, $Q_k = X_k X_k^{\dagger} U_k X_k^{\dagger}$, $z_k = u_k - Q_k x_k$.

This lemma immediately proves one direction of the following result.

Lemma 2 *(Jacobi's Condition).* $\mathcal{F}$ is positive definite if and only if the principal solution of (H) at M has no focal points in $(M, N+1]$.

(Above, the unique solution of (H) satisfying $X_M = 0$ and $U_M = I$ is called the principal solution of (H) at M.) Proofs of these two results may be found in [2, 3] and in Chapter 8 of the textbook [1]. It is remarkable that the continuous version of Lemma 2 (see [8, Theorem 2.4.1]) reads precisely like Lemma 2; however, the definition (FP) of a focal point in the discrete case differs significantly from ($\mathrm{FP_C}$) in the continuous case.

3. Separation Results

The proof of our first separation result combines Lemma 1 and Lemma 2.

Proposition 1. If the principal solution of (H) at M has a focal point in $(M, N+1]$, then so does any conjoined basis of (H).

Proof. We assume that there exists a conjoined basis (X, U) of (H) without focal points in $(M, N+1]$, i.e., with

$$\mathrm{Ker}X_{k+1} \subset \mathrm{Ker}X_k \text{ and } D_k \ge 0 \qquad \text{for all } M \le k \le N.$$

Let (x, u) be admissible with $x_M = x_{N+1} = 0$. Then $x_M \in \mathrm{Im}X_M$ and Picone's identity, Lemma 1, yields

$$\begin{aligned}
\mathcal{F}(x, u) &= \sum_{k=M}^{N} \left\{ x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k \right\} \\
&= \sum_{k=M}^{N} \left\{ \Delta x_k^T Q_k x_k + z_k^T D_k z_k \right\} \\
&= x_{N+1}^T Q_{N+1} x_{N+1} - x_M^T Q_M x_M + \sum_{k=M}^{N} z_k^T D_k z_k \\
&= \sum_{k=M}^{N} z_k^T D_k z_k \ge 0.
\end{aligned}$$

If $\mathcal{F}(x,u)=0$, then $D_k z_k = 0$ for all $M \le k \le N$, and the first assertion of Lemma 1 together with $x_{N+1}=0$ shows $x_N = x_{N-1} = \ldots = x_{M+1} = x_M = 0$. Hence we have that $\mathcal{F}$ is positive definite. This in turn, by Jacobi's condition, Lemma 2, proves that the principal solution of (H) at M has no focal point in $(M, N+1]$.■

Let us remark that at this point we don't care so much about positive definiteness any more; it in fact became a major tool in order to create a theory centering around focal points although it was originally motivated by the examination of discrete variational problems. It is e.g. the proof of Proposition 1 which illustrates these remarks.

Our goal in this section is not only to compare a conjoined basis with the principal solution but to obtain results for two arbitrary conjoined bases. To do so, an extension of Lemma 2 in the following sense is needed (see [5, Lemma 3]).

Lemma 3. The conjoined basis (X,U) of (H) has no focal points in $(M,N+1]$ if and only if

$$\begin{cases} \mathcal{F}(x,u) + x_M^T Q_M x_M > 0 \text{ for all admissible } (x,u) \\ \quad \text{with } x_M \in \operatorname{Im} X_M,\ x_{N+1}=0, \text{ and } x \neq 0. \end{cases}$$

We are now ready to state this section's main result.

Theorem 1 *(Sturm's Separation Theorem).* Let (X,U) and $(\tilde{X},\tilde{U})$ be conjoined bases of (H) with

$$\operatorname{Im}\tilde{X}_M \subset \operatorname{Im}X_M \quad \text{and} \quad \tilde{X}_M^T(\tilde{Q}_M - Q_M)\tilde{X}_M \ge 0.$$

If (X,U) has no focal points in $(M,N+1]$, then neither does $(\tilde{X},\tilde{U})$.

Proof. Assume that (X,U) has no focal points in $(M,N+1]$. Via applying Lemma 3 to $(\tilde{X},\tilde{U})$ we now wish to show that $(\tilde{X},\tilde{U})$ has no focal points in $(M,N+1]$ either. To do so, let (x,u) be admissible with $x_M \in \operatorname{Im}\tilde{X}_M$ and $x_{N+1}=0$. The assumption $\operatorname{Im}\tilde{X}_M \subset \operatorname{Im}X_M$ yields $x_M \in \operatorname{Im}X_M$ so that Lemma 1 becomes applicable. We obtain

$$\begin{aligned}
\mathcal{F}(x,u) + x_M^T\tilde{Q}_M x_M &= \sum_{k=M}^{N}\left\{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\right\} + x_M^T \tilde{Q}_M x_M \\
&= \sum_{k=M}^{N}\left\{\Delta x_k^T Q_k x_k + z_k^T D_k z_k\right\} + x_M^T\tilde{Q}_M x_M \\
&= x_{N+1}^T Q_{N+1} x_{N+1} - x_M^T Q_M x_M + \sum_{k=M}^{N} z_k^T D_k z_k + x_M^T \tilde{Q}_M x_M \\
&= x_M^T(\tilde{Q}_M - Q_M)x_M + \sum_{k=M}^{N} z_k^T D_k z_k \ge 0
\end{aligned}$$

due to our assumption $\tilde{X}_M^T(\tilde{Q}_M - Q_M)\tilde{X}_M \geq 0$. If $\mathcal{F}(x,u) + x_M^T\tilde{Q}_M x_M = 0$, then $D_k z_k = 0$ for all $M \leq k \leq N$ so that the first assertion of Lemma 1 together with $x_{N+1} = 0$ again shows $x = 0$. ∎

4. Comparison Results

In this section we consider two systems (H) and ($\underset{\sim}{H}$) that both satisfy our general assumption (A) and put for $k \in \mathbb{Z}$

$$\mathcal{H}_k = \begin{pmatrix} -C_k & A_k^T \\ A_k & B_k \end{pmatrix}, \; \underset{\sim}{\mathcal{H}}_k = \begin{pmatrix} -\underset{\sim}{C}_k & \underset{\sim}{A}_k^T \\ \underset{\sim}{A}_k & \underset{\sim}{B}_k \end{pmatrix}, \; D_k^* = \underset{\sim}{B}_k(\underset{\sim}{B}_k^\dagger - B_k^\dagger)\underset{\sim}{B}_k.$$

We need the following auxiliary result which is the contents of [5, Lemma 2].

Lemma 4. Let $(x, \underset{\sim}{u})$ be admissible with respect to ($\underset{\sim}{H}$) and suppose that

$$\text{Ker} B_k \subset \text{Ker} \underset{\sim}{B}_k \quad \text{and} \quad \text{Im}(A_k - \underset{\sim}{A}_k) \subset \text{Im}(B_k - \underset{\sim}{B}_k)$$

hold for $k \in \mathbb{Z}$. Then (x, u) with $u_k := B_k^\dagger(\Delta x_k - A_k x_{k+1})$ is admissible with respect to (H) and we have

$$x_{k+1}^T \underset{\sim}{C}_k x_{k+1} + \underset{\sim}{u}_k^T \underset{\sim}{B}_k \underset{\sim}{u}_k = x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k + \hat{z}_k^T(\mathcal{H}_k - \underset{\sim}{\mathcal{H}}_k)\hat{z}_k + z_k^{*T} D_k^* z_k^*,$$

where $\hat{z}_k = \binom{I}{-P_k} x_{k+1}$, $z_k^* = P_k x_{k+1} + \underset{\sim}{u}_k$, $P_k = (B_k - \underset{\sim}{B}_k)^\dagger(A_k - \underset{\sim}{A}_k)$.

For the remainder of this section we will now impose the following comparison assumption on the two systems (H) and ($\underset{\sim}{H}$). We require that

(C) $$\mathcal{H}_k \geq \underset{\sim}{\mathcal{H}}_k, \quad \text{Ker} B_k \subset \text{Ker} \underset{\sim}{B}_k, \text{ and } D_k^* \geq 0$$

hold for all $k \in \mathbb{Z}$. Observe that $\mathcal{H}_k \geq \underset{\sim}{\mathcal{H}}_k$ implies $\text{Im}(A_k - \underset{\sim}{A}_k) \subset \text{Im}(B_k - \underset{\sim}{B}_k)$ by [8, Lemma 3.1.10].

Proposition 2. Assume (C). If the principal solution of ($\underset{\sim}{H}$) at M has a focal point in $(M, N+1]$, then so does any conjoined basis of (H).
Proof. We assume that there exists a conjoined basis (X, U) of (H) without focal points in $(M, N+1]$, i.e., with

$$\text{Ker} X_{k+1} \subset \text{Ker} X_k \text{ and } D_k \geq 0 \qquad \text{for all } M \leq k \leq N.$$

Let $(x, \underset{\sim}{u})$ be admissible with respect to $(\underset{\sim}{H})$ such that $x_M = x_{N+1} = 0$ and define u as in Lemma 4. Then we have by Lemma 4 (and by applying Lemma 1 since $x_M \in \mathrm{Im} X_M$ and (x, u) is admissible with respect to (H))

$$\begin{aligned}
\underset{\sim}{\mathcal{F}}(x, \underset{\sim}{u}) &= \sum_{k=M}^{N} \left\{ x_{k+1}^T \underset{\sim}{C}_k x_{k+1} + \underset{\sim}{u}_k^T \underset{\sim}{B}_k \underset{\sim}{u}_k \right\} \\
&= \sum_{k=M}^{N} \left\{ x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k + \hat{z}_k^T (\mathcal{H}_k - \underset{\sim}{\mathcal{H}}_k) \hat{z}_k + {z_k^*}^T D_k^* z_k^* \right\} \\
&= \sum_{k=M}^{N} \left\{ \Delta x_k^T Q_k x_k + z_k^T D_k z_k \right\} + \sum_{k=M}^{N} \left\{ \hat{z}_k^T (\mathcal{H}_k - \underset{\sim}{\mathcal{H}}_k) \hat{z}_k + {z_k^*}^T D_k^* z_k^* \right\} \\
&= x_{N+1}^T Q_{N+1} x_{N+1} - x_M^T Q_M x_M + \sum_{k=M}^{N} \left\{ z_k^T D_k z_k + \hat{z}_k^T (\mathcal{H}_k - \underset{\sim}{\mathcal{H}}_k) \hat{z}_k + {z_k^*}^T D_k^* z_k^* \right\} \\
&= \sum_{k=M}^{N} z_k^T D_k z_k + \sum_{k=M}^{N} \hat{z}_k^T (\mathcal{H}_k - \underset{\sim}{\mathcal{H}}_k) \hat{z}_k + \sum_{k=M}^{N} {z_k^*}^T D_k^* z_k^* \geq 0.
\end{aligned}$$

If $\underset{\sim}{\mathcal{F}}(x, \underset{\sim}{u}) = 0$, then $D_k z_k = 0$ for all $M \leq k \leq N$ and hence $x = 0$ as before. Thus $\underset{\sim}{\mathcal{F}}$ is positive definite and Lemma 2 yields our desired assertion. ■

As before we can obtain a more general result by employing our Lemma 3 from the previous section.

Theorem 2 *(Sturm's Comparison Theorem).* Suppose (C). Let be given two conjoined bases (X, U) and $(\underset{\sim}{X}, \underset{\sim}{U})$ of (H) and $(\underset{\sim}{H})$, respectively, with

$$\mathrm{Im} \underset{\sim}{X}_M \subset \mathrm{Im} X_M \quad \text{and} \quad \underset{\sim}{X}_M^T (\underset{\sim}{Q}_M - Q_M) \underset{\sim}{X}_M \geq 0.$$

If (X, U) has no focal points in $(M, N+1]$, then neither does $(\underset{\sim}{X}, \underset{\sim}{U})$.

Proof. Assume that (X, U) has no focal points in $(M, N+1]$. Let $(x, \underset{\sim}{u})$ be admissible with respect to $(\underset{\sim}{H})$ such that $x_M \in \mathrm{Im} \underset{\sim}{X}_M$ and $x_{N+1} = 0$. Hence (x, u) with u defined as in Lemma 4 is admissible with respect to (H) and $x_M \in \mathrm{Im} X_M$ holds. Thus, by Lemma 4 and Lemma 3

$$\begin{aligned}
\underset{\sim}{\mathcal{F}}(x, \underset{\sim}{u}) + x_M^T \underset{\sim}{Q}_M x_M &= \sum_{k=M}^{N} \left\{ x_{k+1}^T \underset{\sim}{C}_k x_{k+1} + \underset{\sim}{u}_k^T \underset{\sim}{B}_k \underset{\sim}{u}_k \right\} + x_M^T \underset{\sim}{Q}_M x_M \\
&= \sum_{k=M}^{N} \left\{ x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k + \hat{z}_k^T (\mathcal{H}_k - \underset{\sim}{\mathcal{H}}_k) \hat{z}_k + {z_k^*}^T D_k^* z_k^* \right\} + x_M^T \underset{\sim}{Q}_M x_M
\end{aligned}$$

$$= \sum_{k=M}^{N} \left\{ \Delta x_k^T Q_k x_k + z_k^T D_k z_k + \hat{z}_k^T (\mathcal{H}_k - \underset{\sim}{\mathcal{H}}_k) \hat{z}_k + z_k^{*^T} D_k^* z_k^* \right\} + x_M^T \underset{\sim}{Q}_M x_M$$

$$= x_M^T (\underset{\sim}{Q}_M - Q_M) x_M + \sum_{k=M}^{N} z_k^T D_k z_k + \sum_{k=M}^{N} \hat{z}_k^T (\mathcal{H}_k - \underset{\sim}{\mathcal{H}}_k) \hat{z}_k + \sum_{k=M}^{N} z_k^{*^T} D_k^* z_k^* \geq 0.$$

If $\underset{\sim}{\mathcal{F}}(x, \underset{\sim}{u}) = 0$, then as before $x = 0$. Hence, our result is shown by Lemma 3. ■

We shall conclude this survey with some remarks.

(i) First, if $B_k = \underset{\sim}{B}_k$ for all $k \in \mathbb{Z}$, then condition (C) reduces to

$$\begin{pmatrix} \underset{\sim}{C}_k - C_k & A_k^T - \underset{\sim}{A}_k^T \\ A_k - \underset{\sim}{A}_k & 0 \end{pmatrix} \geq 0.$$

Of course, if in addition $A_k = \underset{\sim}{A}_k$ and $C_k = \underset{\sim}{C}_k$ for all $k \in \mathbb{Z}$, then condition (C) becomes empty, (H) and ($\underset{\sim}{H}$) are the same systems, and Theorem 2 becomes nothing but Theorem 1.

(ii) Next, if B_k is positive definite for all $k \in \mathbb{Z}$ (which is the case e.g. in [7]; see also the comparison theorem [7, Theorem 3]), then (C) reduces to

$$\begin{pmatrix} \underset{\sim}{C}_k - C_k & A_k^T - \underset{\sim}{A}_k^T \\ A_k - \underset{\sim}{A}_k & B_k - \underset{\sim}{B}_k \end{pmatrix} \geq 0.$$

(iii) When trying to check such conditions, the follwing result provides two useful characterizations (see also [6, page 28] or [8, Lemma 3.1.10]).

Lemma 5. Let Q,R, and S be matrices of size (α, α), (β, β), and (α, β), respectively. Then $\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}$ is positive semidefinite if and only if

$$\text{Ker} R \subset \text{Ker} S, \quad R \geq 0, \text{ and } SR^\dagger S^T - Q \geq 0$$

hold, and this is equivalent to

$$\text{Ker} Q \subset \text{Ker} S^T, \quad Q \geq 0, \text{ and } R - S^T Q^\dagger S \geq 0.$$

(iv) It is common to call a system (H) *disconjugate* (on $[M, N+1] \cap \mathbb{Z}$) if the principal solution of (H) at M has no focal points in $(M, N+1]$. With this terminology, Jacobi's condition reads as follows.

Corollary 1. $\mathcal{F} > 0$ iff (H) is disconjugate.

A special case of Theorem 2, namely with (X, U) and $(\underset{\sim}{X}, \underset{\sim}{U})$ being the principal solution of (H) and $(\underset{\sim}{H})$, respectively, then is the following.

Corollary 2. Suppose (C). If (H) is disconjugate, then so is $(\underset{\sim}{H})$.

(v) Finally, we briefly mention the important case of

$$A_k = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}, \quad B_k = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \frac{1}{r_k^{(n)}} \end{pmatrix},$$

$$\text{and} \qquad C_k = \begin{pmatrix} r_k^{(0)} & & & \\ & r_k^{(1)} & & \\ & & \ddots & \\ & & & r_k^{(n-1)} \end{pmatrix},$$

where the corresponding linear Hamiltonian difference system is equivalent to a Sturm-Liouville difference equation (in the sense described more precisely in [3]). Above we require that $r_k^{(\nu)} \in \mathbb{R}$ for all $k \in \mathbb{Z}$ and $0 \le \nu \le n$ and that $r_k^{(n)} \neq 0$ for all $k \in \mathbb{Z}$. Hence the condition

$$\mathrm{Ker} B_k \subset \mathrm{Ker} \underset{\sim}{B}_k \qquad \text{is automatically satisfied,}$$

where $\underset{\sim}{B}_k$ (and also $\underset{\sim}{A}_k$ and $\underset{\sim}{C}_k$) are matrices of the form as above with $\underset{\sim}{r}_k^{(\nu)} \in \mathbb{R}$ etc.. We furthermore have

$$D_k^* = \underset{\sim}{B}_k (\underset{\sim}{B}_k^\dagger - B_k^\dagger) \underset{\sim}{B}_k = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \frac{\underset{\sim}{r}_k^{(n)} - r_k^{(n)}}{\left\{ \underset{\sim}{r}_k^{(n)} \right\}^2} \end{pmatrix}$$

so that $D_k^* \geq 0$ iff $\underset{\sim}{r}_k^{(n)} \geq r_k^{(n)}$. Since

$$\mathcal{H}_k - \underset{\sim}{\mathcal{H}}_k = \begin{pmatrix} \underset{\sim}{r}_k^{(0)} - r_k^{(0)} & & & & & & \\ & \ddots & & & & & \\ & & \underset{\sim}{r}_k^{(n-1)} - r_k^{(n-1)} & & & & \\ & & & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \\ & & & & & & \frac{1}{r_k^{(n)}} - \frac{1}{\underset{\sim}{r}_k^{(n)}} \end{pmatrix},$$

we have that condition (C) for the Sturm-Liouville case reads

$$(\mathrm{C}_{\mathrm{SL}}) \qquad \underset{\sim}{r}_k^{(\nu)} \geq r_k^{(\nu)} \text{ for all } k \in \mathbb{Z} \text{ and all } 0 \leq \nu \leq n.$$

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