Fite–Hille–Wintner-type oscillation criteria for second-order half-linear dynamic equations with deviating arguments

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Abstract

We study oscillatory behavior of solutions to a class of second-order half-linear dynamic equations with deviating arguments under the assumptions that allow applications to dynamic equations with delayed and advanced arguments. Several improved Fite–Hille–Wintner-type criteria are obtained that do not need some restrictive assumptions required in related results. Illustrative examples and conclusions are presented to show that these criteria are sharp for differential equations and provide sharper estimates for oscillation of corresponding $q$-difference equations.

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1. Introduction

In this paper, we are concerned with the oscillatory behavior of a class of second-order half-linear functional dynamic equations

\[ r(t)\phi_{\alpha}(x^{\Delta}(t))^{\Delta} + q(t)\phi_{\alpha}(x(g(t))) = 0 \] (1.1)

on an arbitrary time scale \( \mathbb{T} \) unbounded above, where \( t \in [t_0, \infty)_\mathbb{T} := [t_0, \infty) \cap \mathbb{T}, t_0 \geq 0, \) \( t_0 \in \mathbb{T}, \phi_{\alpha}(u) := |u|^{\alpha-1}u, \alpha > 0, \) \( r \) is a positive rd-continuous function on \( \mathbb{T} \) such that \( r^{\Delta} \geq 0, \) \( q \) is a positive rd-continuous function on \( \mathbb{T} \), and \( g : \mathbb{T} \to \mathbb{T} \) is a rd-continuous function satisfying \( \lim_{t \to \infty} g(t) = \infty. \) Analysis of qualitative properties of (1.1) is important not only for the sake of further development of the oscillation theory, but for practical reasons too. In fact, the study of half-linear equations has become an important area of research due to the fact that such equations arise in a variety of real world problems such as in the study of \( p \)-Laplace equations, non-Newtonian fluid theory, and the turbulent flow of a polytrophic gas in a porous medium; see Agarwal et al. [2,4].

For an excellent introduction to the calculus on time scales; see Bohner and Peterson [8] and Hilger [18]. We assume that the reader is familiar with the basic facts of time scales and time scale notation. By a solution of (1.1) we mean a nontrivial real-valued function \( x \in C^1_{\text{rd}}[T_x, \infty)_\mathbb{T}, T_x \in [t_0, \infty)_\mathbb{T}, \) such that \( r\phi_{\alpha}(x^{\Delta}) \in C^1_{\text{rd}}[T_x, \infty)_\mathbb{T} \) and \( x \) satisfies (1.1) on \([T_x, \infty)_\mathbb{T}, \) where \( C^1_{\text{rd}} \) is the set of right-dense continuous functions. A solution \( x \) of (1.1) is termed oscillatory if it is neither eventually positive nor eventually negative; otherwise, we call it nonoscillatory. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration.

In what follows, we state some oscillation results for differential equations that will be related to our oscillation results for (1.1) on time scales and explain the important contributions of this paper. Fite [14] studied the oscillatory behavior of solutions to the second-order linear differential equation

\[ x''(t) + q(t)x(t) = 0, \quad q(t) > 0, \] (1.2)

and showed that if

\[ \int_{t_0}^{\infty} q(s)ds = \infty, \] (1.3)

then every solution of equation (1.2) is oscillatory. Hille [19] improved condition (1.3) and proved that if

\[ \liminf_{t \to \infty} t \int_{t}^{\infty} q(s)ds > \frac{1}{4}, \] (1.4)

then all solutions of (1.2) are oscillatory. Erbe [9] generalized the Hille-type condition (1.4) to the delay equation

\[ x''(t) + q(t)x(g(t)) = 0, \quad q(t) > 0, \quad g(t) \leq t, \] (1.5)

and obtained that if

\[ \liminf_{t \to \infty} \int_{t}^{\infty} \frac{g(s)}{s} q(s)ds > \frac{1}{4}, \] (1.6)

then every solution of (1.5) is oscillatory. Ohriska [21] proved that, if

\[ \limsup_{t \to \infty} \int_{t}^{\infty} \frac{g(s)}{s} q(s)ds > 1, \] (1.7)
then all solutions of (1.5) are oscillatory. Condition (1.7) was extended by Agarwal et al. [5] who considered the retarded equation

\[
\left( (x'(t))^\alpha \right)' + q(t)x^\alpha(g(t)) = 0, \quad q(t) > 0, \quad g(t) \leq t, \tag{1.8}
\]
and showed that if

\[
\limsup_{t \to \infty} t^\alpha \int_t^\infty \left( \frac{g(s)}{s} \right)^\alpha g(s) ds > 1, \tag{1.9}
\]
then every solution of (1.8) is oscillatory.

For oscillation of second-order dynamic equations, Erbe et al. [13] studied the half-linear dynamic equation

\[
\left( r(t)(x^\Delta(t))^\alpha \right)^\Delta + q(t)x^\alpha(g(t)) = 0, \quad q(t) > 0, \tag{1.10}
\]
where \( g : \mathbb{T} \to \mathbb{T} \) satisfies \( g(t) \leq t \) for \( t \in \mathbb{T} \) and \( \lim_{t \to \infty} g(t) = \infty \). In particular, a Hille-type oscillation criterion was derived in the special case when \( r(t) = 1 \), namely,

\[
\left( (x^\Delta(t))^\alpha \right)^\Delta + q(t)x^\alpha(g(t)) = 0, \quad q(t) > 0, \tag{1.11}
\]
we present below for the convenience of the reader.

**Theorem 1.1** (See [13]). Let \( \alpha \geq 1 \) be a quotient of odd positive integers and

\[
\int_{t_0}^\infty g^\alpha(s)q(s)\Delta s = \infty. \tag{1.12}
\]
Assume that \( l > 0 \) and

\[
\liminf_{t \to \infty} t^\alpha \int_{\sigma(t)}^{\infty} \left( \frac{g(s)}{\sigma(s)} \right)^\alpha q(s) \Delta s > \frac{\alpha^\alpha}{l^{\alpha^2(\alpha + 1)^{\alpha + 1}}}, \tag{1.13}
\]
where \( l := \liminf_{t \to \infty} t/\sigma(t) \). Then every solution of (1.11) is oscillatory.

Erbe et al. [12] established the following Hille-type oscillation criterion for (1.10).

**Theorem 1.2** (See [12]). Let \( 0 < \alpha \leq 1 \) be a ratio of odd positive integers and assume that (1.12) holds, \( r^\Delta \geq 0 \), and

\[
\int_{t_0}^\infty r^{-\frac{1}{\alpha}}(s)\Delta s = \infty. \tag{1.14}
\]
Suppose further that \( l > 0 \) and

\[
\liminf_{t \to \infty} t^\alpha \int_{\sigma(t)}^{\infty} \left( \frac{g(s)}{\sigma(s)} \right)^\alpha q(s) \Delta s > \frac{\alpha^\alpha}{l^{\alpha^2(\alpha + 1)^{\alpha + 1}}}, \tag{1.15}
\]
where \( l := \liminf_{t \to \infty} t/\sigma(t) \). Then every solution of (1.10) is oscillatory.

Similar discrete analogues of (1.3), (1.4), (1.6), (1.7), and (1.9) are contained in the monograph by Agarwal et al. [1] for an excellent analysis of advances in this direction. We conclude by mentioning that Agarwal et al. [2,3,6], Bohner and Li [7], Erbe et al. [10,11], Hassan [16,17], Li and Saker [20], Şahiner [22], Saker [23], and Zhang and Li [25] established several Kamenev-type and Philos-type oscillation results for different classes of second-order dynamic equations.
It should be noted that research in this paper was strongly motivated by the contributions of Fite [14], Hille [19], and Wintner [24]. The purpose of this paper is to derive some sharp Fite–Hille–Wintner-type oscillation criteria for (1.1) in the cases where \( g(t) \leq t \) and \( g(t) \geq t \). On the other hand, we point out that, contrary to [12,13], we do not need in our oscillation theorems restrictive conditions (1.12) and (1.14). All functional inequalities considered in the sequel are tacitly assumed to hold eventually, that is, they are satisfied for all \( t \) large enough.

2. Main results

We begin this section with two preliminary lemmas.

**Lemma 2.1** (See [10,11]). Assume that either
\[
\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(s) \Delta s = \infty
\] (2.1)
or
\[
\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(s) \Delta s < \infty \text{ and } \int_{t_0}^{\infty} \left[ \frac{1}{r(t)} \int_{t_0}^{t} \beta^\alpha(s) q(s) \Delta s \right]^{\frac{1}{\alpha}} \Delta t = \infty,
\] (2.2)
and (1.1) has a positive solution \( x \) on \([t_0, \infty)_T\), where \( \beta(t) := \int_{s(t)}^{\infty} r^{-1/\alpha}(s) \Delta s \).

Then
\[
x^\Delta(t) > 0 \text{ and } \left[ r(t) \phi_\alpha \left( x^\Delta(t) \right) \right]^\Delta < 0
\] (2.3)
eventually.

**Lemma 2.2.** Assume that
\[
x(t) > 0, \quad x^\Delta(t) > 0, \quad \left[ r(t) \phi_\alpha \left( x^\Delta(t) \right) \right]^\Delta < 0 \text{ on } [t_0, \infty)_T.
\]
Then \( x(t) > (t - t_0) x^\Delta(t) \) and \( x(t)/(t - t_0) \) is strictly decreasing on \([t_0, \infty)_T\).

**Proof.** Since \( r \phi_\alpha(x^\Delta) \) is strictly decreasing and \( r \) is nondecreasing on \([t_0, \infty)_T\), we have, for \( t \in [t_0, \infty)_T\),
\[
x(t) > x(t) - x(t_0)
= \int_{t_0}^{t} \left[ r(s) \phi_\alpha \left( x^\Delta(s) \right) \right]^{\frac{1}{\alpha}} r^{-\frac{1}{\alpha}}(s) \Delta s
\geq \left[ r(t) \phi_\alpha \left( x^\Delta(t) \right) \right]^{\frac{1}{\alpha}} \int_{t_0}^{t} r^{-\frac{1}{\alpha}}(s) \Delta s
\geq \left[ r(t) \phi_\alpha \left( x^\Delta(t) \right) \right]^{\frac{1}{\alpha}} r^{-\frac{1}{\alpha}}(t) \int_{t_0}^{t} \Delta s
= x^\Delta(t) (t - t_0).
\]
Hence, we conclude that
\[
\left( \frac{x(t)}{t - t_0} \right)^\Delta = \frac{1}{(t - t_0)(\sigma(t) - t_0)} \left[ x^\Delta(t) (t - t_0) - x(t) \right] < 0.
\]
This completes the proof. \( \square \)

The following result is a Fite–Wintner-type oscillation criterion for (1.1).
Theorem 2.3. Assume that either (2.1) or (2.2) is satisfied. If
\[ \int_{t_0}^{\infty} q(s) \Delta s = \infty, \]  
(2.4)
then every solution of (1.1) is oscillatory.

Proof. Assume (1.1) has a nonoscillatory solution \( x \) on \([t_0, \infty)_T\). Since \(-x\) is also a solution of (1.1), without loss of generality, we may assume \( x(t) > 0 \) and \( x(g(t)) > 0 \) on \([t_0, \infty)_T\). From Lemma 2.1, there exists a \( t_1 \in [t_0, \infty)_T \) such that (2.3) holds for \( t \in [t_1, \infty)_T \). Since \( x^\Delta > 0 \), we have \( x(t) \geq x(t_1) := c > 0 \) for \( t \in [t_1, \infty)_T \), and so there exists a \( t_2 \in [t_1, \infty)_T \) such that \( x(g(t)) \geq c \) for \( t \in [t_2, \infty)_T \). Integrating (1.1) from \( t_2 \) to \( t \), we obtain
\[ r(t_2)(x^\Delta(t_2))^\alpha \geq r(t_2)(x^\Delta(t_2))^\alpha - r(t)(x^\Delta(t))^\alpha = \int_{t_2}^{t} q(s)x^\alpha(g(s))\Delta s, \]
which implies that
\[ r(t_2)(x^\Delta(t_2))^\alpha \geq c^\alpha \int_{t_2}^{t} q(s)\Delta s. \]
This contradicts (2.4) and the proof is complete. \( \Box \)

In the next results, we use the notation
\[ \varphi(t) := \begin{cases} \left( \frac{g(t)}{t} \right)^\alpha, & g(t) \leq t, \\ 1, & g(t) \geq t \end{cases} \]
and
\[ l := \liminf_{t \to \infty} \frac{t}{\sigma(t)}. \]
Furthermore, we assume in the next theorem that
\[ \int_{t_0}^{\infty} \varphi(s)q(s)\Delta s < \infty. \]

Theorem 2.4. Let \( \alpha \geq 1 \) and assume that either (2.1) or (2.2) holds. If \( l > 0 \) and
\[ \liminf_{t \to \infty} \frac{t^\alpha}{r(t)} \int_{t}^{\infty} \varphi(s)q(s)\Delta s > \frac{\alpha^\alpha}{\Gamma(\alpha+1)^{\alpha+1}}, \]  
(2.5)
then every solution of (1.1) is oscillatory.

Proof. Assume (1.1) has a nonoscillatory solution \( x \) on \([t_0, \infty)_T\). As above, without loss of generality, let \( x(t) > 0 \) and \( x(g(t)) > 0 \) on \([t_0, \infty)_T\). In view of Lemma 2.1, there exists a \( t_1 \in [t_0, \infty)_T \) such that (2.3) holds for \( t \in [t_1, \infty)_T \). Define
\[ w(t) := \frac{r(t)\varphi(t)}{x^\alpha(t)}. \]  
(2.6)
Using the product and quotient rules, we get
\[ w^\Delta(t) = \frac{1}{x^\alpha(t)}[r(t)\varphi(t) x^\Delta(t)]^\Delta + \left( \frac{1}{x^\alpha(t)} \right)^\Delta [r(t)\varphi(t) x^\Delta(t)]^\sigma = \left[ r(t)\varphi(t) x^\Delta(t) \right]^\Delta x^\alpha(t) - \frac{(x^\alpha(t))^\Delta}{x^\alpha(t)x^\sigma(t)} [r(t)\varphi(t) x^\Delta(t)]^\sigma. \]
By virtue of (1.1) and (2.6),
\[ w^\Delta (t) = - \left( x \left( \frac{g(t)}{x(t)} \right)^\alpha \right) q(t) - \frac{(x^\alpha(t))^\Delta}{x^\alpha(t)} w(\sigma(t)). \]

Let \( 0 < k < 1 \) be arbitrary. It follows from Lemma 2.2 that there exists a \( t_k \in [t_1, \infty)_T \) such that
\[ \left( x \left( \frac{g(t)}{x(t)} \right)^\alpha \right) \geq k \phi(t). \]

Hence, we conclude that, for every \( k \) where \( \epsilon > 0 \) and for any \( w > 0 \),
\[ w^\Delta (t) \leq -k \phi(t) q(t) - \frac{(x^\alpha(t))^\Delta}{x^\alpha(t)} w(\sigma(t)). \]

Applications of (2.3) and Pötzsche chain rule (see [8, Theorem 1.90]) yield
\[ (x^\alpha(t))^\Delta = \alpha \left( \int_0^1 [(1 - h) x(t) + h x(\sigma(t))]^{\alpha-1} dh \right) x^\Delta(t) \geq \alpha x^{\alpha-1}(t) x^\Delta(t). \]

Hence, by (2.6),
\[ w^\Delta (t) \leq -k \phi(t) q(t) - \alpha \frac{x^\Delta(t)}{x(t)} w(\sigma(t)) \]
\[ = -k \phi(t) q(t) - \alpha r^{-\frac{1}{\alpha}}(t) w^{\frac{1}{\alpha}}(t) w(\sigma(t)), \]
which implies that \( w^\Delta < 0 \). Integrating (2.7) from \( t \) to \( v \), we have
\[ w(v) - w(t) \leq -k \int_t^v \phi(s) q(s) \Delta s - \alpha \int_t^v r^{-\frac{1}{\alpha}}(s) w^{\frac{1}{\alpha}}(s) w(\sigma(s)) \Delta s. \]

Taking into account that \( w > 0 \) and passing to the limit as \( v \to \infty \), we get
\[ -w(t) \leq -k \int_t^\infty \phi(s) q(s) \Delta s - \alpha \int_t^\infty r^{-\frac{1}{\alpha}}(s) w^{\frac{1}{\alpha}}(s) w(\sigma(s)) \Delta s. \]

Multiplying both sides of (2.8) by \( t^\alpha / r(t) \), we obtain
\[ -\frac{t^\alpha}{r(t)} w(t) \leq -k \frac{t^\alpha}{r(t)} \int_t^\infty \phi(s) q(s) \Delta s - \alpha \frac{t^\alpha}{r(t)} \int_t^\infty r^{-\frac{1}{\alpha}}(s) w^{\frac{1}{\alpha}}(s) w(\sigma(s)) \Delta s \]
\[ = -k \frac{t^\alpha}{r(t)} \int_t^\infty \phi(s) q(s) \Delta s - \alpha \frac{t^\alpha}{r(t)} \int_t^\infty \frac{r(\sigma(s)) s w^{\frac{1}{\alpha}}(s) \sigma^{\alpha}(s) w(\sigma(s))}{s^{\alpha}(s) r^{\frac{1}{\alpha}}(s)} \left( \frac{s}{\sigma(s)} \right)^{\alpha-1} \Delta s. \]

Now, for any \( \epsilon > 0 \), there exists a \( T \in [t_k, \infty)_T \) such that, for \( t \in [T, \infty)_T \),
\[ \frac{t}{\sigma(t)} \geq l - \epsilon \quad \text{and} \quad \frac{t^\alpha w(t)}{r(t)} \geq r_* - \epsilon, \]
where
\[ r_* := \liminf_{t \to \infty} \frac{t^\alpha w(t)}{r(t)}, \quad 0 \leq r_* \leq 1. \]
due to Lemma 2.2 and (2.6). It follows from $r^\Delta \geq 0$ and (2.9) that
\[
- \frac{t^\alpha}{r(t)} w(t) \leq -k \frac{t^\alpha}{r(t)} \int_t^{\infty} \varphi(s) q(s) \Delta s - (l - \varepsilon)^{\alpha - 1} (r_* - \varepsilon)^{1 + \frac{1}{\sigma}} t^\alpha \int_t^{\infty} \frac{\alpha}{s^\alpha \sigma(s)} \Delta s.
\] (2.10)

An application of Pötzsche chain rule implies that
\[
(s^\alpha)^\Delta = \alpha \int_0^1 [(1 - h) s + h \sigma(s)]^\alpha - 1 dh \leq \alpha \sigma^{\alpha - 1}(s).
\] (2.11)

It follows now from (2.11) and the quotient rule that
\[
\left( \frac{-1}{s^\alpha} \right)^\Delta = \frac{(s^\alpha)^\Delta}{s^\alpha \sigma(s)} \leq \frac{\alpha}{s^\alpha \sigma(s)}.
\] (2.12)

Using (2.12) in (2.10), we deduce that
\[
- \frac{t^\alpha}{r(t)} w(t) \leq -k \frac{t^\alpha}{r(t)} \int_t^{\infty} \varphi(s) q(s) \Delta s - (l - \varepsilon)^{\alpha - 1} (r_* - \varepsilon)^{1 + \frac{1}{\sigma}} t^\alpha \int_t^{\infty} \frac{\alpha}{s^\alpha \sigma(s)} \Delta s
\]
which yields
\[
k \frac{t^\alpha}{r(t)} \int_t^{\infty} \varphi(s) q(s) \Delta s \leq \frac{t^\alpha}{r(t)} w(t) - (l - \varepsilon)^{\alpha - 1} (r_* - \varepsilon)^{1 + \frac{1}{\sigma}}.
\]

Taking the lim inf of both sides of the latter inequality as $t \to \infty$, we conclude that
\[
\liminf_{t \to \infty} k \frac{t^\alpha}{r(t)} \int_t^{\infty} \varphi(s) q(s) \Delta s \leq r_* - (l - \varepsilon)^{\alpha - 1} (r_* - \varepsilon)^{1 + \frac{1}{\sigma}}.
\]
Since $0 < k < 1$ and $\varepsilon > 0$ are arbitrary, we arrive at
\[
\liminf_{t \to \infty} t^\alpha \int_t^{\infty} \varphi(s) q(s) \Delta s \leq r_* - t^{\alpha - 1} r_*^{1 + \frac{1}{\sigma}}.
\]

Let
\[
A = t^{\alpha - 1}, \quad B = 1, \quad \text{and} \quad u = r_*.
\]

Using the inequality (see Hardy et al. [15] or Li and Saker [20])
\[
Bu - Au^{\frac{\alpha + 1}{\alpha}} \leq \frac{\alpha^\alpha}{(\alpha + 1)^{\alpha + 1}} B^{\alpha + 1} A^{-\alpha}, \quad A > 0,
\] (2.13)
we have
\[
\liminf_{t \to \infty} t^\alpha \int_t^{\infty} \varphi(s) q(s) \Delta s \leq \frac{\alpha^\alpha}{t^{\alpha (\alpha - 1)} (\alpha + 1)^{\alpha + 1}},
\]
which contradicts (2.5). This completes the proof. \qed

In the following, we utilize the notation
\[
\tilde{\varphi}(t) := \begin{cases} \left( \frac{g(t)}{\sigma(t)} \right)^\alpha, & g(t) \leq \sigma(t), \\ 1, & g(t) \geq \sigma(t) \end{cases}
\]
and assume in the next theorem that
\[ \int_{t_0}^{\infty} \phi(s)q(s)\Delta s < \infty. \]

**Theorem 2.5.** Let \( 0 < \alpha \leq 1 \) and assume that either (2.1) or (2.2) is satisfied. If \( l > 0 \) and

\[ \liminf_{t \to \infty} \frac{t^\alpha}{r(t)} \int_t^\infty \phi(s)q(s)\Delta s > \frac{\alpha}{l^\alpha(1-\alpha)(\alpha+1)^{\alpha+1}}, \]

then every solution of (1.1) is oscillatory.

**Proof.** Suppose to the contrary that \( x \) is a nonoscillatory solution of (1.1) on \([t_0, \infty)_T\). Without loss of generality, we may assume \( x(t) > 0 \) and \( x(g(t)) > 0 \) for \( t \in [t_0, \infty)_T \). By virtue of Lemma 2.1, there exists a \( t_1 \in [t_0, \infty)_T \) such that (2.3) holds for \( t \in [t_1, \infty)_T \). Define a function \( w \) as in (2.6). Using the product and quotient rules, we have

\[ w^\Delta = \frac{1}{x^\alpha(\sigma(t))} \left[ r(t)\phi_{\alpha} \left( x^\Delta(t) \right) \right]^\Delta + \left( \frac{1}{x^\alpha(t)} \right)^\Delta r(t)\phi_{\alpha} \left( x^\Delta(t) \right) \]

\[ = \frac{r(t)\phi_{\alpha}(x^\Delta(t))}{x^\alpha(\sigma(t))} - \frac{(x^\alpha(t))^\Delta}{x^\alpha(t)\sigma(\sigma(t))} r(t)\phi_{\alpha} \left( x^\Delta(t) \right). \]

From (1.1) and (2.6), we conclude that

\[ w^\Delta(t) = -\left( \frac{x^\alpha(g(t))}{x(\sigma(t))} \right)^\alpha q(t) - \frac{(x^\alpha(t))^\Delta}{x^\alpha(\sigma(t))} w(t). \]

Let \( 0 < k < 1 \) be arbitrary. It follows from Lemma 2.2 that there exists a \( t_k \in [t_1, \infty)_T \) such that

\[ \left( \frac{x^\alpha(g(t))}{x(\sigma(t))} \right)^\alpha \geq k\phi(t). \]

Therefore, we have, for every \( k \in (0, 1) \) and for \( t \in [t_k, \infty)_T \),

\[ w^\Delta(t) \leq -k\phi(t)q(t) - \frac{(x^\alpha(t))^\Delta}{x^\alpha(\sigma(t))} w(t). \]

Using Pötzsche's chain rule, we obtain

\[ (x^\alpha(t))^\Delta = \alpha \left( \int_0^1 [(1-h)x(t) + hx(\sigma(t))]^{\alpha-1} \Delta h \right) x^\Delta(t) \geq \alpha x^{\alpha-1}(\sigma(t)) x^\Delta(t). \]

Hence, by virtue of

\[ \frac{x^\Delta(t)}{x(\sigma(t))} = \frac{1}{x(\sigma(t))} \left[ \frac{r(t)\phi_{\alpha} \left( x^\Delta(t) \right)}{r(t)} \right]^{\frac{1}{\alpha}} \]

\[ \geq \frac{1}{x(\sigma(t))} \left[ \frac{r(\sigma(t))\phi_{\alpha}(x^\Delta(\sigma(t)))}{r(t)} \right]^{\frac{1}{\alpha}} = r^{\frac{1}{\alpha}}(t)w^{\frac{1}{\alpha}}(\sigma(t)), \]

we deduce that

\[ w^\Delta(t) \leq -k\phi(t)q(t) - \alpha \frac{x^\Delta(t)}{x(\sigma(t))} w(t) \]

\[ \leq -k\phi(t)q(t) - \alpha r^{\frac{1}{\alpha}}(t)w^{\frac{1}{\alpha}}(\sigma(t)) w(t), \]

(2.15)
which yields \( w^\Delta < 0 \). Integrating (2.15) from \( t \) to \( v \), we arrive at
\[
 w(v) - w(t) \leq -k \int_t^v \tilde{\varphi}(s) q(s) \, \Delta s - \alpha \int_t^v r^{-\frac{1}{\sigma}}(s) w_{+}^{\frac{1}{\sigma}}(\sigma(s)) w(s) \, \Delta s,
\]
and thus
\[
 -w(t) \leq -k \int_t^\infty \tilde{\varphi}(s) q(s) \, \Delta s - \alpha \int_t^\infty r^{-\frac{1}{\sigma}}(s) w_{+}^{\frac{1}{\sigma}}(\sigma(s)) w(s) \, \Delta s \tag{2.16}
\]
due to the fact that \( w > 0 \). Multiplying both sides of (2.16) by \( t^\alpha / r(t) \), we have
\[
 -\frac{t^\alpha}{r(t)} w(t) \leq -k \frac{t^\alpha}{r(t)} \int_t^\infty \tilde{\varphi}(s) q(s) \, \Delta s - \frac{\alpha t^\alpha}{r(t)} \int_t^\infty r^{-\frac{1}{\sigma}}(s) w_{+}^{\frac{1}{\sigma}}(\sigma(s)) w(s) \, \Delta s
\]
\[
 = -k \frac{t^\alpha}{r(t)} \int_t^\infty \tilde{\varphi}(s) q(s) \, \Delta s - \frac{\alpha t^\alpha}{r(t)} \int_t^\infty \frac{r(s)}{s^{\sigma}(s)} \left( \frac{r(\sigma(s))}{r(s)} \right)^{\frac{1}{\sigma}} s^\sigma w(\sigma(s)) w_{+}^{\frac{1}{\sigma}}(\sigma(s)) \left( \frac{s}{\sigma(s)} \right)^{1-\frac{1}{\sigma}} \Delta s.
\]
Now, for any \( \varepsilon > 0 \), there exists a \( T \in [t_k, \infty)_T \) such that (2.9) holds for \( t \in [T, \infty)_T \). On the other hand, by Pötzsche chain rule, we conclude that
\[
 (s^\alpha)^\Delta = \alpha \int_0^1 [(1-h) s + h(\sigma(s))^{\alpha-1}] \, dh \leq \alpha s^{\alpha-1},
\]
and so
\[
 \left( \frac{-1}{s^\alpha} \right)^\Delta \leq \frac{(s^\alpha)^\Delta}{s^\alpha \sigma^\alpha(s)} = \frac{\alpha}{\sigma^\alpha(s)}.
\]
It follows now from \( r^\Delta \geq 0 \) and (2.9) that
\[
 -\frac{t^\alpha}{r(t)} w(t) \leq -k \frac{t^\alpha}{r(t)} \int_t^\infty \tilde{\varphi}(s) q(s) \, \Delta s - (l-\varepsilon)^{1-\alpha} (r_+ - \varepsilon)^{1+\frac{1}{\sigma}} t^\alpha \int_t^\infty \frac{\alpha}{s^{\sigma}(s)} \, \Delta s
\]
\[
 \leq -k \frac{t^\alpha}{r(t)} \int_t^\infty \tilde{\varphi}(s) q(s) \, \Delta s - (l-\varepsilon)^{1-\alpha} (r_+ - \varepsilon)^{1+\frac{1}{\sigma}} t^\alpha \int_t^\infty \left( \frac{-1}{s^\alpha} \right)^\Delta \, \Delta s
\]
\[
 = -k \frac{t^\alpha}{r(t)} \int_t^\infty \tilde{\varphi}(s) q(s) \, \Delta s - (l-\varepsilon)^{1-\alpha} (r_+ - \varepsilon)^{1+\frac{1}{\sigma}},
\]
which implies that
\[
k \frac{t^\alpha}{r(t)} \int_t^\infty \tilde{\varphi}(s) q(s) \, \Delta s \leq \frac{t^\alpha}{r(t)} w(t) - (l-\varepsilon)^{1-\alpha} (r_+ - \varepsilon)^{1+\frac{1}{\sigma}}.
\]
Taking the lim inf of both sides of the latter inequality as \( t \to \infty \), we get
\[
 \liminf_{t \to \infty} k \frac{t^\alpha}{r(t)} \int_t^\infty \tilde{\varphi}(s) q(s) \, \Delta s \leq r_+ - (l-\varepsilon)^{1-\alpha} (r_+ - \varepsilon)^{1+\frac{1}{\sigma}}.
\]
By virtue of the facts that \( 0 < k < 1 \) and \( \varepsilon > 0 \) are arbitrary, we conclude that
\[
 \liminf_{t \to \infty} \frac{t^\alpha}{r(t)} \int_t^\infty \tilde{\varphi}(s) q(s) \, \Delta s \leq r_+ - l^{1-\alpha} r_+^{1+\frac{1}{\sigma}}.
\]
Letting $A = \lambda^{1-a}, B = 1,$ and $u = r_\ast,$ and using inequality (2.13), we obtain
\[
\liminf_{t \to \infty} \int_{t}^{\infty} \hat{\phi}(s) q(s) \, \Delta s \leq \frac{\alpha}{\mu(1-a)(\alpha+1)^{\alpha+1}},
\]
which contradicts (2.14). The proof is complete. □

**Theorem 2.6.** Assume that either (2.1) or (2.2) holds. If
\[
\limsup_{t \to \infty} \int_{t}^{\infty} \phi(s) q(s) \, \Delta s > 1,
\]
then every solution of (1.1) is oscillatory.

**Proof.** Let $x$ be a nonoscillatory solution of (1.1) on $[t_0, \infty)_T.$ Without loss of generality, we may assume $x(t) > 0$ and $x(g(t)) > 0$ for $t \in [t_0, \infty)_T.$ By Lemma 2.1, there exists a $t_1 \in [t_0, \infty)_T$ such that (2.3) holds for $t \in [t_1, \infty)_T.$ It follows from Lemma 2.2 and $x^{\Delta} > 0$ that, for each $0 < k < 1,$ there exists a $t_k \in [t_0, \infty)_T$ such that
\[
x^\alpha(g(s)) \geq k \phi(s) r^\alpha(x^{\Delta}(t))
\]
for $s \in [t_0, \infty)_T$ and $t \in [t_k, \infty)_T.$ Integrating (1.1) from $t$ to $v,$ we obtain
\[
\int_{t}^{v} q(s) x^\alpha(g(s)) \, \Delta s = r(t)(x^{\Delta}(t))^\alpha - r(v)(x^{\Delta}(v))^\alpha \leq r(t)(x^{\Delta}(t))^\alpha.
\]
Using (2.18) in the latter inequality, we conclude that
\[
k \frac{t^\alpha}{r(t)} \int_{t}^{v} \phi(s) q(s) \, \Delta s \leq 1.
\]
Since $0 < k < 1$ is arbitrary, we have
\[
\frac{t^\alpha}{r(t)} \int_{t}^{v} \phi(s) q(s) \, \Delta s \leq 1.
\]
Letting $v \to \infty,$ we obtain
\[
\frac{t^\alpha}{r(t)} \int_{t}^{\infty} \phi(s) q(s) \, \Delta s \leq 1,
\]
and so
\[
\limsup_{t \to \infty} \frac{t^\alpha}{r(t)} \int_{t}^{\infty} \phi(s) q(s) \, \Delta s \leq 1,
\]
which contradicts (2.17). This completes the proof. □

3. **Examples**

The following examples illustrate applications of theoretical results presented in this paper.

**Example 3.1.** For $t \in [t_0, \infty)_T,$ consider the second-order Euler dynamic equations
\[
x^{\Delta\Delta}(t) + \frac{\beta}{t \sigma(t)} x(t) = 0 \quad (3.1)
\]
and
\[
x^{\Delta\Delta}(t) + \frac{\beta}{t \sigma(t)} x(\sigma(t)) = 0, \quad (3.2)
\]
where \( \beta > 0 \) is a constant and \( l = \liminf_{t \to \infty} t/\sigma(t) > 0 \). It is not difficult to deduce that every solution of (3.1) and (3.2) is oscillatory if \( \beta > 1/4 \) when using Theorems 2.4 and 2.5, respectively. It is well known that this condition is the best possible for the second-order Euler differential equation

\[
x''(t) + \frac{\beta}{t^2} x(t) = 0.
\]

**Example 3.2.** For \( t \in [t_0, \infty)_T \), consider a second-order half-linear delay dynamic equation

\[
\left[ \phi_{\alpha}(x^{\Delta}(t)) \right]^{\Delta} + \frac{\alpha \beta}{tg^{\alpha}(t)} \phi_{\alpha}(x(g(t))) = 0, \quad g(t) \leq t,
\]

where \( \beta > 0 \) is a constant and \( l = \liminf_{t \to \infty} t/\sigma(t) > 0 \). Let \( r(t) = 1 \) and \( q(t) = \alpha \beta/(tg^{\alpha}(t)) \).

If \( \alpha \geq 1 \), then

\[
\liminf_{t \to \infty} \frac{t^\alpha}{r(t)} \int_t^\infty \varphi(s) q(s) \Delta s = \beta \liminf_{t \to \infty} t^\alpha \int_t^\infty \frac{\alpha}{s^{\alpha+1}} \Delta s \\
\geq \beta \liminf_{t \to \infty} t^\alpha \int_t^\infty \left( -\frac{1}{s^{\alpha}} \right)^{\Delta} \Delta s = \beta.
\]

An application of Theorem 2.4 implies that every solution of (3.3) is oscillatory if

\[
\alpha \geq 1 \quad \text{and} \quad \beta > \frac{\alpha^\alpha}{l^{\alpha}(\alpha-1)(\alpha+1)^{\alpha+1}}.
\]

If \( 0 < \alpha \leq 1 \), then

\[
\liminf_{t \to \infty} \frac{t^\alpha}{r(t)} \int_t^\infty \bar{\varphi}(s) q(s) \Delta s = \beta \liminf_{t \to \infty} t^\alpha \int_t^\infty \frac{\alpha}{s^{\alpha}(s)} \Delta s \\
\geq \beta \liminf_{t \to \infty} t^\alpha \int_t^\infty \left( -\frac{1}{s^{\alpha}} \right)^{\Delta} \Delta s = \beta.
\]

An application of Theorem 2.5 yields oscillation of every solution of (3.3) for

\[
0 < \alpha \leq 1 \quad \text{and} \quad \beta > \frac{\alpha^\alpha}{l^{\alpha(1-\alpha)}(\alpha+1)^{\alpha+1}}.
\]

### 4. Conclusions

In this paper, several new Fite–Hille–Wintner-type criteria are presented that can be applied to (1.1) on an arbitrary time scale. Our results extend related contributions to the second-order differential equations; see the following details.

1. Condition (2.4) reduces to (1.3) in the case where \( T = \mathbb{R}, \alpha = 1, r(t) = 1, \) and \( g(t) = t \).
2. Conditions (2.5) and (2.14) reduce to (1.4) in the case when \( T = \mathbb{R}, \alpha = 1, r(t) = 1, \) and \( g(t) = t \).
3. Conditions (2.5) and (2.14) reduce to (1.6) assuming that \( T = \mathbb{R}, \alpha = 1, r(t) = 1, \) and \( g(t) \leq t \).
4. Condition (2.17) reduces to (1.7) under the assumptions that \( T = \mathbb{R}, \alpha = 1, r(t) = 1, \) and \( g(t) \leq t \).
5. Condition (2.17) reduces to (1.9) in the case where \( T = \mathbb{R}, r(t) = 1, \) and \( g(t) \leq t \).
We derive several oscillation criteria for (1.1) in both cases $g(t) \leq t$ and $g(t) \geq t$. Contrary to [12,13], we do not impose restrictive conditions (1.12) and (1.14) in our oscillation results which, in a certain sense, is a significant improvement compared to the results in the cited papers. In particular, our results improve those reported in [12,13] when $g(t) \leq t$; see the following details.

1. If $r(t) = 1$ and $g(t) \leq t$, then condition (2.5) reduces to
   \[ \liminf_{t \to \infty} t^\alpha \int_t^\infty \left( \frac{g(s)}{s} \right)^\alpha q(s) \Delta s > \frac{\alpha^\alpha}{l_0(\alpha - 1)(\alpha + 1)^{\alpha + 1}}. \]

   By virtue of
   \[ \frac{\alpha^\alpha}{l_0(\alpha - 1)(\alpha + 1)^{\alpha + 1}} < \frac{\alpha^\alpha}{l_0^2(\alpha + 1)^{\alpha + 1}} \quad \text{for } 0 < l < 1, \]

   Theorem 2.4 improves Theorem 1.1 (condition (2.5) improves (1.13)).

2. If $g(t) \leq t$, then condition (2.14) reduces to
   \[ \liminf_{t \to \infty} \frac{t^\alpha}{r(t)} \int_t^\infty \left( \frac{g(s)}{\sigma(s)} \right)^\alpha q(s) \Delta s > \frac{\alpha^\alpha}{l_0(1 - \alpha)(\alpha + 1)^{\alpha + 1}}. \]

   Since
   \[ \frac{t^\alpha}{r(t)} \int_t^\infty \left( \frac{g(s)}{\sigma(s)} \right)^\alpha q(s) \Delta s \geq \frac{t^\alpha}{r(t)} \int_0^\infty \left( \frac{g(s)}{\sigma(s)} \right)^\alpha q(s) \Delta s \]

   and
   \[ \frac{\alpha^\alpha}{l_0(1 - \alpha)(\alpha + 1)^{\alpha + 1}} < \frac{\alpha^\alpha}{l_0^2(\alpha + 1)^{\alpha + 1}} \]

   for $0 < l < 1$ and $1/2 < \alpha \leq 1$, we conclude that Theorem 2.5 improves Theorem 1.2 in the case when $1/2 < \alpha \leq 1$ (condition (2.14) improves (1.15) in the case where $1/2 < \alpha \leq 1$).

**Remark 4.1.** As demonstrated above, Theorems 2.4 and 2.5 provide sharper estimates for oscillation of (1.1) on the time scale $T := q^\mathbb{Z} = \{ q^k : k \in \mathbb{Z}, q > 1 \} \cup \{ 0 \}$ (which is called second-order half-linear $q$-difference equation with deviating arguments).

**Remark 4.2.** It would be of interest to suggest a different method to further investigate (1.1) assuming that
\[ \int_0^\infty r^{-\frac{1}{\alpha}}(s) \Delta s < \infty. \]

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