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FORMULAS OF BENDIXSON AND ALEKSEEV FOR DIFFERENCE EQUATIONS

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Abstract

A well-known formula of Bendixson states that solutions of first-order differential equations, as functions of their initial conditions, satisfy a certain partial differential equation. A consequence is Alekseev's nonlinear variation of parameters formula. In this paper, corresponding results are proved for difference equations. To achieve this, use is made of the recently introduced concept of alpha derivatives, rather than of differences or of the usual derivatives. This technique allows the results to be generalized to alpha dynamic equations, which include among others ordinary differential and difference equations.

1. *Introduction*

It is well known that for appropriate real-valued functions f, the solution $y(t,s,z)$ of the initial value problem

$$y' = f(t,y), \qquad y(s) = z \tag{1}$$

satisfies the partial differential equation

$$y_s(t,s,z) + f(s,z)y_z(t,s,z) = 0, \tag{2}$$

where y_s and y_z denote the partial derivatives of y with respect to the second and third variables, respectively. This result is due to I. Bendixson (1896). If we consider the discrete analogue of (1), namely

$$\Delta y = f(t,y), \qquad y(s) = z, \tag{3}$$

where Δ is the usual forward difference operator, then formula (2) does not remain true, even if we replace $y_s(t,s,z)$ by $y(t,s+1,z) - y(t,s,z)$. In this paper we show that (2) does remain true if we additionally replace $y_z(t,s,z)$ by $y_\beta(t,s+1,z)$, which is the beta derivative of y with respect to the third variable (and β will be explicitly given). The concept of alpha derivatives has recently been introduced in [**2**], and we will collect some basic results from that theory in the next section. For a general introduction to the theory of time scales, we refer to [**6**, **7**]. A non-empty set $\mathbb{T} \subset \mathbb{R}$ such that every Cauchy sequence in $\mathbb{T}$ converges to a point in $\mathbb{T}$ (with the possible exception of Cauchy sequences that converge to a finite infimum or supremum of $\mathbb{T}$), together with a function α that maps $\mathbb{T}$ into $\mathbb{T}$, is called a *generalized time scale*. A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is said to be *alpha differentiable* at a point $t \in \mathbb{T}$ if there exists a number $f_\alpha(t)$, the so-called *alpha derivative* of f at t, with the property that for every $\varepsilon > 0$ there exists a neighborhood U of t such that

$$|f(\alpha(t)) - f(s) - f_\alpha(t)(\alpha(t) - s)| \leqslant \varepsilon|\alpha(t) - s|$$

is true for all $s \in U$.

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The following theorem offers a discrete version of formula (2).

THEOREM 1.1. *Suppose that solutions to the initial value problem* (3) *exist and are unique. Let*

$$\beta(z) = \beta_s(z) = z + f(s, z). \tag{4}$$

Then

$$y_s(t, s, z) + f(s, z) y_\beta(t, s+1, z) = 0, \tag{5}$$

where $y_s(t, s, z) = y(t, s+1, z) - y(t, s, z)$ *and* $y_\beta(t, s+1, z)$ *is the beta derivative of* $y(t, s+1, z)$ *with respect to the third variable.*

Note that this formula is very simple if $f(s, z) \neq 0$, for in this case

$$\begin{aligned} y_\beta(t, s+1, z) &= \frac{y(t, s+1, \beta(z)) - y(t, s+1, z)}{\beta(z) - z} \\ &= \frac{y(t, s, z) - y(t, s+1, z)}{f(s, z)} \\ &= -\frac{y_s(t, s, z)}{f(s, z)} \end{aligned}$$

since

$$\begin{aligned} \beta(z) &= z + f(s, z) \\ &= y(s, s, z) + f(s, y(s, s, z)) \\ &= y(s+1, s, z) \end{aligned}$$

implies (because of the unique solvability of the initial value problem (3)) that

$$y(t, s+1, \beta(z)) = y(t, s, z).$$

In Section 3 of this paper we prove a generalization of formulas (2) and (5) for the initial value problem

$$y_\alpha = f(t, y), \qquad y(s) = z, \tag{6}$$

where $s, t \in \mathbb{T}$ and $z \in \mathbb{R}$, and where $(\mathbb{T}, \alpha)$ is a generalized time scale. Note that the cases (3) (that is, $\mathbb{T} = \mathbb{Z}$ and $\alpha(t) = t + 1$ for all $t \in \mathbb{T}$) and (1) (that is, $\mathbb{T} = \mathbb{R}$ and $\alpha(t) = t$ for all $t \in \mathbb{T}$) are special cases of our result. In Theorem 1.1, we replace (4) by

$$\beta(z) = \beta_s(z) = z + \mu_\alpha(s) f(s, z), \tag{7}$$

where the *generalized graininess* μ_α is given by

$$\mu_\alpha(t) = \alpha(t) - t \quad \text{for all } t \in \mathbb{T},$$

while the conclusion (5) is then replaced by

$$y_{\alpha_s}(t, s, z) + f(s, z) y_\beta(t, \alpha(s), z) = 0, \tag{8}$$

where $y_{\alpha_s}(t, s, z)$ is the alpha derivative of $y(t, s, z)$ with respect to the second variable, while $y_\beta(t, \alpha(s), z)$ is the beta derivative of $y(t, \alpha(s), z)$ with respect to the third variable. Note that $\mu_\alpha(s) = 0$ in (7) implies that $\mu_\beta(z) = 0$, so that (8) indeed resembles (2) in case (1).

In Section 3 we finally also offer an alpha-dynamic version of Alekseev's nonlinear variation of parameters formula [**5**] (for a discrete version see [**1**, Theorem 5.7.1]), which states that if y solves (1) and x solves (1) with f replaced by $f+g$, then their difference satisfies

$$x(t,s,z)-y(t,s,z)=\int_s^t y_\gamma(t,\alpha(\tau),\beta(x(\tau,s,z)))g(\tau,x(\tau,s,z))d_\alpha\tau, \qquad (9)$$

where the integral is the Cauchy integral with respect to alpha differentiation, and where $y_\gamma(t,\alpha(\tau),\beta(x(\tau,s,z)))$ is the gamma derivative of $y(t,\alpha(\tau),\beta(x(\tau,s,z)))$ with respect to the third variable, with

$$\gamma(\beta(z))=\beta(z)+\mu_\alpha(s)g(s,z)=z+\mu_\alpha(s)[f(s,z)+g(s,z)]. \qquad (10)$$

Note that (9) reduces to Alekseev's formula in case (1), since then (7) and (10) imply that $\beta(z)=\gamma(z)=z$ for all $z\in\mathbb{R}$.

Concluding this introduction, we remark that the dynamic versions of these results to be found in [**8**, Section 2.7] are different from ours, due to a partly incorrect chain rule given without proof in [**8**, Theorem 1.2.3 (iv)]. The chain rule for alpha derivatives given in [**2**, Theorem 2.7] plays a central rôle in our investigations, and is presented in the next section.

2. *Alpha derivatives*

We consider a generalized time scale $(\mathbb{T},\alpha)$, as introduced in the previous section. The *interior* of $\mathbb{T}$ relative to α is defined to be the set

$$\mathbb{T}^\kappa=\{t\in\mathbb{T}:\text{either } \alpha(t)\neq t, \text{ or } \alpha(t)=t \text{ and } t \text{ is not isolated}\}.$$

Let $f:\mathbb{T}\longrightarrow\mathbb{R}$ be a function. We put $f^\alpha=f\circ\alpha$, and we collect the following results from [**2**] (see also [**3**, **6**]).

LEMMA 2.1 [**6**, Theorem 8.3.1]. *If f is alpha differentiable at t, then it is continuous at t.*

LEMMA 2.2 [**6**, Formula (8.7)]. *If f is alpha differentiable at $t\in\mathbb{T}^\kappa$, then $f^\alpha(t)=f(t)+\mu_\alpha(t)f_\alpha(t)$.*

LEMMA 2.3 [**6**, Theorem 8.33 and Exercise 8.34]. *If $f,g:\mathbb{T}\longrightarrow\mathbb{R}$ are alpha differentiable at $t\in\mathbb{T}^\kappa$, then fg is alpha differentiable at t, and the formulas*

$$(fg)_\alpha=f_\alpha g^\alpha+fg_\alpha=f_\alpha g+f^\alpha g_\alpha$$

hold at t. If $g(t)g(\alpha(t))\neq 0$, then f/g is alpha differentiable at t with

$$\left(\frac{f}{g}\right)_\alpha=\frac{f_\alpha g-fg_\alpha}{gg^\alpha}.$$

LEMMA 2.4 [**6**, Theorem 8.35]. *Let $(\mathbb{T},\alpha)$ and $(\tilde{\mathbb{T}},\tilde\alpha)$ be generalized time scales related by a function $g:\mathbb{T}\longrightarrow\tilde{\mathbb{T}}$. Let $F:\tilde{\mathbb{T}}\longrightarrow\mathbb{R}$, and suppose that $t\in\mathbb{T}^\kappa$ is such that $g(\alpha(t))=\tilde\alpha(g(t))$. If $g_\alpha(t)$ and $F_{\tilde\alpha}(g(t))$ exist, then $(F\circ g)_\alpha(t)$ exists and satisfies at t the chain rule*

$$(F\circ g)_\alpha=(F_{\tilde\alpha}\circ g)g_\alpha.$$

LEMMA 2.5 (Proof as in [**6**, Theorem 1.90]). *If f is defined and continuously differentiable on all of $\mathbb{R}$, then*

$$f_\alpha(t) = \int_0^1 f'(h\alpha(t) + (1-h)t)dh.$$

In what follows, we will always assume that the occurring functions satisfy the assumptions of Lemma 2.5. In order to illustrate the above rules as well as Theorem 1.1, we now conclude this section with an example.

EXAMPLE 2.6. We consider problem (3) when $1 + ty \neq 0$, with

$$f(t,y) = \frac{ty(1-y)}{1+ty}.$$

It is easy to verify that the solution of (3) is given by

$$y(t,s,z) = \frac{t!/s!}{1/z + t!/s! - 1}.$$

Also, we find β from Theorem 1.1, as $\beta(z) = z + f(s,z) = (1+s)z/(1+sz)$, so that

$$\begin{aligned} y(t, s+1, \beta(z)) &= \frac{t!/(s+1)!}{1/\beta(z) + t!/(s+1)! - 1} \\ &= \frac{t!/(s+1)!}{(1+sz)/((1+s)z) + t!/(s+1)! - 1} \\ &= y(t,s,z). \end{aligned}$$

Applications of Lemma 2.3 and Lemma 2.4 yield

$$\begin{aligned} f(s,z)y_\beta(t,s+1,z) &= \frac{t!/(z\beta(z)(s+1)!)f(s,z)}{(1/z + t!/(s+1)! - 1)(1/\beta(z) + t!/(s+1)! - 1)} \\ &= \frac{s/(s+1)(1/z-1)y(t,s+1,\beta(z))}{1/z + t!/(s+1)! - 1} \\ &= \frac{s(1/z-1)t!/(s+1)!}{(1/z + t!/(s+1)! - 1)(1/z + t!/s! - 1)} \\ &= \frac{t!/s!(1/z + t!/(s+1)! - 1) - t!/(s+1)!(1/z + t!/s! - 1)}{(1/z + t!/(s+1)! - 1)(1/z + t!/s! - 1)} \\ &= y(t,s,z) - y(t,s+1,z) \\ &= -y_s(t,s,z), \end{aligned}$$

which establishes formula (5) for this example. This version of the discrete logistic equation, as well as its dynamic analogues, is discussed in more detail in [**4**].

3. *Main results*

Let $(\mathbb{T}, \alpha)$ be a generalized time scale. In the sequel we will assume that the order of differentiation of y with respect to α_s and α_t, and also with respect to β and α_s, is irrelevant; that is, $y_{\alpha_s\alpha_t} = y_{\alpha_t\alpha_s}$ and $y_{\beta\alpha_s} = y_{\alpha_s\beta}$. This assumption is satisfied, for example, for discrete time scales with $\mu_\alpha(t) > 0$ for all $t \in \mathbb{T}$.

THEOREM 3.1. *Suppose that solutions to the initial value problem* (6) *exist and are unique. Define* β *by* (7), *let* $\tilde{\alpha}(y(t,s,z)) = y(t,\alpha(s),z)$ *for* t *and* z *fixed, and let* $\tilde{\beta}(y(t,\alpha(s),z)) = y(t,\alpha(s),\beta(z))$ *for* t *and* s *fixed. Assume also that*

$$\big(\{y(t,s,z) : s \in \mathbb{T}\}, \tilde{\alpha}\big) \quad \text{and} \quad \big(\{y(t,\alpha(s),z) : z \in \mathbb{R}\}, \tilde{\beta}\big)$$

are generalized time scales. Then

$$v(t) = y_{\alpha_s}(t,s,z) \quad \text{and} \quad w(t) = y_\beta(t,\alpha(s),z)$$

are solutions of the initial value problems

$$v_\alpha = f_{\tilde{\alpha}}(t, y(t,s,z))v, \qquad v(\alpha(s)) = -f(s,z) \tag{11}$$

and

$$w_\alpha = f_{\tilde{\beta}}(t, y(t,\alpha(s),z))w, \qquad w(\alpha(s)) = 1, \tag{12}$$

respectively, where $f_{\tilde{\alpha}}$ *and* $f_{\tilde{\beta}}$ *indicate the* $\tilde{\alpha}$ *and* $\tilde{\beta}$ *derivatives, respectively, of* f *with respect to the second variable. Moreover,*

$$f_{\tilde{\alpha}}(t, y(t,s,z)) = f_{\tilde{\beta}}(t, y(t,\alpha(s),z)),$$

and formula (8) *holds* (*that is,* $v(t) + f(s,z)w(t) = 0$).

Proof. Let $F(y) = f(t,y)$, $G(s) = y(t,s,z)$, and $H(z) = y(t,\alpha(s),z)$. For each

$$G(s) \in \{y(t,s,z) : s \in \mathbb{T}\} =: G(\mathbb{T})$$

and each

$$H(z) \in \{y(t,\alpha(s),z) : z \in \mathbb{R}\} =: H(\mathbb{R}),$$

we define $\tilde{\alpha}$ and $\tilde{\beta}$ as above, so that

$$\tilde{\alpha} : G(\mathbb{T}) \longrightarrow G(\mathbb{T}) \quad \text{and} \quad \tilde{\beta} : H(\mathbb{R}) \longrightarrow H(\mathbb{R}),$$

and $(G(\mathbb{T}), \tilde{\alpha})$ and $(H(\mathbb{R}), \tilde{\beta})$ are generalized time scales. We use Lemma 2.4 to derive

$$v_\alpha = (F \circ G)_\alpha = (F_{\tilde{\alpha}} \circ G)G_\alpha = (F_{\tilde{\alpha}} \circ G)v$$

and

$$w_\alpha = (F \circ H)_\beta = (F_{\tilde{\beta}} \circ H)H_\beta = (F_{\tilde{\beta}} \circ H)w,$$

since

$$\tilde{\alpha}(G(s)) = y(t,\alpha(s),z) = H(z)$$

and

$$\tilde{\beta}(H(z)) = y(t,\alpha(s),\beta(z)) = y(t,s,z) = G(s),$$

where the last equality follows from the unique solvability together with

$$\beta(z) = z + \mu_\alpha(s)f(s,z) = y(s,s,z) + \mu_\alpha(s)f(s,y(s,s,z)) = y(\alpha(s),s,z)$$

when we apply Lemma 2.2. An application of Lemma 2.5, using the change of variables $k = 1 - h$, implies that $F_{\tilde{\alpha}}(G(s)) = F_{\tilde{\beta}}(H(z))$. By the definition of the beta derivative and

$$|y(\alpha(s),\alpha(s),\beta(z)) - y(\alpha(s),\alpha(s),c) - 1 \cdot (\beta(z) - c)| = 0,$$

we find that $w(\alpha(s)) = 1$, and similarly (with some more calculation) $v(\alpha(s)) = -f(s,z)$. Hence v and w satisfy the initial value problems (11) and (12), respectively, from which formula (8) follows immediately. □

Now, besides (6), we consider another initial value problem (subject to the same assumptions):

$$x_\alpha = f(t,x) + g(t,x), \qquad x(s) = z, \tag{13}$$

and we denote its unique solution by $x(t,s,z)$. Essentially the same technique as in the proof of Theorem 3.1 is used to derive the following nonlinear variation of parameters formula. As before, we also assume that the order of differentiation of

$$h(\tau) = y(t,\tau,x(\tau,s,z)) \tag{14}$$

with respect to τ and t, as well as the order of differentiation of y with respect to γ and α_t, is irrelevant.

THEOREM 3.2. *Suppose that solutions to the initial value problems* (6) *and* (13) *exist and are unique. Define* β *and* γ *by* (7) *and* (10)*, respectively, define* h *by* (14)*, let* $\bar{\alpha}(h(\tau)) = h(\alpha(\tau))$*, and let* $\bar{\gamma}(y(t,\alpha(\tau),z)) = y(t,\alpha(\tau),\gamma(z))$*. Assume also that*

$$(\{h(\tau) : \tau \in \mathbb{T}\}, \bar{\alpha}) \quad \text{and} \quad (\{y(t,\alpha(\tau),z) : z \in \mathbb{R}\}, \bar{\gamma})$$

are generalized time scales. Then formula (9) *holds, and*

$$h_\alpha(\tau) = y_\gamma(t,\alpha(\tau),\beta(x(\tau,s,z)))g(\tau,x(\tau,s,z)). \tag{15}$$

Proof. Note that (15) implies that (9) holds, as

$$h(t) = y(t,t,x(t,s,z)) = x(t,s,z) \quad \text{and} \quad h(s) = y(t,s,x(s,s,z)) = y(t,s,z).$$

We now show that formula (15) holds. Let

$$v(t) = h_\alpha(\tau) \quad \text{and} \quad w(t) = y_\gamma(t,\alpha(\tau),\beta(x(\tau,s,z))).$$

With $F(y) = f(t,y)$ as in the proof of Theorem 3.1, we find by Lemma 2.4, that

$$v_\alpha = (F \circ h)_\alpha = (F_{\bar{\alpha}} \circ h)h_\alpha = (F_{\bar{\alpha}} \circ h)v,$$

where $\bar{\alpha}(h(\tau)) = h(\alpha(\tau))$, and, with $k(z) = y(t,\alpha(\tau),z)$,

$$\begin{aligned} w_\alpha &= (F \circ k)_\gamma(\beta(x(\tau,s,z))) \\ &= [(F_{\bar{\gamma}} \circ k)k_\gamma](\beta(x(\tau,s,z))) \\ &= (F_{\bar{\gamma}} \circ k)(\beta(x(\tau,s,z)))w, \end{aligned}$$

where $\bar{\gamma}(k(z)) = k(\gamma(z))$. Since

$$\begin{aligned} \bar{\gamma}(k(\beta(x(\tau,s,z)))) &= k(\gamma(\beta(x(\tau,s,z)))) \\ &= k(x(\alpha(\tau),s,z)) \\ &= h(\alpha(\tau)) \\ &= \bar{\alpha}(h(\tau)) \end{aligned}$$

(note that (7) and (10) imply that $\gamma(\beta(x(\tau,s,z))) = x(\alpha(\tau),s,z)$), we find that

$$F_{\bar{\alpha}} \circ h = (F_{\bar{\gamma}} \circ k)(\beta(x(\tau,s,z))).$$

As in the proof of Theorem 3.1, $v(\alpha(\tau)) = g(\tau, x(\tau, s, z))$ and $w(\alpha(\tau)) = 1$ imply that

$$v(t) - g(\tau, x(\tau, s, z))w(t) = 0;$$

that is, formula (15) holds. □

REMARK 3.3. Note that h_α could also have been calculated using the chain rule (use a proof as in [**9**, Theorem 1] and Lemma 2.5)

$$h_\alpha(\tau) = y_{\gamma\circ\beta}(t, \alpha(\tau), x(\tau, s, z))x_{\alpha_t}(\tau, s, z) + y_{\alpha_s}(t, \tau, x(\tau, s, z)). \tag{16}$$

Because of Theorem 3.1, and because $x_\alpha(\tau) = f(\tau, x(\tau)) + g(\tau, x(\tau))$, the relation

$$\begin{aligned} &y_{\gamma\circ\beta}(t, \alpha(\tau), x(\tau, s, z))\,[f(\tau, x(\tau, s, z)) + g(\tau, x(\tau, s, z))] \\ &\quad = y_\beta(t, \tau, x(\tau, s, z))f(\tau, x(\tau, s, z)) + y_\gamma(t, \alpha(\tau), \beta(x(\tau, s, z)))g(\tau, x(\tau, s, z)) \end{aligned}$$

can be derived from (15) and (16).

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