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Euler-type Boundary Value Problems in Quantum Calculus

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ABSTRACT

We study a boundary value problem consisting of a second-order q-difference equation together with Dirichlet boundary conditions. Separation of variables leads us to an eigenvalue problem for a second-order Euler q-difference equation. We determine the exact number of eigenvalues.

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1 Introduction

While in ordinary calculus we study differential equations and in discrete calculus we study difference equations, we study so-called q-difference equations in quantum calculus. Let

$$q > 1 \quad \text{and} \quad \mathbb{T} = \{q^k : \ k \in \mathbb{N}_0\}. \tag{1.1}$$

For a function $u : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, we define the *Jackson derivatives* of u with respect to the first and the second variable, respectively, by

$$u_x(x,t) = \frac{u(qx,t) - u(x,t)}{(q-1)x} \quad \text{and} \quad u_t(x,t) = \frac{u(x,qt) - u(x,t)}{(q-1)t}.$$

Let $N \in \mathbb{N}$. In this paper, we consider the boundary value problem

$$x^2 u_{xx} = t^2 u_{tt}, \quad u(1,t) = u(q^N, t) = 0, \tag{1.2}$$

which is a second-order q-difference equation together with Dirichlet boundary conditions. For material on quantum calculus we refer to the monograph by Kac and Cheung (Kac and Cheung 2002), the paper by Bohner and Ünal (Bohner and Ünal 2005), and the books about dynamic equations on time scales by Bohner and Peterson (Bohner and Peterson 2001, Bohner and Peterson 2003).

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Now we rewrite the second-order q-difference equation in (1.2) as a second-order q-recursion relation. By the definition of the Jackson derivative, the second-order partial of u with respect to x is given by

$$u_{xx}(x,t) = \frac{u_x(qx,t) - u_x(x,t)}{(q-1)x}. \tag{1.3}$$

When we expand equation (1.3), we obtain

$$u_{xx}(x,t) = \frac{\frac{u(q^2x,t)-u(qx,t)}{(q-1)qx} - \frac{u(qx,t)-u(x,t)}{(q-1)x}}{(q-1)x}.$$

Similarly we can compute the partial u_{tt}. Thus, the second-order q-difference equation in (1.2) is equivalent to

$$x^2\left(\frac{u(q^2x,t) - (q+1)u(qx,t) + qu(x,t)}{x^2}\right) = t^2\left(\frac{u(x,q^2t) - (q+1)u(x,qt) + qu(x,t)}{t^2}\right),$$

i.e.,

$$u(q^2x,t) - (q+1)u(qx,t) + qu(x,t) = u(x,q^2t) - (q+1)u(x,qt) + qu(x,t). \tag{1.4}$$

The setup of this paper is as follows. In the next section, we use separation of variables to arrive at a certain eigenvalue problem. Finally, in Section 3 we determine the eigenvalues and the number of eigenvalues of the resulting eigenvalue problem. An example is given as well.

2 Separation of Variables

We let $u(x,t) = f(x)g(t)$ so that $u(qx,t) = f(qx)g(t)$ and $u(q^2x,t) = f(q^2x)g(t)$. This is also applied to the terms $u(x,qt)$ and $u(x,q^2t)$. When we substitute these values into the partial q-difference equation (1.4), we get

$$f(q^2x)g(t) - (q+1)f(qx)g(t) + qf(x)g(t) = f(x)g(q^2t) - (q+1)f(x)g(qt) + qf(x)g(t). \tag{2.1}$$

Now we divide each side of (2.1) by $f(x)g(t)$ to gather like terms and then set both sides equal to a constant λ to arrive at

$$\frac{f(q^2x) - (q+1)f(qx) + qf(x)}{f(x)} = \frac{g(q^2t) - (q+1)g(qt) + qg(t)}{g(t)} = \lambda. \tag{2.2}$$

Hence, from (1.2) and (2.2), the eigenvalue problem for f is

$$f(q^2x) - (q+1)f(qx) + (q-\lambda)f(x) = 0, \quad f(1) = f(q^N) = 0. \tag{2.3}$$

The second-order q-difference equation in (2.3) is an Euler–Cauchy q-difference equation as studied in (Bohner and Ünal 2005). We let $f(x) = \alpha^{\log_q x}$, which in return gives us

$$f(qx) = \alpha^{\log_q qx} = \alpha f(x) \quad \text{and} \quad f(q^2x) = f(q(qx)) = \alpha f(qx) = \alpha^2 f(x).$$

Now we make these substitutions into the Euler–Cauchy equation in (2.3) and get

$$\alpha^2 f(x) - (q+1)\alpha f(x) + (q-\lambda)f(x) = 0.$$

The characteristic equation therefore reads

$$\alpha^2 - (q+1)\alpha + (q-\lambda) = 0. \tag{2.4}$$

We solve (2.4) for α and get

$$\alpha = \frac{q+1 \pm \sqrt{(q+1)^2 - 4(q-\lambda)}}{2} = \frac{q+1}{2} \pm \sqrt{\left(\frac{q-1}{2}\right)^2 + \lambda}.$$

Hence we let

$$\alpha_1 = \frac{q+1}{2} + \sqrt{\left(\frac{q-1}{2}\right)^2 + \lambda} \quad \text{and} \quad \alpha_2 = \frac{q+1}{2} - \sqrt{\left(\frac{q-1}{2}\right)^2 + \lambda}.$$

We distinguish the following three cases:

$$\begin{aligned} \text{Case I.} \quad & \lambda > -\left(\frac{q-1}{2}\right)^2; \\ \text{Case II.} \quad & \lambda = -\left(\frac{q-1}{2}\right)^2; \\ \text{Case III.} \quad & \lambda < -\left(\frac{q-1}{2}\right)^2. \end{aligned}$$

The general solution of the Euler–Cauchy equation in (2.3) for each case is found in (Bohner and Ünal 2005) as follows:

$$\text{Case I:} \quad f(x) = c_1 \alpha_1^{\log_q x} + c_2 \alpha_2^{\log_q x}, \tag{2.5}$$

$$\text{Case II:} \quad f(x) = (c_1 \ln x + c_2)\left(\frac{q+1}{2}\right)^{\log_q x}, \tag{2.6}$$

$$\text{Case III:} \quad f(x) = |\alpha|^{\log_q x}(c_1 \cos(\theta \log_q x) + c_2 \sin(\theta \log_q x)), \tag{2.7}$$

where $\theta = \arccos\left(\frac{\operatorname{Re}\alpha}{|\alpha|}\right)$ and c_1, $c_2 \in \mathbb{R}$.

3 Finding Eigenvalues

Our next step is to look at the three different cases and thus find the eigenvalues of (2.3).

Case I

We apply the first Dirichlet condition $f(1) = 0$ to (2.5) to obtain

$$f(1) = c_1 \alpha_1^{\log_q 1} + c_2 \alpha_2^{\log_q 1} = c_1 + c_2 = 0 \quad \text{so that} \quad c := c_1 = -c_2.$$

Now we use the relationship between c_1 and c_2 and apply it to the general solution and then use the other Dirichlet condition $f(q^N) = 0$ to find

$$0 = f(q^N) = c\left(\alpha_1^{\log_q q^N} - \alpha_2^{\log_q q^N}\right) = c\left(\alpha_1^N - \alpha_2^N\right).$$

Since $c = 0$ results in the trivial solution, we shall discuss

$$\alpha_1^N = \alpha_2^N.$$

This can occur if $\alpha_1 = \alpha_2$ or (for even N) if $\alpha_1 = -\alpha_2$. First, $\alpha_1 = \alpha_2$ implies

$$\frac{q+1}{2} + \sqrt{\left(\frac{q-1}{2}\right)^2 + \lambda} = \frac{q+1}{2} - \sqrt{\left(\frac{q-1}{2}\right)^2 + \lambda},$$

i.e.,

$$2\sqrt{\left(\frac{q-1}{2}\right)^2 + \lambda} = 0 \quad \text{so that} \quad \lambda = -\left(\frac{q-1}{2}\right)^2,$$

which is not a valid value for λ in Case I. Next, $\alpha_1 = -\alpha_2$ implies

$$\frac{q+1}{2} + \sqrt{\left(\frac{q-1}{2}\right)^2 + \lambda} = -\frac{q+1}{2} + \sqrt{\left(\frac{q-1}{2}\right)^2 + \lambda}$$

which results in $q = -1$, contradicting (1.1). Thus there are no eigenvalues in this case.

Case II

Now we look at the case where $\lambda = -(\frac{q-1}{2})^2$ and use equation (2.6) with the first Dirichlet condition to find

$$0 = f(1) = (c_1 \ln 1 + c_2)\left(\frac{q+1}{2}\right)^{\log_q 1} = c_2.$$

We let $c := c_1$ and apply the second Dirichlet condition, which gives

$$0 = f(q^N) = c \ln q^N \left(\frac{q+1}{2}\right)^{\log_q q^N} = cN \ln q \left(\frac{q+1}{2}\right)^N.$$

For this to be true either $c = 0$, $N = 0$, $q = 1$, or $q = -1$, which would all not lead to any eigenvalues. Hence there are no eigenvalues in this case also.

Case III

Finally we look at the case where $\lambda < -(\frac{q-1}{2})^2$ and use equation (2.7) with the first Dirichlet condition to find

$$0 = f(1) = |\alpha|^{\log_q 1}(c_1 \cos(\theta \log_q 1) + c_2 \sin(\theta \log_q 1)) = c_1.$$

We let $c := c_2$ and apply the other Dirichlet condition to obtain

$$0 = f(q^N) = |\alpha|^{\log_q q^N}(c \sin(\theta \log_q q^N)) = c|\alpha|^N \sin(\theta N). \tag{3.1}$$

Note now that

$$|\alpha| = \sqrt{q-\lambda} \quad \text{and} \quad \theta = \arccos\left(\frac{\operatorname{Re}\alpha}{|\alpha|}\right) = \arccos\left(\frac{q+1}{2\sqrt{q-\lambda}}\right).$$

Therefore we obtain from (3.1) that

$$0 = f(q^N) = c\sqrt{q-\lambda}^N \sin(\theta N). \tag{3.2}$$

When looking at (3.2), we can see that $\lambda = q$ would work, but this is not in the range of values we are looking at for this case. So we consider the only other possible solution, which is $\sin(\theta N) = 0$. This leads us to $\theta_m N = m\pi$, which gives us the values λ_m, $m \in \mathbb{N}_0$, where

$$\arccos\left(\frac{q+1}{2\sqrt{q-\lambda_m}}\right) = \frac{m\pi}{N}. \tag{3.3}$$

Solving for λ_m provides

$$\lambda_m = q - \left(\frac{q+1}{2\cos\left(\frac{m\pi}{N}\right)}\right)^2 \tag{3.4}$$

for $m = 1, \ldots, (N-2)/2$ if N is even and $m = 1, \ldots, (N-1)/2$ if N is odd. Hence we arrive at the following main result of this paper.

Theorem 3.1. *Let $N \in \mathbb{N}$. The problem* (2.3) *has exactly*

$$\left\lfloor \frac{N-1}{2} \right\rfloor, \quad \textit{where} \quad \lfloor \cdot \rfloor \textit{ denotes the greatest integer function},$$

eigenvalues, and they can be calculated from the formula (3.4)*. The corresponding eigenfunctions are given by*

$$f(t) = \sqrt{q-\lambda_m}^{\,\log_q t} \sin\left(\frac{m\pi \log_q t}{N}\right). \tag{3.5}$$

Example 3.2. Let $N = 6$. For $m = 0$, f given by (3.5) is trivial, so this case does not lead to an eigenvalue. For $m = 1$, the eigenvalue and corresponding eigenfunction is

$$\lambda_1 = \frac{-q^2+q-1}{3} \quad \text{and} \quad f_1(t) = \left(\frac{q+1}{\sqrt{3}}\right)^{\log_q t} \sin\left(\frac{\pi \log_q t}{6}\right).$$

For $m = 2$, the eigenvalue and corresponding eigenfunction is

$$\lambda_2 = -(q^2+q+1) \quad \text{and} \quad f_2(t) = (q+1)^{\log_q t} \sin\left(\frac{\pi \log_q t}{3}\right).$$

Next, $m = 3$ would imply by (3.3) that $q = -1$, which therefore does not lead to an eigenvalue. Similarly, (3.3) implies that $m = 4$ and $m = 5$ leads to $q < -1$. Further values of m result in repetition of the above arguments. Hence there are only two eigenvalues in this case as given above. In particular, if $q = 2$, then the eigenvalues are -1 and -7.

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