Existence of at Least One Homoclinic Solution for a Nonlinear Second-Order Difference Equation

Abstract: This paper presents sufficient conditions for the existence of at least one homoclinic solution for a nonlinear second-order difference equation with $p$-Laplacian. Our technical approach is based on variational methods. An example is offered to demonstrate the applicability of our main results.

Keywords: homoclinic solution, difference equations, discrete $p$-Laplacian, variational methods

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1 Introduction

In the present work, we consider the nonlinear second-order difference problem

\[
\begin{align*}
-\Delta (a(k)\phi_p(\Delta u(k-1))) + b(k)\phi_p(u(k)) &= f(k, u(k)) \\
\text{for all } k \in \mathbb{Z} \\
u(k) &\to 0 \quad \text{as } |k| \to \infty.
\end{align*}
\]

(P)

Here, $p > 1$ is a real number, $\phi_p(t) = |t|^{p-2}t$ for all $t \in \mathbb{R}$, $a, b : \mathbb{Z} \to (0, +\infty)$, while $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Moreover, the forward difference operator is defined as $\Delta u(k) = u(k+1) - u(k)$. We say any function $u$ is homoclinic if $\lim_{|k| \to \infty} u(k) = 0$.

The theory of nonlinear difference equations has been widely used to examine discrete models appearing in many fields such as computing, computer science, economics, neural networks, biology, ecology, cybernetics, physics, and so on. Discrete boundary value problems received some attention lately. Let us mention, far from being exhaustive, the recent papers [1–3] on discrete BVPs investigated via critical point theory. The tools employed cover the Morse theory, mountain pass methodology, and linking arguments, i.e., methods usually applied for continuous problems.

There is an increasing interest in the existence of solutions to boundary value problems for difference equations with $p$-Laplacian operator. Their applications in many fields such as biological neural networks, economics, optimal control, and other areas of study have led to the rapid development of the theory of difference equations, see the monograph of Agarwal [4]. Recently, the study of discrete problems subject to various boundary value conditions has been widely approached by using different abstract methods such as fixed point theory, lower and upper solutions method, critical point theory, variational methods, Morse theory, and the mountain pass theorem.

For background and recent results, we refer the reader to [5–20] and the references therein for details. For example, Liang and Weng in [17], employing critical point theory, obtained the existence of multiple solutions for a second-order difference boundary value problem. In [9], Bonanno et al. critical point theory, studied the existence of at least three solutions for some periodic and Neumann boundary value problems involving the discrete $p$-Laplacian operator. In [21], Iannizzotto and Tersian introduced a variational framework for a second-order difference equation driven by the discrete $p$-Laplacian and involving a coercive weight function and a positive parameter $\lambda$, and by means of critical point theory, proved the existence of at least two nontrivial homoclinic solutions for $\lambda$ big enough. Stegliński in [18–20], using variational methods and critical point theory, studied the existence of multiple homoclinic solutions for discrete problems.

Motivated by the above facts, in the present paper, we study the existence of at least one solution for the problem (P) under an asymptotical behaviour of the nonlinear datum at zero, see Theorem 3.1. We present Example 3.2 in which the hypotheses of Theorem 3.1 are fulfilled. Also, in Theorem 3.3, a parametric version of this result is successively studied in which, for small values of the parameter, the existence of at least one solution is ensured. Requiring an additional asymptotical behaviour of the potential at
zero if \( f(k, 0) = 0 \) for all \( k \in \mathbb{Z} \), we show that the solution is nontrivial, see Remark 3.7. Moreover, we deduce the existence of solutions for small positive values of the parameter \( \lambda \) such that the corresponding solutions have smaller and smaller energies as the parameter goes to zero; see Theorem 3.8.

The rest of this paper is organized as follows. In Section 2, some definitions and lemmas which are essential to prove our main results are stated. In Section 3, we give proof our main results are stated. In Section 3, we give

\section{Preliminaries}

We shall prove the existence of at least one nontrivial solution to the problem (\( P_f \)) applying the following variant of Ricceri’s variational principle [22, Theorem 2.1] as given by Bonanno and Molica Bisci in [23], which its first version has been obtained in [24] (see [24, Theorem 3.1 and Remark 3.1]).

\begin{theorem}
Let \( X \) be a reflexive real Banach space, let \( \Phi, \Psi : X \rightarrow \mathbb{R} \) be two Gâteaux differentiable functionals such that \( \Phi \) is sequentially weakly lower semicontinuous, strongly continuous and coercive in \( X \) and \( \Psi \) is sequentially weakly upper semicontinuous in \( X \). Let \( I_{\lambda} \) be the functional defined as \( I_{\lambda} := \Phi - \lambda \Psi \), \( \lambda \in \mathbb{R} \), and for every \( r > \inf_X \Phi \), let \( \varphi \) be the function defined as

\[ \varphi(r) := \inf_{u \in \Phi^{-1}(-\infty, r)} \sup_{u \in \Phi^{-1}(-\infty, r)} \frac{\Psi(u) - \Psi(u)}{r - \Phi(u)}. \]

Then, for every \( r > \inf_X \Phi \) and every \( \lambda \in \left( 0, \frac{1}{\varphi(r)} \right) \), the restriction of the functional \( I_{\lambda} \) to \( \Phi^{-1}(-\infty, r) \) admits a global minimum, which is a critical point (precisely a local minimum) of \( I_{\lambda} \) in \( X \).
\end{theorem}

We refer the interested reader to the papers [25–30] in which Theorem 2.1 has been successfully employed to prove the existence of at least one nontrivial solution for boundary value problems.

For \( b : \mathbb{Z} \rightarrow \mathbb{R} \) and the continuous mapping \( f : \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R} \), consider the condition:

\( B(k) \geq \alpha > 0 \) for all \( k \in \mathbb{Z} \), \( b(k) \rightarrow +\infty \) as \( |k| \rightarrow +\infty \).

For all \( 1 \leq p < +\infty \), we denote by \( \ell^p \) the set of all functions \( u : \mathbb{Z} \rightarrow \mathbb{R} \) such that

\[ \|u\|_p^p = \sum_{k \in \mathbb{Z}} |u(k)|^p < +\infty. \]

Moreover, we denote by \( \ell^\infty \) the set of all functions \( u : \mathbb{Z} \rightarrow \mathbb{R} \) such that

\[ \|u\|_\infty = \sup_{k \in \mathbb{Z}} |u(k)| < +\infty. \]

\begin{lemma} \text{(See [20, Lemma 2.1]).} \ Let \( f : \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function satisfying

\[ \sup_{|t| \leq T} |f(t, t)| \in \ell^1 \text{ for all } T > 0. \tag{1} \]

Then the functional \( \Psi : \ell^p \rightarrow \mathbb{R} \) defined by

\[ \Psi(u) := \sum_{k \in \mathbb{Z}} F(k, u(k)) \text{ for all } u \in \ell^p, \tag{2} \]

where \( F(k, s) = \int_0^s f(k, t)\,dt \) for every \( k, s \in \mathbb{Z} \times \mathbb{R} \), is continuously differentiable.
\end{lemma}

Now, we set

\[ X = \left\{ u : \mathbb{Z} \rightarrow \mathbb{R} : \sum_{k \in \mathbb{Z}} \left[ a(k)|u(k-1)|^p + b(k)|u(k)|^p \right] < \infty \right\} \]

and

\[ \|u\| = \left( \sum_{k \in \mathbb{Z}} \left[ a(k)|u(k-1)|^p + b(k)|u(k)|^p \right] \right)^{1/p}. \]

Clearly, we have

\[ \|u\|_\infty \leq \|u\|_p \leq \alpha^{-1/p}\|u\| \text{ for all } u \in X. \tag{3} \]

As is shown in [21, Propositions 3], \( (X, \| \cdot \|) \) is a reflexive Banach space and the embedding \( X \hookrightarrow \ell^p \) is compact. See also [31, Lemma 2.2].

Let \( I : X \rightarrow \mathbb{R} \) be the functional associated with problem (\( P_f \)) defined by

\[ I(u) = \Phi(u) - \Psi(u), \]

where

\[ \Phi(u) := \frac{1}{p} \sum_{k \in \mathbb{Z}} \left[ a(k)|u(k-1)|^p + b(k)|u(k)|^p \right] \text{ for all } u \in X \]

and \( \Psi \) is given by (2).

\begin{proposition} \text{(See [20, Proposition 2.2]).} \ Assume that \( (A_1) \) and (1) are satisfied. Then

\end{proposition}
3 Main results

We formulate our main result as follows.

**Theorem 3.1.** Assume that

\[ \sup_{\gamma > 0} \frac{\gamma^p}{\sum_{k \in \mathbb{Z}} \max_{|t| \leq \tilde{\gamma}} F(k, t)} > \frac{p}{\alpha} \quad (D_{\mathcal{F}}) \]

Then, the problem \((P_f)\) admits at least one solution in \(X\).

**Proof.** Our aim is to apply Theorem 2.1 to the problem \((P_f)\). Let \(\Phi, \Psi\) and \(I\) be as in Section 2. The functionals \(\Phi\) and \(\Psi\) satisfy the regularity assumptions of Theorem 2.1. It is well known that \(\Psi\) is a differentiable functional whose differential at the point \(u \in X\) is:

\[ \Psi'(u)(v) = \sum_{k \in \mathbb{Z}} f(k, u(k)) v(k) \]

for every \(v \in X\), as well as is sequentially weakly upper semi-continuous. The functional \(\Psi\) is weakly continuous since it has compact derivative (see [32]). From Proposition 2.3, it is also easy to verify that \(\Phi : X \rightarrow \mathbb{R}\) is sequentially weakly lower semi-continuous and coercive. Moreover, \(\Phi\) is differentiable and sequentially weakly lower semi-continuous and its derivative is the functional \(\Phi'(u) \in X^*\), given by

\[ \Phi'(u)(v) = \sum_{k \in \mathbb{Z}} a(k) \Delta u(k-1) |v(k-1)|^{p-2} v(k-1) + \sum_{k \in \mathbb{Z}} b(k) |u(k)|^{p-2} u(k) v(k) \]

for every \(v \in X\). Therefore, we observe that the regularity assumptions on \(\Phi\) and \(\Psi\), as requested in Theorem 2.1, are verified. We now look on the existence of a critical point of the functional \(I\) in \(X\). By using the condition \((D_{\mathcal{F}})\), there exists \(\tilde{\gamma} > 0\) such that:

\[ \frac{\tilde{\gamma}^p}{\sum_{k \in \mathbb{Z}} \max_{|t| \leq \tilde{\gamma}} F(k, t)} > \frac{p}{\alpha} \quad (4) \]

Set

\[ r = \frac{\alpha}{p} \tilde{\gamma}^p. \]

From the definition of \(\Phi\) and in view of (3), for every \(r > 0\), one has

\[ \Phi^{-1}(-\infty, r) \subseteq \left\{ u \in X : \|u\|^p \leq pr \right\} \subseteq \left\{ u \in X : \alpha \|u\|_p \leq pr \right\} = \left\{ u \in X : \|u\|_{\infty} \leq \tilde{\gamma} \right\}, \]

which implies that

\[ \Psi(u) \leq \sup_{u \in \Phi^{-1}(-\infty, r)} \sum_{k \in \mathbb{Z}} F(k, u(k)) \leq \sum_{k \in \mathbb{Z}} \max_{|t| \leq \tilde{\gamma}} F(k, t) \]

for every \(u \in X\) such that \(\Phi(u) < r\). Then

\[ \sup_{\Phi(u) < r} \Psi(u) \leq \frac{p}{\alpha} \sum_{k \in \mathbb{Z}} \max_{|t| \leq \tilde{\gamma}} F(k, t). \]

Bearing in mind that \(\Phi(0) = \Psi(0) = 0\), it follows

\[ \varphi(r) \leq \frac{\sup_{u \in \Phi^{-1}(-\infty, r)} \Psi(u)}{r} \leq \frac{p}{\alpha} \sum_{k \in \mathbb{Z}} \max_{|t| \leq \tilde{\gamma}} F(k, t). \]

Consequently, by (4), one has \(\varphi(r) < 1\). Hence, since \(1 \in \left(0, \frac{1}{\varphi(r)}\right)\), applying Theorem 2.1, the functional \(I\) admits at least one critical point (local minima) \(\tilde{u} \in \Phi^{-1}(-\infty, r)\), and since the critical points of the functional \(I\) are the solutions of the problem \((P_f)\), we have the result. 

Here we present an example in which the hypotheses of Theorem 3.1 are satisfied.

**Example 3.2.** Let \(p = 3\). Consider the problem

\[ \begin{cases} -\Delta (a(k) \Delta u(k-1)) + b(k) u(k) = f(k, u(k)) \\ \text{for all } k \in \mathbb{Z} \\ u(k) \rightarrow 0 \text{ as } |k| \rightarrow \infty, \end{cases} \]

where \(a(k) = k^2 + 1\) for all \(k \in \mathbb{Z}\), \(b(k) = |k| + 1\) for all \(k \in \mathbb{Z}\), and

\[ f(k, t) = \frac{t^3 + 5t^2 - 5}{100 \cdot 2^{|k|}} \]

for every \((k, t) \in \mathbb{Z} \times \mathbb{R}\). By the expression of \(f\) we have

\[ F(k, t) = \frac{t^3 + 5t^2 - 5}{100 \cdot 2^{|k|}} \]

for every \((k, t) \in \mathbb{Z} \times \mathbb{R}\). Since

\[ \sup_{\gamma > 0} \frac{\gamma^3}{\sum_{k \in \mathbb{Z}} \max_{|t| \leq \gamma} F(k, t)} > 3, \]
we observe that all assumptions of Theorem 3.1 are fulfilled. Hence, Theorem 3.1 implies that the problem (5) admits at least one solution in

$$
\begin{align*}
\{ u : \mathbb{Z} \to \mathbb{R} : \sum_{k \in \mathbb{Z}} (a(k)|\Delta u(k-1)|^3 + b(k)|u(k)|^3) < \infty \}
\end{align*}
$$

We note that Theorem 3.1 can be exploited showing the existence of at least one solution for the parametric version

$$
\begin{align*}
\{ \Delta (a(k)f_p(\Delta u(k-1))) + b(k)f_p(u(k)) = \lambda f(k, u(k)) \\
\quad \text{for all } k \in \mathbb{Z} \\
u(k) \to 0 \quad \text{as } |k| \to \infty,
\end{align*}
$$

where \( \lambda > 0 \) is a parameter. More precisely, we have the following existence result.

**Theorem 3.3.** For every

$$
\lambda \in \left( 0, \frac{\alpha}{\sup_{\gamma > 0} \sum_{k \in \mathbb{Z}} \max_{|f|_{L^p}} F(k, t)} \right),
$$

the problem \((P_{\lambda}^f)\) admits at least one solution \( u_\lambda \in X \).

**Proof.** Fix \( \lambda \) as in the statement. Let \( \Phi \) and \( \Psi \) be as in Section 2, and put \( I_\lambda(u) = \Phi(u) - \lambda \Psi(u) \) for every \( u \in X \). Let us pick

$$
0 < \lambda < \frac{\alpha}{\sup_{\gamma > 0} \sum_{k \in \mathbb{Z}} \max_{|f|_{L^p}} F(k, t)}.
$$

Hence, there exists \( \gamma_0 > 0 \) such that

$$
\lambda p \frac{\gamma_0}{\alpha} < \sum_{k \in \mathbb{Z}} \max_{|f|_{L^p}} F(k, t).
$$

Choosing \( r = \frac{\alpha}{\lambda} \gamma_0 \) and arguing as in the proof of Theorem 3.1, one has

$$
\varphi(r) \leq \frac{\sup_{|\nu| \leq \gamma_0} \Psi(\nu)}{r} \leq \frac{p}{\alpha} \sum_{k \in \mathbb{Z}} \max_{|f|_{L^p}} F(k, t) \frac{\gamma_0}{\lambda} = \frac{1}{\lambda}.
$$

Hence, since \( \lambda \in \left( 0, \frac{1}{\varphi(r)} \right) \), Theorem 2.1 ensures that the functional \( I_\lambda \) admits at least one critical point (local minima) \( u_\lambda \in \Phi^{-1}(\infty, r) \), and since the critical points of the functional \( I_\lambda \) are the solutions of the problem \((P_{\lambda}^f)\), we have the conclusion. \( \square \)

Now, we list some remarks of our results.

**Remark 3.4.** We note that, in general, \( I_1 \) can be unbounded from below in \( X \). Indeed, for example, in the case when \( f(k, \xi) = 1 + |\xi|^{\gamma-p} \) for every \( (k, \xi) \in \mathbb{Z} \times \mathbb{R} \) with \( \gamma > p \), for any fixed \( u \in X \setminus \{ 0 \} \) and \( k \in \mathbb{R} \), we obtain

$$
\begin{align*}
I_1(\Phi) = \Phi(\Phi) - \lambda \sum_{k \in \mathbb{Z}} F(k, \Phi(k)) &\leq \frac{\lambda}{p} \Phi(\Phi)^p \\
- \lambda k \sum_{k \in \mathbb{Z}} |\Phi(k)| - \lambda \sum_{k \in \mathbb{Z}} |\Phi(k)|^\gamma &\to -\infty
\end{align*}
$$

as \( \lambda \to +\infty \). Therefore, the condition \([33, (I_2)] \) in Theorem 2.2] is not satisfied, and we cannot use direct minimization to find critical points of the functional \( I_1 \). Next, the energy functional \( I_1 \) associated with the problem \((P_{\lambda}^f)\) is non coercive. Indeed, when \( F(k, \xi) = |\xi|^{s} \) with \( s \in (p, +\infty) \) for every \( (k, \xi) \in \mathbb{Z} \times \mathbb{R} \), for any fixed \( u \in X \setminus \{ 0 \} \) and \( k \in \mathbb{R} \), we have

$$
\begin{align*}
I_1(\Phi) = \Phi(\Phi) - \lambda \sum_{k \in \mathbb{Z}} F(k, \Phi(k)) &\leq \frac{\lambda}{p} \Phi(\Phi)^p \\
- \lambda k \sum_{k \in \mathbb{Z}} \Phi(k) &\to -\infty
\end{align*}
$$

as \( \lambda \to +\infty \).

**Remark 3.5.** For fixed \( \gamma > 0 \), let \( \frac{\gamma p}{\alpha} > \sum_{k \in \mathbb{Z}} \max_{|f|_{L^p}} F(k, t) \). Then the result of Theorem 3.3 holds with \( \|u_\lambda\|_{\infty} \leq \gamma \).

**Remark 3.6.** If in Theorem 3.1, \( f(k, \xi) \geq 0 \) for every \( (k, \xi) \in \mathbb{Z} \times \mathbb{R} \), then the condition \((D_F)\) takes the following more simple and significant form

$$
\sup_{\gamma > 0} \frac{\gamma p}{\alpha} \sum_{k \in \mathbb{Z}} F(k, \gamma) > \frac{p}{\alpha}.
$$

Moreover, if the assumption

$$
\lim_{\gamma \to +\infty} \frac{\gamma p}{\alpha} \sum_{k \in \mathbb{Z}} F(k, \gamma) > \frac{p}{\alpha}
$$

holds, then the condition \((D_{F}')\) is automatically verified.

**Remark 3.7.** If in Theorem 3.3, \( f(k, 0) = 0 \) for all \( k \in \mathbb{Z} \), then the ensured solution is obviously nontrivial. On the other hand, the nontriviality of the solution can be achieved also in the case \( f(k, 0) = 0 \) for all \( k \in \mathbb{Z} \) requiring an extra condition at zero, namely, there are discrete intervals \([1, T_1]\) and \([1, T_2]\) with \( T_1, T_2 \geq 2 \) such that

$$
\lim_{\xi \to 0^+} \frac{\inf_{k \in [1, T_2]} F(k, \xi)}{|\xi|^p} = +\infty
$$

(6)
Indeed, arguing as in the proof of [34, Theorem 3.5], let $0 < \lambda < \lambda^*$, where
\[ \lambda^* = \alpha \sup_{p > 0} \frac{\gamma^p}{\sum_{k \in Z} \max_{|t| \leq p} F(k, t)}. \]

Then, there exists $\bar{T} > 0$ such that
\[ \frac{p\lambda}{\alpha} < \sum_{k \in Z} \max_{|t| \leq \bar{T}} F(k, t). \]

Thanks to Theorem 2.1, for every $\lambda \in (0, \lambda^*)$, there exists a critical point of $I_1 = \Phi - \lambda^*p$ such that $u_1 \in \Phi^{-1}(-\infty, r_1)$, where $r_1 = \frac{\alpha}{\lambda^*}$. In particular, $u_1$ is a global minimum of the restriction of $I_1$ to $\Phi^{-1}(-\infty, r_1)$. We will prove that the function $u_1$ cannot be trivial. Let us show that
\[ \lim_{|u| \to 0^+} \frac{\Psi(u)}{\Phi(u)} = +\infty. \]

Using (6) and (7), a sequence $\{\xi_n\} \subset \mathbb{R}^+$ converging to zero and two constants $c, \kappa$ (with $c > 0$) can be considered such that
\[ \lim_{n \to +\infty} \inf_{k \in [1, T_1]} F(k, \xi_n) = +\infty \]

and
\[ \inf_{k \in [1, T_1]} F(k, \xi) \geq \kappa |\xi|^p \]

for every $\xi \in [0, \varepsilon]$. We consider a discrete interval $[1, T_3] \subset [1, T_2]$ with $T_3 \geq 2$ and a function $\nu \in X$ such that
\begin{align*}
(a_1) & \quad \nu(k) \in [0, 1] \text{ for every } k \in \mathbb{Z}, \\
(a_2) & \quad \nu(k) = 1 \text{ for every } k \in [1, T_1], \\
(a_3) & \quad \nu(k) = 0 \text{ for every } k \in [T_1 + 1, T_1 + 1 + 1] \text{ for every } l \in \mathbb{Z}^+. 
\end{align*}

Hence, fix $M > 0$ and consider a real positive number $\eta$ with
\[ M < \frac{\eta T_3 + \kappa \sum_{k=T_3+1}^{T_3} |\nu(k)|^p}{\frac{1}{p} \|\nu\|^p}. \]

Then, there exists $n_0 \in \mathbb{N}$ such that $\xi_n < \varepsilon$ and
\[ \inf_{k \in [1, T_2]} F(k, \xi_n) \geq \eta |\xi_n|^p \]

for every $n > n_0$. Now, for every $n > n_0$, by recalling the properties of the function $\nu$ (that is, $0 \leq \xi_n \nu(k) < \varepsilon$ for $n$ large enough), we have
\[ \Psi(\xi_n, \nu) = \sum_{k=1}^{T_3} F(k, \xi_n) + \sum_{k=T_3+1}^{T_1} F(k, \xi_n \nu(k)) \]

\[ \Phi(\xi_n, \nu) = \frac{\eta T_3 + \kappa \sum_{k=T_3+1}^{T_1} |\nu(k)|^p}{\frac{1}{p} \|\nu\|^p} > M. \]

Since $M$ could be taken arbitrarily large, it follows that
\[ \lim_{n \to +\infty} \frac{\Psi(\xi_n, \nu)}{\Phi(\xi_n, \nu)} = +\infty, \]

from which (8) clearly follows. Hence, there exists a sequence $\{w_n\} \subset X$ converging to zero such that, for $n$ large enough, $w_n \in \Phi^{-1}(-\infty, r)$ and
\[ I_1(w_n) = \Phi(w_n) - \lambda \Psi(w_n) < 0. \]

Since $u_1$ is a global minimum of the restriction of $I_1$ to $\Phi^{-1}(-\infty, r)$, we obtain
\[ I_1(u_1) < 0, \]

so that $u_1$ is not trivial.

**Theorem 3.8.** Assume that there are discrete intervals $[1, T_1]$ and $[1, T_2] \subset [1, T_1]$, where $T_1, T_2 \geq 2$, such that (6) and (7) are satisfied. Then, for each
\[ \lambda \in \Lambda = \left( 0, \frac{\alpha}{\lambda^*} \sup_{p > 0} \frac{\gamma^p}{\sum_{k \in Z} \max_{|t| \leq p} F(k, t)} \right) \]

the problem $(P')_{\lambda}^{\nu}$ admits at least one nontrivial solution $u_\lambda \in X$. Moreover, one has
\[ \lim_{\lambda \to 0^+} \|u_\lambda\|_a = 0 \]

and the real function $\lambda \to I_1$ is negative and strictly decreasing in $\Lambda$.

**Proof.** By Theorem 3.3 and Remark 3.7, we obtain that $u_\lambda$ is a nontrivial solution of the problem $(P')_{\lambda}^{\nu}$ such that
\[ I_1(u_\lambda) < 0. \]

From (9), we easily observe that the map
\[ (0, \lambda^*) \ni \lambda \mapsto I_1(u_\lambda) \]

is negative. Also, one has
Indeed, bearing in mind that $\Phi$ is coercive and for all $\lambda \in (0, \lambda^*)$, $u_\lambda \in \Phi^{-1}(-\infty, r)$, one has that there exists a positive constant $L$ such that $\|u_\lambda\| \leq L$ for every $\lambda \in (0, \lambda^*)$. After that, it is easy to see that there exists a positive constant $N$ such that

$$\sum_{k \in \mathbb{Z}} f(k, u_\lambda(k)) u_\lambda(k) \leq N\|u_\lambda\| \leq NL$$

for every $\lambda \in (0, \lambda^*)$. Since $u_\lambda$ is a critical point of $I_\lambda$, we have $I'_\lambda(u_\lambda)(v) = 0$ for every $v \in X$ and every $\lambda \in (0, \lambda^*)$. In particular $I'_\lambda(u_\lambda)(u_\lambda) = 0$, that is,

$$\Phi'(u_\lambda)(u_\lambda) = \lambda \sum_{k \in \mathbb{Z}} f(k, u_\lambda(k)) u_\lambda(k)$$

for every $\lambda \in (0, \lambda^*)$. We have

$$0 \leq \|u_\lambda\|^p \leq \Phi'(u_\lambda)(u_\lambda),$$

from (12), we obtain

$$0 \leq \|u_\lambda\|^p \leq \lambda \sum_{k \in \mathbb{Z}} f(k, u_\lambda(k)) u_\lambda(k)$$

for any $\lambda \in (0, \lambda^*)$. Letting $\lambda \to 0^*$, by (13) together with (11), we get

$$\lim_{\lambda \to 0^*} \|u_\lambda\| = 0.$$

Then, we have obviously the desired conclusion. Finally, we have to show that the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $(0, \lambda^*)$. For our goal, we see that for any $u \in X$, one has

$$I_\lambda(u) = \lambda \left( \frac{\Phi(u)}{\lambda} - \Psi(u) \right).$$

Now, let us fix $0 < \lambda_1 < \lambda_2 < \lambda^*$ and let $u_{\lambda_i}$ be the global minimum of the functional $I_{\lambda_i}$ restricted to $\Phi(-\infty, r)$ for $i = 1, 2$. Also, set

$$m_{\lambda_i} = \left( \frac{\Phi(u_{\lambda_i})}{\lambda_i} - \Psi(u_{\lambda_i}) \right) = \inf_{v \in \Phi^{-1}(-\infty, r)} \left( \frac{\Phi(v)}{\lambda_i} - \Psi(v) \right)$$

for every $i = 1, 2$. Clearly, (10) in conjunction with (14) and the positivity of $\lambda$ implies that

$$m_{\lambda_i} < 0 \text{ for } i = 1, 2.$$


