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Eigenvalues and Eigenfunctions of Discrete Conjugate Boundary Value Problems

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Abstract—We consider the following boundary value problem:

$$(-1)^{n-p}\Delta^n y = \lambda F\left(k, y, \Delta y, \dots, \Delta^{n-1} y\right), \qquad n \ge 2, \quad 0 \le k \le m,$$
$$\Delta^i y(0) = 0, \quad 0 \le i \le p-1; \qquad \Delta^i y(m+n-i) = 0, \quad 0 \le i \le n-p-1,$$

where $1 \le p \le n-1$ is fixed and $\lambda > 0$. A characterization of the values of λ is carried out so that the boundary value problem has a positive solution. Next, for $\lambda = 1$, criteria are developed for the existence of two positive solutions of the boundary value problem. In addition, for particular cases we also offer upper and lower bounds for these positive solutions. Several examples are included to dwell upon the importance of the results obtained.

Keywords—Eigenvalues, Positive solutions, Difference equations.

1. INTRODUCTION

Let c, d $(d > c)$ be integers. We shall define the discrete interval $[c, d] = \{c, c+1, \dots, d\}$. All other interval notation will carry its standard meaning, e.g., $(0, \infty)$ denotes the set of positive real numbers. For a nonnegative integer n, we also define the factorial expression $k^{(n)} = \prod_{i=0}^{n-1}(k-i)$ with $k^{(0)} = 1$. Let $\Delta y(k) = y(k+1) - y(k)$ and for $n \ge 2$, $\Delta^n y(k) = \Delta(\Delta^{n-1} y(k))$.

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In this paper, we shall consider the following conjugate boundary value problem:

$$\begin{aligned} &(-1)^{n-p}\Delta^n y = \lambda F\left(k, y, \Delta y, \dots, \Delta^{n-1} y\right), \qquad k \in [0,m], \\ &\Delta^i y(0) = 0, \quad 0 \le i \le p-1, \qquad \Delta^i y(m+n-i) = 0, \quad 0 \le i \le n-p-1, \end{aligned} \tag{1.1}$$

where $\lambda > 0$ and n, p, m are fixed integers with $n \ge 2$, $1 \le p \le n-1$, and $m \ge p$. Throughout, it is assumed that there exist continuous functions $f : (0,\infty) \to (0,\infty)$ and $u, v : [0,m] \to \mathbb{R}$ such that

(A1) f is nondecreasing;
(A2) for $x \in (0,\infty)$,

$$u(k) \le \frac{F(k, x, x_1, \dots, x_{n-1})}{f(x)} \le v(k); \qquad \text{and}$$

(A3) $u(k)$ is nonnegative and is not identically zero on $[0,m]$; also there exists $0 < k_0 \le 1$ with $u(k) \ge k_0 v(k)$ for $k \in [0,m]$.

By a *positive solution* y of (1.1), we mean a nontrivial $y : [0, m+n] \to [0,\infty)$ satisfying (1.1). If, for a particular λ, the boundary value problem (1.1) has a positive solution y, then λ is called an *eigenvalue* and y a corresponding *eigenfunction* of (1.1). We let E be the set of eigenvalues of the boundary value problem (1.1), i.e.,

$$E = \{\lambda > 0 \mid (1.1) \text{ has a positive solution}\}.$$

Further, we introduce the notations

$$f_0 = \lim_{x \to 0^+} \frac{f(x)}{x}, \qquad f_\infty = \lim_{x \to \infty} \frac{f(x)}{x}.$$

Our first task is the characterization of the values of λ so that the boundary value problem (1.1) has a positive solution. Specifically, we shall show that the set E is an interval and establish criteria for E to be an unbounded interval or a bounded (open or half-closed) interval. In addition, without the monotonicity condition (A1), explicit eigenvalue intervals are derived in terms of f_0 and f_∞.

Next, for the case $\lambda = 1$, we shall develop criteria for the existence of two positive solutions of (1.1). Further, we shall consider the following special cases of (1.1) ($n = 2$, $p = 1$):

$$\Delta^2 y + a(k)\ \left(y^\alpha + y^\beta\right) = 0, \quad k \in [0,m], \qquad y(0) = y(m+2) = 0 \tag{1.2}$$

and

$$\Delta^2 y + a(k)\ e^{\sigma y} = 0, \quad k \in [0,m], \qquad y(0) = y(m+2) = 0. \tag{1.3}$$

It is assumed that $0 \le \alpha < 1 < \beta$, $\sigma > 0$, and $a(k)$ is nonnegative on $[0,m]$ and is not identically zero on $[p,m]$. In addition to providing existence criteria for two positive solutions of (1.2) and (1.3), we also establish upper and lower bounds for these positive solutions.

The motivation for the present work stems from many recent investigations. In fact, when $n = 2$ the boundary value problem (1.1) is a discrete model of a wide spectrum of nonlinear phenomena such as gas diffusion through porous media, nonlinear diffusion generated by nonlinear sources, thermal self-ignition of a chemically active mixture of gases in a vessel, catalysis theory, chemically reacting systems, adiabatic tubular reactor processes, as well as concentration in chemical or biological problems, where only positive solutions are meaningful, e.g., see [1–7]. For the special case $\lambda = 1$, (1.1) and its particular and related cases have been the subject matter of many recent publications on singular boundary value problems, for this we refer to [8–15]. Further, in the case of second-order boundary value problems, (1.1) occurs in applications involving nonlinear elliptic problems in annular regions, e.g., see [16–19]. Once again in all these applications, it is frequent that only solutions that are positive are useful.

It is noted that the importance of (1.2) and of its continuous version have been well illustrated in [20,21], respectively. With $a(k)$ being a constant function, the boundary value problem (1.3) actually arises in applications involving the diffusion of heat generated by positive temperature-dependent sources [22]. For instance, if $\sigma = 1$ the boundary value problem occurs in the analysis of Joule losses in electrically conducting solids as well as in frictional heating.

Recently, several eigenvalue characterizations for related continuous systems have been carried out. To cite a few examples, Fink, Gatica and Hernandez [23] have dealt with the boundary value problem

$$y'' + \lambda q(t) f(y) = 0, \quad t \in (0,1), \qquad y(0) = y(1) = 0.$$

A more general problem, namely,

$$y^{(n)} + q(t) f(y) = 0, \quad t \in (0,1), \qquad y^{(i)}(0) = y(1) = 0, \quad 0 \le i \le n-2$$

has been discussed in [24]. Further, Eloe and Henderson [25] have considered a special case of the continuous version of (1.1). As for twin positive solutions, several studies on boundary value problems different from (1.1) can be found in [20,26–29]. Our results not only generalize and extend the known theorems for all the above eigenvalue problems, but also complement the work of many authors [9,15,30–38], as well as include several other known criteria offered in [39]. Note also that our approach to the discrete problem (1.1) is similar to the methods we have used in the continuous case [40].

The plan of the paper is as follows. In Section 2, we shall state a fixed-point theorem due to [41], obtain the explicit expression of a certain Green's function, and develop some properties of this Green's function for later use. By defining an appropriate Banach space and cone, in Section 3, we shall characterize the set E. Explicit eigenvalue intervals in terms of f_0 and f_∞ are established in Section 4. We shall investigate the existence of double positive solutions in Section 5. Finally, the boundary value problems (1.2) and (1.3) are treated, respectively, in Sections 6 and 7.

2. PRELIMINARIES

THEOREM 2.1. *(See [41].) Let B be a Banach space, and let $C(\subset B)$ be a cone. Assume Ω_1, Ω_2 are open subsets of B with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, and let*

$$S : C \cap \left(\bar{\Omega}_2 \backslash \Omega_1\right) \to C$$

be a completely continuous operator such that, either

(a) $\|Sy\| \le \|y\|$, $y \in C \cap \partial\Omega_1$, *and* $\|Sy\| \ge \|y\|$, $y \in C \cap \partial\Omega_2$, *or*
(b) $\|Sy\| \ge \|y\|$, $y \in C \cap \partial\Omega_1$, *and* $\|Sy\| \le \|y\|$, $y \in C \cap \partial\Omega_2$.

Then, S has a fixed point in $C \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

To obtain a solution for (1.1), we require a mapping whose kernel $G(k,\ell)$ is the Green's function of the boundary value problem

$$\Delta^n y = 0, \quad \Delta^i y(0) = 0, \qquad 0 \le i \le p-1, \qquad \Delta^i y(m+n-i) = 0, \qquad 0 \le i \le n-p-1,$$

or equivalently,

$$\Delta^n y = 0, \quad \Delta^i y(0) = 0, \qquad 0 \le i \le p-1, \qquad \Delta^i y(m+p+1) = 0, \qquad 0 \le i \le n-p-1, \tag{2.1}$$

where $1 \le p \le n-1$ is fixed. We shall find the explicit expression of the Green's function $G(k,\ell)$. For this, it is known from [42] that

$$y(k) - H(k) = \sum_{\ell=0}^{m} G(k,\ell) \Delta^n y(\ell), \tag{2.2}$$

where H is the two-point Hermite interpolating polynomial of degree $(n-1)$ satisfying

$$\begin{aligned} \Delta^i H(0) &= \Delta^i y(0), & 0 \le i \le p-1, \\ \Delta^i H(m+p+1) &= \Delta^i y(m+p+1), & 0 \le i \le n-p-1. \end{aligned} \tag{2.3}$$

We shall need some notations. Let

$$\underset{\sim}{H}(k,\ell) = \begin{pmatrix} \dfrac{(k-\ell)^{(n-p)}}{(n-p)!} & \cdots & \dfrac{(k-\ell)^{(n-1)}}{(n-1)!} \\ \vdots & & \vdots \\ \Delta^{p-1}\dfrac{(k-\ell)^{(n-p)}}{(n-p)!} & \cdots & \Delta^{p-1}\dfrac{(k-\ell)^{(n-1)}}{(n-1)!} \end{pmatrix}$$

and

$$\tilde{H}(k,\ell) = \begin{pmatrix} \dfrac{(k-\ell)^{(p)}}{p!} & \cdots & \dfrac{(k-\ell)^{(n-1)}}{(n-1)!} \\ \vdots & & \vdots \\ \Delta^{n-p-1}\dfrac{(k-\ell)^{(p)}}{p!} & \cdots & \Delta^{n-p-1}\dfrac{(k-\ell)^{(n-1)}}{(n-1)!} \end{pmatrix},$$

and put

$$\underset{\sim}{H}(k) = \underset{\sim}{H}(k, m+p+1)\underset{\sim}{H}^{-1}(0, m+p+1) \quad \text{and} \quad \tilde{H}(k) = \tilde{H}(k,0)\tilde{H}^{-1}(m+p+1,0).$$

Then, there exist $c_0(k), \dots, c_{p-1}(k), d_0(k), \dots, d_{n-p-1}(k)$ with

$$\underset{\sim}{H}(k) = \begin{pmatrix} c_0(k) & \cdots & c_{p-1}(k) \\ \vdots & & \vdots \\ \Delta^{p-1}c_0(k) & \cdots & \Delta^{p-1}c_{p-1}(k) \end{pmatrix}$$

and

$$\tilde{H}(k) = \begin{pmatrix} d_0(k) & \cdots & d_{n-p-1}(k) \\ \vdots & & \vdots \\ \Delta^{n-p-1}d_0(k) & \cdots & \Delta^{n-p-1}d_{n-p-1}(k) \end{pmatrix}.$$

We have $\underset{\sim}{H}(0) = \tilde{H}(m+p+1) = I$. If $n-2p+1 > 0$, then $\underset{\sim}{H}(m+p+1) = 0$ and if $n-2p+1 \le 0$, then the number of rows of $\underset{\sim}{H}(m+p+1)$ is $p \ge n-p+1$ and the first $(n-p)$ rows of $\underset{\sim}{H}(m+p+1)$ have only zero entries. If $2p-n+1 > 0$, then $\tilde{H}(0) = 0$ and if $2p-n+1 \le 0$, then the number of rows of $\tilde{H}(0)$ is $n-p \ge p+1$ and the first p rows of $\tilde{H}(0)$ have only zero entries. This implies that

$$H(k) = \sum_{i=0}^{p-1} c_i(k)\Delta^i y(0) + \sum_{j=0}^{n-p-1} d_j(k)\Delta^j y(m+p+1)$$

is a polynomial of degree $(n-1)$ satisfying (2.3) and hence is our required Hermite interpolating polynomial. In [42], the explicit expressions of c_i and d_j are given by

$$c_i(k) = (m+n-k)^{(n-p)} \sum_{r=0}^{p-i-1} \binom{n-p+r-1}{r} \frac{k^{(i+r)}}{i!(m+n-i)^{(n-p+r)}} \tag{2.4}$$

and

$$d_j(k) = (-1)^j k^{(p)} \sum_{r=0}^{n-p-j-1} \binom{p+r-1}{r} \frac{(m+p+j+r-k)^{(j+r)}}{j!(m+p+1+j+r)^{(p+r)}}. \tag{2.5}$$

Now, we have by the discrete version of Taylor's theorem [39, Theorem 1.7.5],

$$\begin{pmatrix} (y-H)(k) \\ \vdots \\ \Delta^{p-1}(y-H)(k) \end{pmatrix}$$

$$= \underset{\sim}{H}(k,m+p+1)\begin{pmatrix} \Delta^{n-p}(y-H)(m+p+1) \\ \vdots \\ \Delta^{n-1}(y-H)(m+p+1) \end{pmatrix} - \begin{pmatrix} \sum_{\ell=k}^{m+p} \frac{(k-\ell-1)^{(n-1)}}{(n-1)!}\Delta^n y(\ell) \\ \vdots \\ \sum_{\ell=k}^{m+p} \frac{(k-\ell-1)^{(n-p)}}{(n-p)!}\Delta^n y(\ell) \end{pmatrix}$$

so that

$$\begin{pmatrix} \Delta^{n-p}(y-H)(m+p+1) \\ \vdots \\ \Delta^{n-1}(y-H)(m+p+1) \end{pmatrix} = \underset{\sim}{H}^{-1}(0,m+p+1)\begin{pmatrix} \sum_{\ell=0}^{m+p} \frac{(-\ell-1)^{(n-1)}}{(n-1)!}\Delta^n y(\ell) \\ \vdots \\ \sum_{\ell=0}^{m+p} \frac{(-\ell-1)^{(n-p)}}{(n-p)!}\Delta^n y(\ell) \end{pmatrix},$$

and hence,

$$\begin{pmatrix} (y-H)(k) \\ \vdots \\ \Delta^{p-1}(y-H)(k) \end{pmatrix}$$

$$= \underset{\sim}{H}(k)\begin{pmatrix} \sum_{\ell=0}^{m+p} \frac{(-\ell-1)^{(n-1)}}{(n-1)!}\Delta^n y(\ell) \\ \vdots \\ \sum_{\ell=0}^{m+p} d\frac{(-\ell-1)^{(n-p)}}{(n-p)!}\Delta^n y(\ell) \end{pmatrix} - \begin{pmatrix} \sum_{\ell=k}^{m+p} \frac{(k-\ell-1)^{(n-1)}}{(n-1)!}\Delta^n y(\ell) \\ \vdots \\ \sum_{\ell=k}^{m+p} \frac{(k-\ell-1)^{(n-p)}}{(n-p)!}\Delta^n y(\ell) \end{pmatrix}.$$

Therefore,

$$y(k) - H(k) = \sum_{\ell=0}^{m+p}\sum_{i=0}^{p-1} c_i(k)\frac{(-\ell-1)^{(n-i-1)}}{(n-i-1)!}\Delta^n y(\ell) - \sum_{\ell=k}^{m+p}\frac{(k-\ell-1)^{(n-1)}}{(n-1)!}\Delta^n y(\ell). \tag{2.6}$$

Similarly, we obtain

$$\begin{pmatrix} (y-H)(k) \\ \vdots \\ \Delta^{n-p-1}(y-H)(k) \end{pmatrix} = \tilde{H}(k,0)\begin{pmatrix} \Delta^{p}(y-H)(0) \\ \vdots \\ \Delta^{n-1}(y-H)(0) \end{pmatrix} + \begin{pmatrix} \sum_{\ell=0}^{k-1} \frac{(k-\ell-1)^{(n-1)}}{(n-1)!}\Delta^n y(\ell) \\ \vdots \\ \sum_{\ell=0}^{k-1} \frac{(k-\ell-1)^{(p)}}{p!}\Delta^n y(\ell) \end{pmatrix}$$

$$= -\tilde{H}(k)\begin{pmatrix} \sum_{\ell=0}^{m} \frac{(m+p-\ell)^{(n-1)}}{(n-1)!}\Delta^n y(\ell) \\ \vdots \\ \sum_{\ell=0}^{m} \frac{(m+p-\ell)^{(p)}}{p!}\Delta^n y(\ell) \end{pmatrix} + \begin{pmatrix} \sum_{\ell=0}^{k-1} \frac{(k-\ell-1)^{(n-1)}}{(n-1)!}\Delta^n y(\ell) \\ \vdots \\ \sum_{\ell=0}^{k-1} \frac{(k-\ell-1)^{(p)}}{p!}\Delta^n y(\ell), \end{pmatrix},$$

and therefore,

$$y(k) - H(k) = \sum_{\ell=0}^{k-1}\frac{(k-\ell-1)^{(n-1)}}{(n-1)!}\Delta^n y(\ell) - \sum_{\ell=0}^{m}\sum_{j=0}^{n-p-1} d_j(k)\frac{(m+p-\ell)^{(n-j-1)}}{(n-j-1)!}\Delta^n y(\ell). \tag{2.7}$$

Plugging (2.4) and (2.5) into (2.6) and (2.7), respectively, and then using (2.2), we find that the Green's function $G(k,\ell)$ for (2.1) is given by

$$G(k,\ell)=\begin{cases} \displaystyle\sum_{j=0}^{p-1}\left[\sum_{i=0}^{p-j-1}\binom{n-p+i-1}{i}\frac{k^{(j+i)}}{(m+n-j)^{(n-p+i)}}\right] \\ \displaystyle\times\frac{(-\ell-1)^{(n-j-1)}}{j!(n-j-1)!}(m+n-k)^{(n-p)}, & 0\le \ell\le k-1, \\ \displaystyle-\sum_{j=0}^{n-p-1}\left[\sum_{i=0}^{n-p-j-1}\binom{p+i-1}{i}\frac{(m+p+j+i-k)^{(j+i)}}{(m+p+1+j+i)^{(p+i)}}\right] \\ \displaystyle\times(-1)^j\frac{(m+p-\ell)^{(n-j-1)}}{j!(n-j-1)!}k^{(p)}, & k\le \ell\le m. \end{cases} \tag{2.8}$$

Further, it is known that [39,43]

$$(-1)^{n-p}G(k,\ell)>0,\qquad (k,\ell)\in[p,m+p]\times[0,m]. \tag{2.9}$$

For each $\ell\in[0,m]$, we shall denote

$$\|G(\cdot,\ell)\|=\max_{k\in[0,m+n]}|G(k,\ell)|=\max_{k\in[0,m+n]}(-1)^{n-p}G(k,\ell). \tag{2.10}$$

LEMMA 2.1. *(See [43].) Let $\delta\in[p,m+p]$ be given. For $(k,\ell)\in[\delta,m+p]\times[0,m]$, we have*

$$(-1)^{n-p}G(k,\ell)\ge K_\delta\|G(\cdot,\ell)\|, \tag{2.11}$$

where $0<K_\delta<1$ is a constant given by

$$K_\delta=\min\left\{\frac{\min_{k\in[\delta,m+p]}h(p+1,k)}{\max_{k\in[\delta,m+p]}h(p+1,k)},\ \frac{\min_{k\in[\delta,m+p]}h(p,k)}{\max_{k\in[\delta,m+p]}h(p,k)}\right\} \tag{2.12}$$

and

$$h(x,k)=k^{(x-1)}(m+n-k)^{(n-x)}.$$

LEMMA 2.2. *For $(k,\ell)\in[0,m+n]\times[0,m]$, we have*

$$(-1)^{n-p}G(k,\ell)=|G(k,\ell)|\le\max\{q(\ell),r(\ell)\}\equiv\phi(\ell), \tag{2.13}$$

where

$$q(\ell)=\sum_{j=0}^{p-1}\left[\sum_{i=0}^{p-j-1}\binom{n-p+i-1}{i}\frac{(m+n)^{(j+i)}}{(m+n-j)^{(n-p+i)}}\right]\frac{(\ell+n-j-1)^{(n-j-1)}}{j!(n-j-1)!}(m+n-\ell-1)^{(n-p)}$$

and

$$r(\ell)=\sum_{j=0}^{n-p-1}\left[\sum_{i=0}^{n-p-j-1}\binom{p+i-1}{i}\frac{(m+p+j+i)^{(j+i)}}{(m+p+1+j+i)^{(p+i)}}\right]\frac{(m+p-\ell)^{(n-j-1)}}{j!(n-j-1)!}\ell^{(p)}.$$

PROOF. It is clear from (2.8) that

$$|G(k,\ell)|\le\left\{\begin{matrix} q(\ell), & 0\le\ell\le k-1 \\ r(\ell), & k\le\ell\le m\end{matrix}\right\}\le\phi(\ell).$$

For a nontrivial $y:[0,m+n]\to[0,\infty)$, we shall denote

$$a=\sum_{\ell=0}^{m}\phi(\ell)v(\ell)f(y(\ell))\quad\text{and}\quad b=\sum_{\ell=0}^{m}\|G(\cdot,\ell)\|u(\ell)f(y(\ell)). \tag{2.14}$$

In view of Lemma 2.2, (A2) and (A3), it is clear that $a\ge b\ge 0$. Further, we define the constant

$$\gamma=K_p\,k_0\min_{\ell\in[0,m]}\frac{\|G(\cdot,\ell)\|}{\phi(\ell)}, \tag{2.15}$$

where K_p is given in (2.12). It is noted that $0<\gamma<1$.

3. CHARACTERIZATION OF EIGENVALUES

Let B be the Banach space defined by

$$B = \{y \mid y : [0, m+n] \to \mathbb{R}\}$$

with norm $\|y\| = \max_{k\in[0,m+n]} |y(k)|$, and let

$$C = \left\{ y \in B \mid y(k) \geq 0,\ k \in [0, m+n];\ \min_{k\in[p,m+p]} y(k) \geq \gamma\|y\| \right\}.$$

We note that C is a cone in B. Further, we denote

$$C(N) = \{y \in C \mid \|y\| \leq N\}.$$

Let the operator $S : C \to B$ be defined by

$$Sy(k) = \sum_{\ell=0}^{m} (-1)^{n-p} G(k,\ell) F\left(\ell, y(\ell), \Delta y(\ell), \dots, \Delta^{n-1} y(\ell)\right), \qquad k \in [0, m+n]. \tag{3.1}$$

To obtain a positive solution of (1.1), we shall seek a fixed point of the operator λS in the cone C.

It is clear from (A2) and (2.9) that

$$\sum_{\ell=0}^{m} (-1)^{n-p} G(k,\ell) u(\ell) f(y(\ell)) \leq Sy(k) \leq \sum_{\ell=0}^{m} (-1)^{n-p} G(k,\ell) v(\ell) f(y(\ell)), \quad k \in [0, m+n]. \tag{3.2}$$

THEOREM 3.1. *There exists $c > 0$ such that the interval $(0, c] \subseteq E$.*

PROOF. Let $N > 0$ be given. Define

$$c = \frac{N}{f(N)} \left[\sum_{\ell=0}^{m} \phi(\ell) v(\ell) \right]^{-1}. \tag{3.3}$$

Let $\lambda \in (0, c]$. We shall prove that λS maps $C(N)$ into $C(N)$. For this, let $y \in C(N)$. We shall first show that $\lambda Sy \in C$. From (3.2) and (A3), it is clear that

$$(\lambda Sy)(k) \geq \lambda \sum_{\ell=0}^{m} (-1)^{n-p} G(k,\ell) u(\ell) f(y(\ell)) \geq 0, \qquad k \in [0, m+n]. \tag{3.4}$$

Further, it follows from (3.2), Lemma 2.2, and (2.14) that

$$Sy(k) \leq \sum_{\ell=0}^{m} (-1)^{n-p} G(k,\ell) v(\ell) f(y(\ell)) \leq \sum_{\ell=0}^{m} \phi(\ell) v(\ell) f(y(\ell)) = a, \qquad k \in [0, m+n].$$

Thus,

$$\|Sy\| \leq a. \tag{3.5}$$

Now, in view of (3.2), Lemma 2.1, (2.14), (3.5), and (2.15), we find for $k \in [p, m+p]$,

$$(\lambda Sy)(k) \geq \lambda \sum_{\ell=0}^{m} K_p \|G(\cdot,\ell)\| u(\ell) f(y(\ell)) = \lambda K_p b \geq \lambda\gamma\|Sy\| = \gamma\|\lambda Sy\|.$$

Therefore,

$$\min_{k\in[p,m+p]} (\lambda Sy)(k) \geq \gamma\|\lambda Sy\|. \tag{3.6}$$

Coupling (3.4) and (3.6), we see that $\lambda Sy \in C$.

Next, we shall verify that $\|\lambda Sy\| \leq N$. For this, on using (3.2), Lemma 2.2, (A1), and (3.3) successively, we get for $k \in [0, m+n]$,

$$(\lambda Sy)(k) \leq \lambda \sum_{\ell=0}^{m} \phi(\ell) v(\ell) f(y(\ell)) \leq \lambda \sum_{\ell=0}^{m} \phi(\ell) v(\ell) f(N) \leq c \sum_{\ell=0}^{m} \phi(\ell) v(\ell) f(N) = N.$$

Hence, $\|\lambda Sy\| \leq N$. We have shown that $(\lambda S)(C(N)) \subseteq C(N)$. Also, the standard arguments yield that λS is completely continuous. By Schauder's fixed-point theorem, λS has a fixed point in $C(N)$. Clearly, this fixed point is a positive solution of (1.1), and therefore, λ is an eigenvalue of (1.1). Since $\lambda \in (0, c]$ is arbitrary, it follows immediately that the interval $(0, c] \subseteq E$.

THEOREM 3.2. *If $\lambda_0 \in E$, then $(0, \lambda_0] \subseteq E$. So E is an interval.*

PROOF. The proof makes use of the monotonicity and compactness of the operator S on the cone C. We refer to [23, Theorem 3.2] for details.

THEOREM 3.3. *Let λ be an eigenvalue of (1.1) and $y \in C$ be a corresponding eigenfunction.*

(a) *Suppose that $(n-p)$ is odd and $\Delta^i y(0) = b_i$, $p \le i \le n-1$ with $b_{n-1} > 0$. Then, λ satisfies*

$$\max_{0 \le j \le p-1} A(j,\eta) \le \lambda \le \min_{0 \le j \le p-1} B(j,\eta), \qquad \text{if } n-p \ge p \tag{3.7}$$

and

$$\max_{0 \le j \le n-p-1} A(j,\theta) \le \lambda \le \min_{0 \le j \le n-p-1} B(j,\theta), \qquad \text{if } n-p < p, \tag{3.8}$$

where

$$\eta(k) = \sum_{i=0}^{p-1} b_{n-p+i} \frac{k^{(i)}}{i!}, \qquad \theta(k) = \sum_{i=0}^{n-p-1} b_{p+i} \frac{k^{(i+2p-n)}}{(i+2p-n)!},$$

$$\rho(x,k) = \sum_{\ell=0}^{k-1} x(\ell) \frac{(k-1-\ell)^{(p-1)}}{(p-1)!}, \qquad D = \max_{k \in [0,m+n]} \sum_{i=0}^{n-p-1} b_{p+i} \frac{k^{(i+p)}}{(i+p)!},$$

$$A(j,x) = \left[\sum_{\ell=0}^{m+p} x(\ell) \frac{(\ell+n-p-1-j)^{(n-p-1-j)}}{(n-p-1-j)!}\right] \left[f(D) \sum_{\ell=0}^{m+p} \rho(v,\ell) \frac{(\ell+n-p-1-j)^{(n-p-1-j)}}{(n-p-1-j)!}\right]^{-1},$$

and

$$B(j,x) = \left[\sum_{\ell=0}^{m+p} x(\ell) \frac{(\ell+n-p-1-j)^{(n-p-1-j)}}{(n-p-1-j)!}\right] \left[f(0) \sum_{\ell=0}^{m+p} \rho(u,\ell) \frac{(\ell+n-p-1-j)^{(n-p-1-j)}}{(n-p-1-j)!}\right]^{-1}.$$

(b) *Suppose that $(n-p)$, $(p-1)$ are even and $\Delta^i y(m+n-i) = a_i$, $n-p \le i \le n-1$ with $a_{n-1} > 0$. Then, λ satisfies*

$$\max_{0 \le j \le n-p-1} \bar{A}(j,\xi) \le \lambda \le \min_{0 \le j \le n-p-1} \bar{B}(j,\xi), \qquad \text{if } n-p \le p \tag{3.9}$$

and

$$\max_{0 \le j \le p-1} \bar{A}(j,\psi) \le \lambda \le \min_{0 \le j \le p-1} \bar{B}(j,\psi), \qquad \text{if } n-p > p, \tag{3.10}$$

where

$$\xi(k) = \sum_{i=0}^{n-p-1} (-1)^{i+1} a_{p+i} \frac{(m-k+n-p)^{(i)}}{i!},$$

$$\psi(k) = \sum_{i=0}^{p-1} (-1)^i a_{n-p+i} \frac{(m-k+n-p)^{(i+n-2p)}}{(i+n-2p)!},$$

$$\beta(x,k) = \sum_{\ell=k}^{m} x(\ell) \frac{(\ell-k+n-p-1)^{(n-p-1)}}{(n-p-1)!},$$

$$\bar{D} = \max_{k \in [0,m+n]} \sum_{i=0}^{p-1} (-1)^i a_{n-p+i} \frac{(m+n-k)^{(n-p+i)}}{(n-p+i)!},$$

$$\bar{A}(j,x) = \left[\sum_{\ell=0}^{m+p} x(\ell) \frac{(m+p-\ell)^{(p-1-j)}}{(p-1-j)!}\right] \left[f(\bar{D}) \sum_{\ell=0}^{m+p} \beta(v,\ell) \frac{(m+p-\ell)^{(p-1-j)}}{(p-1-j)!}\right]^{-1},$$

and

$$\bar{B}(j,x) = \left[\sum_{\ell=0}^{m+p} x(\ell) \frac{(m+p-\ell)^{(p-1-j)}}{(p-1-j)!}\right] \left[f(0) \sum_{\ell=0}^{m+p} \beta(u,\ell) \frac{(m+p-\ell)^{(p-1-j)}}{(p-1-j)!}\right]^{-1}.$$

PROOF.

(a) It is clear that y is the unique solution of the initial value problem

$$\begin{aligned}(-1)^{n-p}\Delta^n y &= \lambda F\left(k, y, \Delta y, \dots, \Delta^{n-1} y\right), \qquad k \in [0, m],\\ \Delta^i y(0) = 0, \quad 0 \le i \le p-1, &\qquad \Delta^i y(0) = b_i, \quad p \le i \le n-1.\end{aligned} \tag{3.11}$$

We shall obtain an upper bound for y. For this, from (A2) we have

$$(-1)^{n-p}\Delta^n y(k) = \lambda F\left(k, y(k), \Delta y(k), \dots, \Delta^{n-1} y(k)\right) \ge \lambda u(k) f(y(k)) \ge 0. \tag{3.12}$$

Hence, $\Delta^{n-1} y$ is nonincreasing and consequently,

$$\Delta^{n-1} y(k) \le \Delta^{n-1} y(0) = b_{n-1}, \qquad k \in [0, m+1]. \tag{3.13}$$

Using the relation

$$\Delta^i y(k) = b_i + \sum_{\ell=0}^{k-1} \Delta^{i+1} y(\ell), \qquad p \le i \le n-2, \quad k \in [0, m+n-i] \tag{3.14}$$

and also (3.13), we find

$$\Delta^{n-2} y(k) = b_{n-2} + \sum_{\ell=0}^{k-1} \Delta^{n-1} y(\ell) \le b_{n-2} + b_{n-1} k, \qquad k \in [0, m+2].$$

Applying the above inequality and continuing summing, we get

$$\Delta^p y(k) \le \sum_{i=0}^{n-p-1} b_{p+i} \frac{k^{(i)}}{i!}, \qquad k \in [0, m+n-p]. \tag{3.15}$$

Next, noting that

$$\Delta^i y(k) = \sum_{\ell=0}^{k-1} \Delta^{i+1} y(\ell), \qquad 0 \le i \le p-1, \quad k \in [0, m+n-i], \tag{3.16}$$

successive summation of (3.15) yields

$$y(k) \le \sum_{i=0}^{n-p-1} b_{p+i} \frac{k^{(i+p)}}{(i+p)!} \le D, \qquad k \in [0, m+n]. \tag{3.17}$$

Now, it follows from (3.11), (A2), (A1), and (3.17) that

$$\lambda u(k) f(0) \le (-1)^{n-p}\Delta^n y(k) \le \lambda v(k) f(D), \qquad k \in [0, m]. \tag{3.18}$$

CASE 1. $n - p \ge p$. In view of the initial conditions $\Delta^i y(0) = b_i$, $(p \le)\ n-p \le i \le n-1$, repeated summation of (3.18) from 0 to $(k-1)$ provides

$$\eta(k) - \lambda f(D)\rho(v, k) \le \Delta^{n-p} y(k) \le \eta(k) - \lambda f(0)\rho(u, k), \qquad k \in [0, m+p]. \tag{3.19}$$

Then, using the boundary conditions $\Delta^i y(m+n-i) = 0$, $0 \le i \le n-p-1$ or equivalently $\Delta^i y(m+p+1) = 0$, $0 \le i \le n-p-1$, we sum (3.19) from k to $(m+p)$ to get

$$Q(j, k) \le (-1)^{j+1}\Delta^j y(k) \le T(j, k), \qquad 0 \le j \le n-p-1, \quad k \in [0, m+n-j], \tag{3.20}$$

where

$$Q(j,k) = \sum_{\ell=k}^{m+p} \eta(\ell)\, \frac{(\ell-k+n-p-1-j)^{(n-p-1-j)}}{(n-p-1-j)!} \\ - \lambda f(D) \sum_{\ell=k}^{m+p} \rho(v,\ell) \frac{(\ell-k+n-p-1-j)^{(n-p-1-j)}}{(n-p-1-j)!}$$

and

$$T(j,k) = \sum_{\ell=k}^{m+p} \eta(\ell)\, \frac{(\ell-k+n-p-1-j)^{(n-p-1-j)}}{(n-p-1-j)!} \\ - \lambda f(0) \sum_{\ell=k}^{m+p} \rho(u,\ell) \frac{(\ell-k+n-p-1-j)^{(n-p-1-j)}}{(n-p-1-j)!}.$$

In order to have $\Delta^j y(0) = 0,\ 0 \le j \le p-1\ (\le n-p-1)$, from inequality (3.20), it is necessary that

$$Q(j,0) \le 0 \quad \text{and} \quad T(j,0) \ge 0, \qquad 0 \le j \le p-1,$$

or equivalently,

$$\lambda \ge A(j,\eta) \quad \text{and} \quad \lambda \le B(j,\eta), \qquad 0 \le j \le p-1.$$

Coupling the above two inequalities, we get (3.7) immediately.

CASE 2. $n-p<p$. Using the initial conditions $\Delta^i y(0) = b_i,\ p \le i \le n-1$, and $\Delta^i y(0) = 0$, $n-p \le i \le p-1$, we sum (3.18) from 0 to $(k-1)$ to get (3.19)$'$ which is (3.19) with η replaced by θ. Next, applying the boundary conditions $\Delta^i y(m+p+1) = 0,\ 0 \le i \le n-p-1$ and summing (3.19)$'$ from k to $(m+p)$, we obtain (3.20)$'$ which is (3.20) with η replaced by θ. In order that $\Delta^j y(0) = 0,\ 0 \le j \le n-p-1$, we follow a similar argument as in Case 1 and obtain (3.8).

PROOF.

(b) Clearly, y is the unique solution of the initial value problem

$$\begin{gathered}(-1)^{n-p}\Delta^n y = \lambda F\left(k, y, \Delta y, \dots, \Delta^{n-1} y\right), \qquad k \in [0,m], \\ \Delta^i y(m+n-i) = 0, \quad 0 \le i \le n-p-1, \qquad \Delta^i y(m+n-i) = a_i, \quad n-p \le i \le n-1.\end{gathered} \tag{3.21}$$

Once again, we shall obtain an upper estimate for y. For this, since $(n-p)$ is even, from (3.12), we see that $\Delta^{n-1} y$ is nondecreasing, and hence,

$$\Delta^{n-1} y(k) \le \Delta^{n-1} y(m+1) = a_{n-1}, \qquad k \in [0, m+1]. \tag{3.22}$$

Since

$$\Delta^i y(k) = a_i - \sum_{\ell=k}^{m+n-i-1} \Delta^{i+1} y(\ell), \qquad n-p \le i \le n-2, \quad k \in [0, m+n-i], \tag{3.23}$$

we find, in view of (3.22),

$$\Delta^{n-2} y(k) = a_{n-2} - \sum_{\ell=k}^{m+1} \Delta^{n-1} y(\ell) \ge a_{n-2} - a_{n-1}(m-k+2), \qquad k \in [0, m+2].$$

Using the above inequality and continuing the process, we get, on noting that p is odd,

$$\Delta^{n-p} y(k) \le \sum_{i=0}^{p-1} (-1)^i a_{n-p+i}\, \frac{(m-k+p)^{(i)}}{i!}, \qquad k \in [0, m+p]. \tag{3.24}$$

Next, applying the relation

$$\Delta^i y(k) = -\sum_{\ell=k}^{m+p} \Delta^{i+1} y(\ell), \qquad 0 \le i \le n-p-1, \quad k \in [0, m+n-i], \tag{3.25}$$

and summing (3.24), we finally obtain, on noting that $(n-p)$ is even,

$$y(k) \le \sum_{i=0}^{p-1} (-1)^i a_{n-p+i} \frac{(m+n-k)^{(n-p+i)}}{(n-p+i)!} \le \bar{D}, \qquad k \in [0, m+n]. \tag{3.26}$$

Subsequently, it follows from (3.21), (A2), (A1), and (3.26) that

$$\lambda u(k) f(0) \le (-1)^{n-p} \Delta^n y(k) \le \lambda v(k) f\left(\bar{D}\right), \qquad k \in [0, m]. \tag{3.27}$$

CASE 1. $n-p \le p$. Using the initial conditions $\Delta^i y(m+n-i) = a_i$, $(n-p \le)\ p \le i \le n-1$, successive summation of (3.27) from k to $(m+i)$, $0 \le i \le n-p-1$ gives

$$-\xi(k) + \lambda f(0) \beta(u,k) \le \Delta^p y(k) \le -\xi(k) + \lambda f\left(\bar{D}\right) \beta(v,k), \qquad k \in [0, m+n-p]. \tag{3.28}$$

Next, noting the initial conditions $\Delta^i y(0) = 0$, $0 \le i \le p-1$, we sum (3.28) from 0 to $(k-1)$ to get

$$\bar{Q}(j,k) \le \Delta^j y(k) \le \bar{T}(j,k), \qquad 0 \le j \le p-1, \quad k \in [0, m+n-j], \tag{3.29}$$

where

$$\bar{Q}(j,k) = -\sum_{\ell=0}^{k-1} \xi(\ell) \frac{(k-1-\ell)^{(p-1-j)}}{(p-1-j)!} + \lambda f(0) \sum_{\ell=0}^{k-1} \beta(u,\ell) \frac{(k-1-\ell)^{(p-1-j)}}{(p-1-j)!}$$

and

$$\bar{T}(j,k) = -\sum_{\ell=0}^{k-1} \xi(\ell) \frac{(k-1-\ell)^{(p-1-j)}}{(p-1-j)!} + \lambda f(\bar{D}) \sum_{\ell=0}^{k-1} \beta(v,\ell) \frac{(k-1-\ell)^{(p-1-j)}}{(p-1-j)!}.$$

Hence, in order to have $\Delta^j y(m+n-j) = 0$, $0 \le j \le n-p-1$ $(\le p-1)$, or equivalently, $\Delta^j y(m+p+1) = 0$, $0 \le j \le n-p-1$, from inequality (3.29), it is necessary that

$$\bar{Q}(j, m+p+1) \le 0 \quad \text{and} \quad \bar{T}(j, m+p+1) \ge 0, \qquad 0 \le j \le n-p-1$$

or

$$\lambda \le \bar{B}(j,\xi) \quad \text{and} \quad \lambda \ge \bar{A}(j,\xi), \qquad 0 \le j \le n-p-1.$$

A combination of the above two inequalities leads to (3.9) immediately.

CASE 2. $n-p > p$. Noting the initial conditions $\Delta^i y(m+n-i) = a_i$, $n-p \le i \le n-1$, and $\Delta^i y(m+n-i) = 0$, $p \le i \le n-p-1$, successive summation of (3.27) gives (3.28)$'$ which is the same as (3.28) with ξ replaced by ψ. Next, using the boundary conditions $\Delta^i y(0) = 0$, $0 \le i \le p-1$ and summing (3.28)$'$ from 0 to $(k-1)$, we obtain (3.29)$'$ which is (3.29) with ξ replaced by ψ. In order that $\Delta^j y(m+n-j) = 0$, $0 \le j \le p-1$, we follow a similar argument as in Case 1 and obtain (3.10).

THEOREM 3.4. *Let λ be an eigenvalue of (1.1) and $y \in C$ be a corresponding eigenfunction. Further, let $d = \|y\|$. Then,*

$$\lambda \ge \frac{d}{f(d)} \left[\sum_{\ell=0}^{m} \phi(\ell) v(\ell) \right]^{-1}, \tag{3.30}$$

and for all $z \in [p, m+p]$,

$$\lambda \le \frac{d}{f(\gamma d)} \left[\sum_{\ell=p}^{m} (-1)^{n-p} G(z,\ell) u(\ell) \right]^{-1}. \tag{3.31}$$

PROOF. First, we shall prove (3.30). For this, let $k_0 \in [p, m+p]$ be such that $d = \|y\| = y(k_0)$. Then, applying (3.2), Lemma 2.2, and (A1), we find

$$d = y(k_0) = (\lambda S y)(k_0) \le \lambda \sum_{\ell=0}^{m} \phi(\ell) v(\ell) f(y(\ell)) \le \lambda \sum_{\ell=0}^{m} \phi(\ell) v(\ell) f(d)$$

from which (3.30) is immediate.

Next, using (3.2) and (A1), we have for any $z \in [p, m+p]$,

$$d \ge y(z) \ge \lambda \sum_{\ell=p}^{m} (-1)^{n-p} G(z,\ell) u(\ell) f(y(\ell)) \ge \lambda \sum_{\ell=p}^{m} (-1)^{n-p} G(z,\ell) u(\ell) f(\gamma d),$$

which is exactly (3.31).

THEOREM 3.5. *Let*

$$F_B = \left\{ f \,\middle|\, \frac{x}{f(x)} \text{ is bounded for } x \in (0,\infty) \right\},$$
$$F_0 = \left\{ f \,\middle|\, \lim_{x\to\infty} \frac{x}{f(x)} = 0 \right\}, \quad \text{and} \quad F_\infty = \left\{ f \,\middle|\, \lim_{x\to\infty} \frac{x}{f(x)} = \infty \right\}.$$

(a) *If* $f \in F_B$, *then* $E = (0,c)$ *or* $(0,c]$ *for some* $c \in (0,\infty)$.
(b) *If* $f \in F_0$, *then* $E = (0,c]$ *for some* $c \in (0,\infty)$.
(c) *If* $f \in F_\infty$, *then* $E = (0,\infty)$.

PROOF.

(a) This is immediate from (3.31).
(b) Since $F_0 \subseteq F_B$, it follows from Case (a) that $E = (0,c)$ or $(0,c]$ for some $c \in (0,\infty)$. In particular, $c = \sup E$. Let $\{\lambda_m\}_{m=1}^{\infty}$ be a monotonically increasing sequence in E which converges to c, and let $\{y_m\}_{m=1}^{\infty}$ in C be a corresponding sequence of eigenfunctions. Further, let $d_m = \|y_m\|$. Then, (3.31) implies that no subsequence of $\{d_m\}_{m=1}^{\infty}$ can diverge to infinity. Thus, there exists $L > 0$ such that $d_m \le L$ for all m. So y_m is uniformly bounded. Hence, there is a subsequence of $\{y_m\}_{m=1}^{\infty}$, relabeled as the original sequence, which converges uniformly to some $y \in C$. Since $\lambda_m S y_m = y_m$, it follows that

$$cSy_m = \frac{c}{\lambda_m} y_m. \tag{3.32}$$

Further, noting that $\{cSy_m\}_{m=1}^{\infty}$ is relatively compact, y_m converges to y and λ_m converges to c, we let $m \to \infty$ in (3.32) to obtain $cSy = y$, i.e., $c \in E$. This completes the proof for Case (b).
(c) This follows from Theorem 3.2 and (3.30).

EXAMPLE 3.1. Consider the boundary value problem

$$\Delta^4 y(k) = \lambda \frac{(y+2)^r}{[k(k-1)(11-k)(12-k)+2]^r}, \quad k \in [0,8], \qquad y(0) = \Delta y(0) = y(12) = \Delta y(11) = 0,$$

where $\lambda > 0$ and $r \geq 0$. Here, $n = 4$, $p = 2$, and $m = 8$. By taking $f(y) = (y+2)^r$, we may choose

$$u(k) = v(k) = \frac{F\left(k, y, \Delta y, \Delta^2 y, \Delta^3 y\right)}{f(y)} = \frac{1}{[k(k-1)(11-k)(12-k)+2]^r}.$$

All the Hypotheses (A1)–(A3) are satisfied.

CASE 1. $0 \leq r < 1$. We have $f \in F_\infty$. Therefore, by Theorem 3.5(c) the set $E = (0, \infty)$. For instance, when $\lambda = 24$, the boundary value problem has a positive solution given by $y(k) = k(k-1)(11-k)(12-k)$.

CASE 2. $r = 1$. Since $f \in F_B$, by Theorem 3.5(a) the set E is an open or a half-closed interval. Further, from Case 1 and Theorem 3.2, we note that E contains the interval $(0, 24]$.

CASE 3. $r > 1$. Clearly, $f \in F_0$. Thus, by Theorem 3.5(b), the set E is a half-closed interval. Again, as in Case 2, it is noted that $(0, 24] \subseteq E$.

4. EIGENVALUE INTERVALS

In this section, we shall not require the monotonicity condition (A1). Further, let the integer $z^* \in [p, m+p]$ be defined by

$$\sum_{\ell=p}^{m}(-1)^{n-p}G(z^*, \ell)u(\ell) = \max_{k\in[0,m+n]} \sum_{\ell=p}^{m}(-1)^{n-p}G(k, \ell)u(\ell). \tag{4.1}$$

THEOREM 4.1. *Suppose that (A2) and (A3) hold. Then, $(L, R) \subseteq E$ where*

$$L = \left[K_p f_\infty \sum_{\ell=p}^{m}(-1)^{n-p}G(z^*, \ell)u(\ell)\right]^{-1} \quad \text{and} \quad R = \left[f_0 \sum_{\ell=0}^{m} \phi(\ell)v(\ell)\right]^{-1}.$$

PROOF. Let $\lambda \in (L, R)$. Noting that $\gamma \leq K_p$, we let $\epsilon > 0$ be such that

$$\left[\gamma(f_\infty - \epsilon)\sum_{\ell=p}^{m}(-1)^{n-p}G(z^*, \ell)u(\ell)\right]^{-1} \leq \lambda \leq \left[(f_0+\epsilon)\sum_{\ell=0}^{m}\phi(\ell)v(\ell)\right]^{-1}. \tag{4.2}$$

Next, we choose $w > 0$ so that

$$f(x) \leq (f_0 + \epsilon)x, \qquad 0 < x \leq w. \tag{4.3}$$

Let $y \in C$ be such that $\|y\| = w$. Then, applying (3.2), Lemma 2.2, (4.3), and (4.2) successively, we find for $k \in [0, m+n]$,

$$(\lambda S y)(k) \leq \lambda \sum_{\ell=0}^{m}\phi(\ell)v(\ell)(f_0+\epsilon)y(\ell) \leq \lambda \sum_{\ell=0}^{m}\phi(\ell)v(\ell)(f_0+\epsilon)\|y\| \leq \|y\|.$$

Hence,

$$\|\lambda S y\| \leq \|y\|. \tag{4.4}$$

If we set $\Omega_1 = \{y \in B \mid \|y\| < w\}$, then (4.4) holds for $y \in C \cap \partial\Omega_1$.

Further, we pick $T > 0$ so that

$$f(x) \geq (f_\infty - \epsilon)x, \qquad x \geq T. \tag{4.5}$$

Let $y \in C$ be such that $\|y\| = T' \equiv \max\{2w,\ T/\gamma\}$. Then, for $k \in [p, m+p]$, $y(k) \geq \gamma\|y\| \geq \gamma \cdot T/\gamma = T$. This, in view of (4.5), leads to

$$f(y(k)) \geq (f_\infty - \epsilon)y(k), \qquad k \in [p, m+p]. \tag{4.6}$$

Using (3.2), (4.6), and (4.2), we find

$$\begin{aligned}(\lambda Sy)(z^*) &\geq \lambda \sum_{\ell=p}^{m}(-1)^{n-p}G(z^*,\ell)u(\ell)f(y(\ell)) \ \geq \ \lambda \sum_{\ell=p}^{m}(-1)^{n-p}G(z^*,\ell)u(\ell)(f_\infty - \epsilon)y(\ell) \\ &\geq \lambda \sum_{\ell=p}^{m}(-1)^{n-p}G(z^*,\ell)u(\ell)(f_\infty - \epsilon)\gamma\|y\| \ \geq \ \|y\|.\end{aligned}$$

Therefore,

$$\|\lambda Sy\| \geq \|y\|. \tag{4.7}$$

By setting $\Omega_2 = \{y \in B \mid \|y\| < T'\}$, we see that (4.7) holds for $y \in C \cap \partial\Omega_2$.

Now that we have obtained (4.4) and (4.7), it follows from Theorem 2.1 that λS has a fixed point $y \in C \cap (\bar{\Omega}_2 \backslash \Omega_1)$ such that $w \leq \|y\| \leq T'$. Obviously, this y is a positive solution of (1.1).

THEOREM 4.2. *Suppose that (A2) and (A3) hold. Then, $(L', R') \subseteq E$ where*

$$L' = \left[K_p f_0 \sum_{\ell=p}^{m}(-1)^{n-p}G(z^*,\ell)u(\ell)\right]^{-1} \quad \text{and} \quad R' = \left[f_\infty \sum_{\ell=0}^{m}\phi(\ell)v(\ell)\right]^{-1}.$$

PROOF. Let $\lambda \in (L', R')$. Again, in view of the inequality $\gamma \leq K_p$, we pick $\epsilon > 0$ so that

$$\left[\gamma(f_0 - \epsilon)\sum_{\ell=p}^{m}(-1)^{n-p}G(z^*,\ell)u(\ell)\right]^{-1} \leq \lambda \leq \left[(f_\infty + \epsilon)\sum_{\ell=0}^{m}\phi(\ell)v(\ell)\right]^{-1}. \tag{4.8}$$

Let $\bar{w} > 0$ be such that

$$f(x) \geq (f_0 - \epsilon)x, \qquad 0 < x \leq \bar{w}. \tag{4.9}$$

Further, let $y \in C$ with $\|y\| = \bar{w}$. Then, on using (3.2), (4.9), and (4.8) successively, we get

$$(\lambda Sy)(z^*) \geq \lambda \sum_{\ell=p}^{m}(-1)^{n-p}G(z^*,\ell)u(\ell)(f_0 - \epsilon)y(\ell) \geq \|y\|.$$

Therefore, inequality (4.7) follows immediately. If we set $\Omega_1 = \{y \in B \mid \|y\| < \bar{w}\}$, then (4.7) holds for $y \in C \cap \partial\Omega_1$.

Next, we may choose $\bar{T} > 0$ such that

$$f(x) \leq (f_\infty + \epsilon)x, \qquad x \geq \bar{T}. \tag{4.10}$$

There are two cases to consider, namely, f is bounded and f is unbounded.

CASE 1. Suppose that f is bounded. Then, there exists some $M > 0$ such that

$$f(x) \leq M, \qquad x \in (0, \infty). \tag{4.11}$$

We define $T_1 = \max\{2\bar{w}, \ \lambda M \sum_{\ell=0}^{m}\phi(\ell)v(\ell)\}$. Let $y \in C$ be such that $\|y\| = T_1$. From (3.2), Lemma 2.2, and (4.11), we find for $k \in [0, m+n]$,

$$(\lambda Sy)(k) \leq \lambda \sum_{\ell=0}^{m}\phi(\ell)v(\ell)f(y(\ell)) \leq \lambda \sum_{\ell=0}^{m}\phi(\ell)v(\ell)M \leq T_1 = \|y\|.$$

Hence, (4.4) holds.

CASE 2. Suppose that f is unbounded. So there exists $T_1 > \max\{2\bar{w}, \bar{T}\}$ such that

$$f(x) \leq f(T_1), \qquad 0 < x \leq T_1. \tag{4.12}$$

Let $y \in C$ be such that $\|y\| = T_1$. Then, applying (3.2), Lemma 2.2, (4.12), (4.10), and (4.8) successively, we obtain for $k \in [0, m+n]$,

$$(\lambda S y)(k) \leq \lambda \sum_{\ell=0}^{m} \phi(\ell) v(\ell) f(T_1) \leq \lambda \sum_{\ell=0}^{m} \phi(\ell) v(\ell)(f_\infty + \epsilon) T_1 \leq T_1 = \|y\|.$$

Thus, (4.4) follows immediately.

In both Cases 1 and 2, if we set $\Omega_2 = \{y \in B \mid \|y\| < T_1\}$, then (4.4) holds for $y \in C \cap \partial\Omega_2$.

Now that we have obtained (4.7) and (4.4), it follows from Theorem 2.1 that λS has a fixed point $y \in C \cap (\bar{\Omega}_2 \backslash \Omega_1)$ such that $\bar{w} \leq \|y\| \leq T_1$. It is clear that this y is a positive solution of (1.1).

REMARK 4.1. If f is superlinear (i.e., $f_0 = 0$ and $f_\infty = \infty$) or sublinear (i.e., $f_0 = \infty$ and $f_\infty = 0$), then we conclude from Theorems 4.1 and 4.2 that $E = (0, \infty)$, i.e., the boundary value problem (1.1) has a positive solution for any $\lambda > 0$.

EXAMPLE 4.1. Consider the boundary value problem

$$\begin{gathered} -\Delta^3 y = \lambda \, \frac{1}{[ak(k-1)(m+3-k)+b]^r} \, (ay+b)^r, \qquad k \in [0, m], \\ y(0) = \Delta y(0) = y(m+3) = 0, \end{gathered}$$

where λ, a, $b > 0$, and $r \leq 1$. In this example, $n = 3$ and $p = 2$. Choosing $f(y) = (ay+b)^r$, we may take $u(k) = v(k) = [ak(k-1)(m+3-k)+b]^{-r}$. The Hypotheses (A2) and (A3) are satisfied.

CASE 1. $r < 1$. It is clear that f is sublinear. Hence, in view of Remark 4.1, for any $\lambda > 0$ the boundary value problem has a positive solution. Further, we remark that for the special case $0 < r < 1$, Hypothesis (A1) is fulfilled and $f \in F_\infty$. Thus, by Theorem 3.5(c), we also have $E = (0, \infty)$. In fact, it is noted that when $\lambda = 6$, the corresponding eigenfunction is given by $y(k) = k(k-1)(m+3-k)$.

CASE 2. $r = 1$. Here, $f_0 = \infty$ and $f_\infty = a$. As a specific example, we let $m = 6$, $a = 0.1$, and $b = 150$. By direct computation, we obtain from Lemma 2.2 that $\phi(\ell) = q(\ell)$, $\ell \in [0, 6]$. Hence, by Theorem 4.2, we have

$$E \supseteq \left(0,\ 10 \left[\sum_{\ell=0}^{6} q(\ell) v(\ell)\right]^{-1}\right) \supseteq (0, 7.91).$$

As an example, when $\lambda = 6 \in (0, 7.91)$, the corresponding eigenfunction is given by $y(k) = k(k-1)(9-k)$.

EXAMPLE 4.2. Consider the boundary value problem

$$\Delta^4 y = \lambda[(k+2)(2y+1) - y - 1], \quad k \in [0, 8], \qquad y(0) = y(12) = \Delta y(11) = \Delta^2 y(10) = 0,$$

where $\lambda > 0$. Here, $n = 4$, $p = 1$, and $m = 8$. Let $f(y) = y + 1$. Then, $f_0 = \infty$, $f_\infty = 1$ and we may take $u(k) = k + 1$ and $v(k) = 2k + 3$. Hypotheses (A2) and (A3) are satisfied. From Lemma 2.2, we obtain $\phi(0) = q(0)$ and $\phi(\ell) = r(\ell)$, $\ell \in [1, 8]$. Hence, $\sum_{\ell=0}^{8} \phi(\ell) v(\ell) = 10764$ and we conclude by Theorem 4.2 that $(0, 9.29 \times 10^{-5}) \subseteq E$.

5. EXISTENCE OF TWO POSITIVE SOLUTIONS

In this section, let $\lambda = 1$ in (1.1). Further, we shall not require the monotonicity condition (A1).

THEOREM 5.1. *Let $w > 0$ be given. Suppose that f satisfies*

$$0 < f(x) \le w \left[\sum_{\ell=0}^{m} \phi(\ell)v(\ell)\right]^{-1}, \qquad 0 < x \le w. \tag{5.1}$$

(a) *If $f_0 = \infty$, then there exists a positive solution y_1 of (1.1) with $0 < \|y_1\| \le w$.*
(b) *If $f_\infty = \infty$, then there exists a positive solution y_2 of (1.1) with $\|y_2\| \ge w$.*
(c) *If $f_0 = f_\infty = \infty$, then there exist two positive solutions y_1 and y_2 of (1.1) with*

$$0 < \|y_1\| \le w \le \|y_2\|.$$

PROOF. Let

(a)

$$A = \left[\gamma \sum_{\ell=p}^{m} (-1)^{n-p} G(p,\ell)u(\ell)\right]^{-1}. \tag{5.2}$$

Since $f_0 = \infty$, there exists $r \in (0, w)$ such that

$$f(x) \ge Ax, \qquad 0 < x \le r. \tag{5.3}$$

Let $y \in C$ be such that $\|y\| = r$. Then, on using (3.2), (5.3), and (5.2) successively, we find

$$Sy(p) \ge \sum_{\ell=p}^{m} (-1)^{n-p} G(p,\ell)u(\ell)A\, y(\ell) \ge \sum_{\ell=p}^{m} (-1)^{n-p} G(p,\ell)u(\ell)A\gamma\|y\| = \|y\|. \tag{5.4}$$

This immediately implies that

$$\|Sy\| \ge \|y\|. \tag{5.5}$$

If we set $\Omega_1 = \{y \in B \mid \|y\| < r\}$, then (5.5) holds for $y \in C \cap \partial\Omega_1$.

Next, let $y \in C$ be such that $\|y\| = w$. Then, in view of (3.2), Lemma 2.2, and (5.1), we find for $k \in [0, m+n]$,

$$Sy(k) \le \sum_{\ell=0}^{m} \phi(\ell)v(\ell)f(y(\ell)) \le w = \|y\|.$$

Hence,

$$\|Sy\| \le \|y\|. \tag{5.6}$$

By setting $\Omega_2 = \{y \in B \mid \|y\| < w\}$, we see that (5.6) holds for $y \in C \cap \partial\Omega_2$.

Having obtained (5.5) and (5.6), it follows from Theorem 2.1 that S has a fixed point $y_1 \in C \cap (\bar{\Omega}_2 \backslash \Omega_1)$ such that $r \le \|y_1\| \le w$. Clearly, this y_1 is a positive solution of (1.1).

(b) As in Case (a), condition (5.1) gives rise to (5.6). Hence, if we set $\Omega_1 = \{y \in B \mid \|y\| < w\}$, then (5.6) holds for $y \in C \cap \partial\Omega_1$.

Next, let A be defined as in (5.2). Since $f_\infty = \infty$, we may choose $T > w$ such that

$$f(x) \ge Ax, \qquad x \ge T. \tag{5.7}$$

Let $y \in C$ be such that $\|y\| = T/\gamma$. Then, for $k \in [p, m+p]$, $y(k) \geq \gamma\|y\| = \gamma \cdot T/\gamma = T$, which, in view of (5.7), leads to

$$f(y(k)) \geq A\, y(k), \qquad k \in [p, m+p]. \tag{5.8}$$

Using (3.2), (5.8), and (5.2), we find (5.4) and so (5.5) holds. If we set $\Omega_2 = \{y \in B \mid \|y\| < T/\gamma\}$, then (5.5) holds for $y \in C \cap \partial\Omega_2$.

Now that we have obtained (5.6) and (5.5), it again follows from Theorem 2.1 that S has a fixed point $y_2 \in C \cap (\bar{\Omega}_2 \backslash \Omega_1)$ such that $w \leq \|y_2\| \leq T/\gamma$. It is clear that this y_2 is a positive solution of (1.1).

(c) This is a direct consequence of Cases (a) and (b).

THEOREM 5.2. *Let $w > 0$ be given. Suppose that f satisfies*

$$f(x) \geq w \left[\sum_{\ell=p}^{m} (-1)^{n-p} G(p,\ell) u(\ell)\right]^{-1}, \qquad \gamma w \leq u \leq w. \tag{5.9}$$

(a) *If $f_0 = 0$, then there exists a positive solution y_1 of (1.1) with $0 < \|y_1\| \leq w$.*
(b) *If $f_\infty = 0$, then there exists a positive solution y_2 of (1.1) with $\|y_2\| \geq w$.*
(c) *If $f_0 = f_\infty = 0$, then there exist two positive solutions y_1 and y_2 of (1.1) with*

$$0 < \|y_1\| \leq w \leq \|y_2\|.$$

PROOF.

(a) Let

$$\epsilon = \left[\sum_{\ell=0}^{m} \phi(\ell) v(\ell)\right]^{-1}. \tag{5.10}$$

Since $f_0 = 0$, there exists $\delta \in (0, w)$ such that

$$f(x) \leq \epsilon\, x, \qquad 0 < x \leq \delta. \tag{5.11}$$

Let $y \in C$ be such that $\|y\| = \delta$. Then, applying (3.2), Lemma 2.2, (5.11), and (5.10) successively yields

$$Sy(k) \leq \sum_{\ell=0}^{m} \phi(\ell) v(\ell) \epsilon\, y(\ell) \leq \sum_{\ell=0}^{m} \phi(\ell) v(\ell) \epsilon\, \|y\| = \|y\|, \qquad k \in [0, m+n]. \tag{5.12}$$

Hence, (5.6) follows. If we set $\Omega_1 = \{y \in B \mid \|y\| < \delta\}$, then (5.6) holds for $y \in C \cap \partial\Omega_1$.

Next, let $y \in C$ be such that $\|y\| = w$. Then, it follows from (3.2) and (5.9) that

$$Sy(p) \geq \sum_{\ell=p}^{m} (-1)^{n-p} G(p,\ell) u(\ell) f(y(\ell)) \geq w = \|y\|.$$

Thus, (5.5) holds. By setting $\Omega_2 = \{y \in B \mid \|y\| < w\}$, we see that (5.5) holds for $y \in C \cap \partial\Omega_2$.

Having obtained (5.6) and (5.5), it follows from Theorem 2.1 that S has a fixed point $y_1 \in C \cap (\bar{\Omega}_2 \backslash \Omega_1)$ such that $\delta \leq \|y_1\| \leq w$. Clearly, this y_1 is a positive solution of (1.1).

PROOF.

(b) As in Case (a), condition (5.9) gives rise to (5.5). So if we set $\Omega_1 = \{y \in B \mid \|y\| < w\}$, then (5.5) holds for $y \in C \cap \partial\Omega_1$.

Next, let ϵ be defined as in (5.10). Since $f_\infty = 0$, we may choose $M > w$ such that

$$f(x) \leq \epsilon\, x, \qquad x \geq M. \tag{5.13}$$

Let $y \in C$ be such that $\|y\| = M$. Then, by (3.2), Lemma 2.2, and (5.13) we obtain (5.12) and so (5.6) holds. If we set $\Omega_2 = \{y \in B \mid \|y\| < M\}$, then (5.6) holds for $y \in C \cap \partial\Omega_2$.

Now that we have obtained (5.5) and (5.6), once again it follows from Theorem 2.1 that S has a fixed point $y_2 \in C \cap (\bar{\Omega}_2 \backslash \Omega_1)$ such that $w \le \|y_2\| \le M$. It is clear that this y_2 is a positive solution of (1.1).

PROOF.

(c) This is immediate from Cases (a) and (b).

EXAMPLE 5.1. Consider the boundary value problem

$$\Delta^3 y(k) = \frac{6}{[k(8-k)(7-k)]^2 + M}\left(y^2 + M\right), \quad k \in [0,5], \qquad y(0) = \Delta y(7) = y(8) = 0,$$

where $M > 0$. Here, $n = 3$, $p = 1$, and $m = 5$. By taking $f(y) = y^2 + M$, we may pick $u(k) = v(k) = 6\{[k(8-k)(7-k)]^2 + M\}^{-1}$. Clearly, Hypotheses (A2) and (A3) are satisfied. Further, from Lemma 2.2, it is computed that $\phi(0) = q(0)$ and $\phi(\ell) = r(\ell)$, $\ell \in [1,5]$.

It is obvious that $f_0 = f_\infty = \infty$. We aim to find some $w > 0$ such that condition (5.1) is fulfilled. To begin, we see that

$$\sum_{\ell=0}^{m} \phi(\ell) v(\ell) \le \sum_{\ell=0}^{m} \phi(\ell)\, \frac{6}{M} = \frac{244.5}{M}.$$

This implies that

$$\left[\sum_{\ell=0}^{m} \phi(\ell) v(\ell)\right]^{-1} \ge \frac{M}{244.5}.$$

Thus, noting that $f(y) = y^2 + M \le w^2 + M$ for $0 < y \le w$, condition (5.1) is fulfilled provided

$$f(y) \le w^2 + M \le w\, \frac{M}{244.5} \le w \left[\sum_{\ell=0}^{m} \phi(\ell) v(\ell)\right]^{-1}, \qquad 0 < y \le w.$$

The above relation reduces to $244.5w^2 - wM + 244.5M \le 0$. Clearly, this quadratic inequality holds for some $w > 0$ if and only if $M \ge 239121$.

As an example, take $M = 239121$. Then, in order that (5.1) is satisfied, we set

$$f(y) \le w^2 + 239121 \le w \left[\sum_{\ell=0}^{m} \phi(\ell) v(\ell)\right]^{-1} \le 989w, \qquad 0 < y \le w.$$

This gives $421 \le w \le 568$. Hence, (5.1) holds for any $w \in [421, 568]$. By Theorem 5.1(c), there exist two positive solutions y_1 and y_2 with $0 < \|y_1\| \le w \le \|y_2\|$. Since $w \in [421, 568]$, it is clear that

$$0 < \|y_1\| \le 421 \quad \text{and} \quad \|y_2\| \ge 568.$$

In fact, one positive solution is given by $y(k) = k(8-k)(7-k)$ and we note that $\|y\| = 60$ is within one of the ranges given above.

6. POSITIVE SOLUTIONS OF (1.2)

THEOREM 6.1. *Let $w > 0$ be given. Suppose that*

$$\frac{1}{m+2} \sum_{\ell=0}^{m} (m+1-\ell)(\ell+1) a(\ell) \le \frac{w}{w^\alpha + w^\beta}. \tag{6.1}$$

Then, the boundary value problem (1.2) has two positive solutions y_1 and y_2 such that

$$0 < \|y_1\| \leq w \leq \|y_2\|.$$

PROOF. In (1.2), $F(k,y) = a(k)\ (y^\alpha + y^\beta)$. If we take $f(x) = x^\alpha + x^\beta$, then we may pick $u(k) = v(k) = a(k)$. It is also noted that $f_0 = f_\infty = \infty$. Next, since $f(x) \leq w^\alpha + w^\beta$ for $0 < x \leq w$, it follows that (5.1) is fulfilled provided that

$$w^\alpha + w^\beta \leq w \left[\sum_{\ell=0}^{m} \phi(\ell) v(\ell)\right]^{-1}. \tag{6.2}$$

Noting that $n = 2$ and $p = 1$, from Lemma 2.2, we compute that

$$\phi(\ell) = q(\ell) = \frac{(m+1-\ell)(\ell+1)}{m+2}, \qquad \ell \in [0, m].$$

Therefore, inequality (6.2) is exactly (6.1). The conclusion is now clear from Theorem 5.1(c).

REMARK 6.1. In [15], we have also discussed the boundary value problem (1.2). The condition corresponding to (6.1) is obtained as [15]

$$\sum_{\ell=0}^{m} (m+1-\ell) a(\ell) \leq \frac{w}{w^\alpha + w^\beta}.$$

Clearly, this is a stronger condition than (6.1), and hence, (6.1) is an improvement.

EXAMPLE 6.1. Consider the boundary value problem (1.2) with $m = 7$. Let $w = 1$ be given. Then, condition (6.1) reduces to

$$\frac{1}{9} \sum_{\ell=0}^{7} (8-\ell)(\ell+1) a(\ell) \leq \frac{1}{2}. \tag{6.3}$$

By Theorem 6.1, for those $a(k)$ which fulfill (6.3), the boundary value problem has double positive solutions y_1 and y_2 such that $0 < \|y_1\| \leq 1 \leq \|y_2\|$. Some examples of such $a(k)$ are $a(k) = [8(k+1)]^{-1}$, $(k+2)/147$.

Now, we shall establish upper and lower bounds for the two positive solutions of (1.2).

THEOREM 6.2. *We define*

$$A(x) = \max_{\delta \in [1,m]} \frac{\delta}{m+2} \left(\frac{1}{m+2-\delta}\right)^x \sum_{\ell=\delta}^{m} (m+1-\ell) a(\ell),$$
$$w_1 = [A(\alpha)]^{1/1-\alpha}, \quad \text{and} \quad w_2 = [A(\beta)]^{1/1-\beta}.$$

Let $w > 0$ be given. Suppose that (6.1) holds. Then, the boundary value problem (1.2) has two positive solutions y_1 and y_2 such that

(a) *if $w < \min\{w_1, w_2\}$, then $0 < \|y_1\| \leq w \leq \|y_2\| \leq \min\{w_1, w_2\}$;*
(b) *if $\min\{w_1, w_2\} < w < \max\{w_1, w_2\}$, then*

$$\min\{w_1, w_2\} \leq \|y_1\| \leq w \leq \|y_2\| \leq \max\{w_1, w_2\};$$

(c) *if $w > \max\{w_1, w_2\}$, then $\max\{w_1, w_2\} \leq \|y_1\| \leq w \leq \|y_2\|$.*

PROOF. Since (6.1) is satisfied, it follows from Theorem 6.1 that (1.2) has double positive solutions y_3 and y_4 such that

$$0 < \|y_3\| \le w \le \|y_4\|. \tag{6.4}$$

To establish upper and lower bounds for the two positive solutions, for an arbitrary $\delta \in [1, m]$, we let C_δ be a cone in B defined by

$$C_\delta = \left\{ y \in B \;\middle|\; y(k) \text{ is nonnegative on } [0, m+2],\ \min_{k \in [\delta, m+1]} y(k) \ge \frac{1}{m+2-\delta} \|y\| \right\}. \tag{6.5}$$

Define the operator $S : C_\delta \to B$ by

$$Sy(k) = \sum_{\ell=0}^{m} -g(k,\ell)a(\ell) \left[y(\ell)^\alpha + y(\ell)^\beta\right], \qquad k \in [0, m+2],$$

where $g(k,\ell) = G(k,\ell)|_{n=2,\ p=1}$. To obtain a positive solution of (1.2), we shall seek a fixed point of S in the cone C_δ.

First, we shall show that the operator S maps C_δ into itself. For this, let $y \in C_\delta$. It is obvious that $Sy(k)$ is nonnegative on $[0, m+2]$. Further, we have

$$Sy(k) \le \sum_{\ell=0}^{m} \|g(\cdot,\ell)\| a(\ell) \left[y(\ell)^\alpha + y(\ell)^\beta\right], \qquad k \in [0, m+2],$$

which gives

$$\|Sy\| \le \sum_{\ell=0}^{m} \|g(\cdot,\ell)\| a(\ell) \left[y(\ell)^\alpha + y(\ell)^\beta\right]. \tag{6.6}$$

Now, from (2.12) $(n = 2,\ p = 1)$, we compute that

$$K_\delta = \min\left\{\frac{\delta}{m+1}, \frac{1}{m+2-\delta}\right\} = \frac{1}{m+2-\delta}.$$

Thus, applying Lemma 2.1 and (6.6), we find for $k \in [\delta, m+1]$,

$$Sy(k) \ge \sum_{\ell=0}^{m} \frac{1}{m+2-\delta} \|g(\cdot,\ell)\| a(\ell) \left[y(\ell)^\alpha + y(\ell)^\beta\right] \ge \frac{1}{m+2-\delta} \|Sy\|.$$

Consequently,

$$\min_{k \in [\delta, m+1]} Sy(k) \ge \frac{1}{m+2-\delta} \|Sy\|$$

and so $Sy \in C_\delta$. Also, the standard arguments yield that S is completely continuous.

Let $y \in C_\delta$ be such that $\|y\| = w$. Then, in view of Lemma 2.2 and (6.1), we find

$$Sy(k) \le \sum_{\ell=0}^{m} \frac{(m+1-\ell)(\ell+1)}{m+2} a(\ell) \left(w^\alpha + w^\beta\right) \le w = \|y\|, \qquad k \in [0, m+2].$$

Hence, (5.6) follows. If we set $\Omega = \{y \in B \mid \|y\| < w\}$, then (5.6) holds for $y \in C_\delta \cap \partial\Omega$.

Now, let $y \in C_\delta$. It follows that

$$\begin{aligned} \|Sy\| &\ge \sum_{\ell=0}^{m} -g(\delta,\ell)a(\ell) \left[y(\ell)^\alpha + y(\ell)^\beta\right] \ge \sum_{\ell=\delta}^{m} -g(\delta,\ell)a(\ell) \left[y(\ell)^\alpha + y(\ell)^\beta\right] \\ &\ge \sum_{\ell=\delta}^{m} -g(\delta,\ell)a(\ell) \left[\left(\frac{1}{m+2-\delta}\right)^\alpha \|y\|^\alpha + \left(\frac{1}{m+2-\delta}\right)^\beta \|y\|^\beta\right]. \end{aligned}$$

From (2.8), we find that $-g(\delta,\ell) = \delta(m+1-\ell)/(m+2)$. Using this in the above inequality and then taking maximum over δ, it follows immediately that

$$\|Sy\| \geq A(\alpha)\|y\|^{\alpha} + A(\beta)\|y\|^{\beta}. \tag{6.7}$$

Let $y \in C_\delta$ be such that $\|y\| = w_1$. Then, (6.7) provides

$$\|Sy\| \geq A(\alpha)\|y\|^{\alpha} = A(\alpha)\|y\|^{\alpha-1}\|y\| = \|y\|. \tag{6.8}$$

If we set $\Omega_1 = \{y \in B \mid \|y\| < w_1\}$, then (6.8) holds for $y \in C_\delta \cap \partial\Omega_1$. Now that we have obtained (5.6) and (6.8), it follows from Theorem 2.1 that S has a fixed point y_5 such that

$$\min\{w_1, w\} \leq \|y_5\| \leq \max\{w_1, w\}. \tag{6.9}$$

Likewise, if we let $y \in C_\delta$ be such that $\|y\| = w_2$, then from (6.7) we get

$$\|Sy\| \geq A(\beta)\|y\|^{\beta} = A(\beta)\|y\|^{\beta-1}\|y\| = \|y\|. \tag{6.10}$$

By setting $\Omega_2 = \{y \in B \mid \|y\| < w_2\}$, we see that (6.10) holds for $y \in C_\delta \cap \partial\Omega_2$. Having obtained (5.6) and (6.10), once again by Theorem 2.1, we conclude that S has a fixed point y_6 such that

$$\min\{w_2, w\} \leq \|y_6\| \leq \max\{w_2, w\}. \tag{6.11}$$

Now, a combination of (6.4), (6.9), and (6.11) yields our result. To be more precise, in Case (a), we may pick

$$y_1 = y_3 \quad \text{and} \quad y_2 = \begin{cases} y_5, & w_1 \leq w_2, \\ y_6, & w_1 \geq w_2. \end{cases}$$

In Case (b), it is clear that

$$(y_1, y_2) = \begin{cases} (y_5, y_6), & w_1 \leq w_2, \\ (y_6, y_5), & w_1 \geq w_2. \end{cases}$$

Finally, in Case (c), we shall take

$$y_1 = \begin{cases} y_6, & w_1 \leq w_2, \\ y_5, & w_1 \geq w_2, \end{cases} \quad \text{and} \quad y_2 = y_4.$$

EXAMPLE 6.2. Consider the boundary value problem

$$\Delta^2 y + \frac{1}{(2k+3)^2}\left(y^{0.5} + y^{1.1}\right) = 0, \quad k \in [0,6], \qquad y(0) = y(8) = 0.$$

Here, $\alpha = 0.5$, $\beta = 1.1$, and $a(k) = (2k+3)^{-2}$. Condition (6.1) is equivalent to

$$\frac{w}{w^{0.5} + w^{1.1}} \geq \frac{1}{8}\sum_{\ell=0}^{6} \frac{(7-\ell)(\ell+1)}{(2\ell+3)^2} \geq 0.248,$$

which is satisfied for any $w \geq 0.0956$. By direct computation, we find that

$$w_1 = [A(0.5)]^2 = 4.18 \times 10^{-4} \quad \text{and} \quad w_2 = [A(1.1)]^{-10} = 5.45 \times 10^{21}.$$

CASE 1. Let $w \in [0.0956, w_2)$. Then, by Theorem 6.2(b), the boundary value problem has two positive solutions y_1 and y_2 such that $w_1 \leq \|y_1\| \leq w \leq \|y_2\| \leq w_2$. Noting the range of w, this inequality leads to

$$4.18 \times 10^{-4} \leq \|y_1\| \leq 0.0956 \quad \text{and} \quad 5.45 \times 10^{21} - \epsilon \leq \|y_2\| \leq 5.45 \times 10^{21}, \tag{6.12}$$

where $\epsilon > 0$ is small.

CASE 2. Let $w > w_2$. Then, it follows from Theorem 6.2(c) that the boundary value problem has two positive solutions y_3 and y_4 such that $w_2 \leq \|y_3\| \leq w \leq \|y_4\|$. Therefore,

$$5.45 \times 10^{21} \leq \|y_3\| \leq 5.45 \times 10^{21} + \epsilon \quad \text{and} \quad \|y_4\| > 5.45 \times 10^{21} + \epsilon, \tag{6.13}$$

where $\epsilon > 0$ is small.

Combining (6.12) and (6.13), we see that the boundary value problem has (at least) four positive solutions.

7. POSITIVE SOLUTIONS OF (1.3)

THEOREM 7.1. *Let $w > 0$ be given. Suppose that*

$$\frac{1}{m+2}\sum_{\ell=0}^{m}(m+1-\ell)(\ell+1)a(\ell) \le we^{-\sigma w}. \tag{7.1}$$

Then, the boundary value problem (1.3) has two positive solutions y_1 and y_2 such that

$$0 < \|y_1\| \le w \le \|y_2\|.$$

PROOF. In (1.3), $F(k,y) = a(k)\, e^{\sigma y}$. By taking $f(x) = e^{\sigma x}$, we may choose $u(k) = v(k) = a(k)$. Also, it is obvious that $f_0 = f_\infty = \infty$. Noting that $f(x) \le e^{\sigma w}$ for $0 < x \le w$, we see that condition (5.1) is satisfied provided that

$$e^{\sigma w} \le w\left[\sum_{\ell=0}^{m}\phi(\ell)v(\ell)\right]^{-1} = w\left[\sum_{\ell=0}^{m}\frac{(m+1-\ell)(\ell+1)}{m+2}\,a(\ell)\right]^{-1},$$

or equivalently, condition (7.1) holds. The conclusion of the theorem follows immediately from Theorem 5.1(c).

EXAMPLE 7.1. Consider the boundary value problem

$$\Delta^2 y + a(k)\, e^y = 0, \quad k \in [0,9], \qquad y(0) = y(11) = 0.$$

Let $w = 1/2$ be given. Then, condition (7.1) reduces to

$$\frac{1}{11}\sum_{\ell=0}^{9}(10-\ell)(\ell+1)a(\ell) \le \frac{1}{2}\, e^{-1/2}. \tag{7.2}$$

By Theorem 7.1, for those $a(k)$ which fulfill (7.2), the boundary value problem has two positive solutions y_1 and y_2 such that $0 < \|y_1\| \le 1/2 \le \|y_2\|$. Some examples of such $a(k)$ are $a(k) = [6(k^3+3k+1)]^{-1}$, $(k^2+1)/2000$.

The next result offers upper and lower bounds for the two positive solutions of (1.3).

THEOREM 7.2. *Let $i \ne j$ be given integers in the set $\{0, 2, 3, \dots\}$. We define*

$$D(x) = \frac{1}{x!}\max_{\delta\in[1,m]}\frac{\delta}{m+2}\left(\frac{\sigma}{m+2-\delta}\right)^x\sum_{\ell=\delta}^{m}(m+1-\ell)a(\ell),$$
$$w_1 = [D(j)]^{1/1-j}, \quad \text{and} \quad w_2 = [D(i)]^{1/1-i}.$$

Let $w > 0$ be given. Suppose that (7.1) holds. Then, the boundary value problem (1.3) has twin positive solutions y_1 and y_2 such that conclusions (a)–(c) of Theorem 6.2 hold.

PROOF. Since (7.1) is fulfilled, by Theorem 7.1 the boundary value problem (1.3) has double positive solutions y_3 and y_4 such that (6.4) holds.

To establish further upper and lower bounds for the two positive solutions, let $\delta \in [1,m]$ and C_δ be a cone in B defined by (6.5). Further, we define the operator $S : C_\delta \to B$ by

$$Sy(k) = \sum_{\ell=0}^{m} -g(k,\ell)a(\ell)e^{\sigma y(\ell)}, \qquad k \in [0, m+2],$$

where $g(k,\ell) = G(k,\ell)|_{n=2,\ p=1}$. To obtain a positive solution of (1.3), we shall seek a fixed point of S in the cone C_δ. As in the proof of Theorem 6.2, it can be verified that S maps C_δ into itself and S is completely continuous.

Let $y \in C_\delta$ be such that $\|y\| = w$. Then, an application of Lemma 2.2 and (7.1) yields

$$Sy(k) \le \sum_{\ell=0}^{m} \frac{(m+1-\ell)(\ell+1)}{m+2}\, a(\ell)e^{\sigma w} \le w = \|y\|, \qquad k \in [0, m+2].$$

Hence, (5.6) holds. By setting $\Omega = \{y \in B \mid \|y\| < w\}$, we see that (5.6) holds for $y \in C_\delta \cap \partial\Omega$.

Next, let $y \in C_\delta$. We find that

$$\begin{aligned}\|Sy\| &\ge \sum_{\ell=0}^{m} -g(\delta,\ell)a(\ell)e^{\sigma y(\ell)} \ \ge \sum_{\ell=\delta}^{m} -g(\delta,\ell)a(\ell)e^{\sigma y(\ell)} \ \ge \sum_{\ell=\delta}^{m} -g(\delta,\ell)a(\ell)e^{(\sigma/m+2-\delta)\|y\|} \\ &\ge \sum_{\ell=\delta}^{m} -g(\delta,\ell)a(\ell)\left[\left(\frac{\sigma}{m+2-\delta}\right)^j \frac{\|y\|^j}{j!} + \left(\frac{\sigma}{m+2-\delta}\right)^i \frac{\|y\|^i}{i!}\right],\end{aligned}$$

where in the last inequality we have used the relation

$$e^x \ge \frac{x^j}{j!} + \frac{x^i}{i!}, \qquad x > 0.$$

Upon substituting $-g(\delta,\ell) = \delta(m+1-\ell)/(m+2)$ and taking maximum over δ, we get

$$\|Sy\| \ge D(j)\|y\|^j + D(i)\|y\|^i. \tag{7.3}$$

Following a similar technique as in the proof of Theorem 6.2, from (7.3) we obtain (5.5) for $y \in C \cap \partial\Omega_1$ as well as for $y \in C \cap \partial\Omega_2$, where

$$\Omega_1 = \{y \in B \mid \|y\| < w_1\} \quad \text{and} \quad \Omega_2 = \{y \in B \mid \|y\| < w_2\}.$$

Now that we have obtained (5.6) and (5.5), by Theorem 2.1, S has a fixed point y_5 satisfying (6.9) and also a fixed point y_6 such that (6.11) holds. As in the proof of Theorem 6.2, a combination of (6.4), (6.9), and (6.11) leads to conclusions (a)–(c) immediately.

EXAMPLE 7.2. Consider the boundary value problem

$$\Delta^2 y + \frac{2}{e^{(8k-k^2)/10000}}\, e^{y/10000} = 0, \qquad k \in [0,6], \qquad y(0) = y(8) = 0.$$

Let $i = 3$ and $j = 0$ be given. Here, $\sigma = 1/10000$ and $a(k) = 2e^{-(8k-k^2)/10000}$. By direct computation, condition (7.1) is satisfied provided that $21.1 \le w \le 82809$. Further, we compute that

$$w_1 = D(0) = 7.49 \quad \text{and} \quad w_2 = [D(3)]^{-1/2} = 5660249.$$

Since $w \in (w_1, w_2)$, it follows from Theorem 7.2(b) that the boundary value problem has two positive solutions y_1 and y_2 such that $w_1 \le \|y_1\| \le w \le \|y_2\| \le w_2$. Further, since $w \in [21.1, 82809]$, we may conclude that

$$7.49 \le \|y_1\| \le 21.1 \quad \text{and} \quad 82809 \le \|y_2\| \le 5660249. \tag{7.4}$$

In fact, a positive solution is given by $y(k) = k(8-k)$ and we note that $\|y\| = 16$ is within the range obtained in (7.4).

REFERENCES

1. D. Aronson, M.G. Crandall and L.A. Peletier, Stabilization of solutions of a degenerate nonlinear diffusion problem, *Nonlinear Analysis* **6**, 1001–1022, (1982).
2. Y.S. Choi and G.S. Ludford, An unexpected stability result of the near-extinction diffusion flame for non-unity Lewis numbers, *Q. J. Mech. Appl. Math.* **42** (1), 143–158, (1989).
3. D.S. Cohen, Multiple stable solutions of nonlinear boundary value problems arising in chemical reactor theory, *SIAM J. Appl. Math.* **20**, 1–13, (1971).
4. E.N. Dancer, On the structure of solutions of an equation in catalysis theory when a parameter is large, *J. Differential Equations* **37**, 404–437, (1980).
5. H. Fujita, On the nonlinear equations $\Delta u + e^u = 0$ and $\frac{\partial v}{\partial t} = \Delta v + e^v$, *Bull. Amer. Math. Soc.* **75**, 132–135, (1969).
6. I.M. Gel'fand, Some problems in the theory of quasilinear equations, *Uspehi Mat. Nauka* **14**, 87–158, (1959); English translation, *Trans. Amer. Math. Soc.* **29**, 295–381, (1963).
7. S. Parter, Solutions of differential equations arising in chemical reactor processes, *SIAM J. Appl. Math.* **26**, 687–714, (1974).
8. R.P. Agarwal and P.J.Y. Wong, Existence of solutions for singular boundary value problems for higher order differential equations, *Rendiconti del Seminario Mathematico e Fisico di Milano* **55**, 249–264, (1995).
9. R.P. Agarwal and P.J.Y. Wong, *Advanced Topics in Difference Equations*, Kluwer Academic, Dordrecht, (1997).
10. C.J. Chyan and J. Henderson, Positive solutions for singular higher order nonlinear equations, *Diff. Eqs. Dyn. Sys.* **2**, 153–160, (1994).
11. J.A. Gatica, V. Oliker and P. Waltman, Singular nonlinear boundary value problems for second-order ordinary differential equations, *J. Differential Equations* **79**, 62–78, (1989).
12. J. Henderson, Singular boundary value problems for difference equations, *Dynamic Systems and Applications*, 271–282, (1992).
13. J. Henderson, Singular boundary value problems for higher order difference equations, In *Proceedings of the First World Congress on Nonlinear Analysts*, (Edited by V. Lakshmikantham), pp. 1139–1150, Walter de Gruyter, (1994).
14. D. O'Regan, *Theory of Singular Boundary Value Problems*, World Scientific, Singapore, (1994).
15. P.J.Y. Wong and R.P. Agarwal, Double positive solutions of (n, p) boundary value problems for higher order difference equations, *Computers Math. Applic.* **32** (8), 1–21, (1996).
16. C. Bandle, C.V. Coffman and M. Marcus, Nonlinear elliptic problems in annular domains, *J. Differential Equations* **69**, 322–345, (1987).
17. C. Bandle and M.K. Kwong, Semilinear elliptic problems in annular domains, *J. Appl. Math. Phys.* **40**, 245–257, (1989).
18. X. Garaizar, Existence of positive radial solutions for semilinear elliptic problems in the annulus, *J. Differential Equations* **70**, 69–72, (1987).
19. H. Wang, On the existence of positive solutions for semilinear elliptic equations in the annulus, *J. Differential Equations* **109**, 1–7, (1994).
20. F. Atici and A.C. Peterson, Bounds for positive solutions for a focal boundary value problem, *Computers Math. Applic.* **36** (10–12), 99–107, (1998).
21. D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, CA, (1988).
22. R.P. Agarwal, *Boundary Value Problems for Higher Order Differential Equations*, World Scientific, Singapore, (1984).
23. A.M. Fink, J.A. Gatica and G.E. Hernandez, Eigenvalues of generalized Gel'fand models, *Nonlinear Analysis* **20**, 1453–1468, (1993).
24. P.W. Eloe and J. Henderson, Positive solutions for $(n-1, 1)$ boundary value problems, *Nonlinear Analysis* **28**, 1669–1680, (1997).
25. P.W. Eloe and J. Henderson, Positive solutions and nonlinear $(k, n-k)$ conjugate eigenvalue problems, *Diff. Eqs. Dyn. Sys.* (to appear).
26. L.H. Erbe, S. Hu and H. Wang, Multiple positive solutions of some boundary value problems, *J. Math. Anal. Appl.* **184**, 640–648, (1994).
27. F. Merdivenci, Two positive solutions of a boundary value problem for difference equations, *J. Difference Eqns. Appl.* **1**, 262–270, (1995).
28. F. Merdivenci, Green's matrices and positive solutions of a discrete boundary value problem, *PanAmerican Math. J.* **5**, 25–42, (1995).
29. F. Merdivenci, Positive solutions for focal point problems for $2n^{\text{th}}$ order difference equations, *PanAmerican Math. J.* **5**, 71–82, (1995).
30. L.H. Erbe and H. Wang, On the existence of positive solutions of ordinary differential equations, *Proc. Amer. Math. Soc.* **120**, 743–748, (1994).
31. A.M. Fink, The radial Laplacian Gel'fand problem, In *Delay and Differential Equations*, pp. 93–98, World Sci. Publishing, River Edge, NJ, (1992).
32. D. Hankerson and A.C. Peterson, Comparison of eigenvalues for focal point problems for n^{th} order difference equations, *Differential and Integral Eqns.* **3**, 363–380, (1990).

33. A.C. Peterson, Boundary value problems for an n^{th} order difference equation, *SIAM J. Math. Anal.* **15**, 124–132, (1984).
34. P.J.Y. Wong and R.P. Agarwal, Eigenvalues of boundary value problems for higher order differential equations, *Mathematical Problems in Engineering* **2**, 401–434, (1996).
35. P.J.Y. Wong and R.P. Agarwal, On the existence of positive solutions of higher order difference equations, *Topological Methods in Nonlinear Analysis* **10**, 339–351, (1997).
36. P.J.Y. Wong and R.P. Agarwal, On the eigenvalues of boundary value problems for higher order difference equations, *Rocky Mountain J. Math.* (to appear).
37. P.J.Y. Wong and R.P. Agarwal, Eigenvalue characterization for (n,p) boundary value problems, *J. Austral. Math. Soc. Ser. B* **39**, 386–407, (1998).
38. P.J.Y. Wong and R.P. Agarwal, Eigenvalues of an n^{th} order difference equation with (n,p) type conditions, *Dynamics of Continuous, Discrete and Impulsive Systems* **4**, 149–172, (1998).
39. R.P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York, (1992).
40. R.P. Agarwal, M. Bohner and P.J.Y. Wong, Positive solutions and eigenvalues of conjugate boundary value problems, In *Proceedings of the Edinburgh Mathematical Society* (to appear).
41. M.A. Krasnosel'skii, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, (1964).
42. R.P. Agarwal, Continuous and discrete maximum principles and their applications to boundary value problems, In *Dynamic Systems and Applications*, (Edited by G.S. Ladde and M. Sambandham), Volume 2, pp. 11–18, Dynamic, Atlanta, GA, (1994).
43. P.J.Y. Wong and R.P. Agarwal, Extension of continuous and discrete inequalities due to Eloe and Henderson, *Nonlinear Analysis* **34**, 479–487, (1998).
44. P.J.Y. Wong and R.P. Agarwal, On the existence of solutions of singular boundary value problems for higher order difference equations, *Nonlinear Analysis* **28**, 277–287, (1997).
45. R.P. Agarwal, D. O'Regan and P.J.Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic, Dordrecht (to appear).