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FLOQUET THEORY AND STABILITY OF NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. One of the classical topics in the qualitative theory of differential equations is the Floquet theory. It provides a means to represent solutions and helps in particular for stability analysis. In this paper first we shall study Floquet theory for integro-differential equations (IDE), and then employ it to address stability problems for linear and nonlinear equations.

1. Introduction

The extensive use of mathematical models including integro-differential equations (IDE) is one of the main reasons explaining the fast development of the theory of these equations (see, for example, [2, 7, 24] and the references therein). There are several groups of such models. Integral forms are widely used in the literature on viscoelasticity [21, 27]. Many applications of IDE stability to viscoelasticity can be found in [17, 18, 37].

An important group of processes described by integral PDE is connected with memory effects in phase transitions (see [38] presenting a phenomenological theory of phase transition dynamics with memory). Note that a particular application of the approach proposed here to the Gurtin–Pipkin model consisting of phase transition and energy balance equations was proposed in [11, 12]. The approaches in the study of stability of integro-differential equations are different from classical methods of stability theory for ordinary differential equations, since the classical methods are essentially based on the local character of each term in ordinary differential equations. We can refer, for example, either to methods connected with roots of characteristic equations in the case of autonomous systems or to the Floquet theory in the case of systems with periodic coefficients. However, it is not clear how to

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find analogues of autonomous and periodic objects for the integro-differential equation

(1.1)
$$x'(t) + A(t)x(t) + \int_0^t K(t,s)x(s)\,ds = 0, \quad x:\,[0,\infty) \to \mathbf{R}^n, \quad t \in [0,\infty).$$

with continuous $n \times n$-matrix-valued functions A and K.

With respect to research on the stability of IDE in the case of time-dependent coefficients, note the averaging method developed in [20]. Instead of IDE, a corresponding ordinary differential equation with averaged coefficients was considered, and the stability of this ODE can be interpreted as the stability of IDE. Another approach is a direct Lyapunov method described, for example, in [28] and, more recently, in [16]. In the book [36], Laplace transforms are combined with Lyapunov's technique (see also [14]). In the papers [15, 26], Lyapunov's method is combined with the method of integral estimates. In fact, this method of integral estimates is a particular case of a more general operator approach elaborated by Azbelev et al. (see [1] and the references therein). The idea of this approach can be demonstrated by the equation

$$x'(t) + A(t)x(t) + \int_0^t K(t,s)x(s)\,ds = \xi(t), \quad x:\,[0,\infty) \to \mathbf{R}^n, \quad t \in [0,\infty).$$

For $n = 1$, this is a scalar first-order equation equivalent to the integral equation

$$x(t) = (Tx)(t) + \eta(t), \quad x:\,[0,\infty) \to \mathbf{R}^n, \quad t \in [0,\infty),$$

where the operator $T:\,C \to C$ is defined by

$$(Tx)(t) = -\int_0^t e^{-A(s)(t-s)} \left\{ \int_0^s K(s,\theta)x(\theta)\,d\theta \right\} ds$$

and

$$\eta(t) = \int_0^t e^{-A(s)(t-s)}\xi(s)\,ds + x(0)e^{-pt}.$$

For $\|T\| < 1$, any solution x is bounded for each bounded η, and for a wide class of kernels K (see [1, Theorem 3.5 (p. 95)] the solutions of the homogeneous equation (1.1) satisfy an exponential estimate: There exist positive constants α and β such that $\left|x(t)\right| \leqq \beta e^{-\alpha t}$ for $t \in [0,\infty)$. In this case we also say that the equation is *exponentially stable*. In order to reach the inequality $\|T\| < 1$, one has to require A to be positive and the kernel K to be small.

REMARK 1.1. Notice that the stability conditions obtained using the Lyapunov method and the operator approach coincide (see [15, Theorem 2] and [26, Theorem 3.1]). The essence of such results is as follows: The differential part of IDE (1.1)

$$x'(t) + A(t)x(t) = 0, \quad x : [0, \infty) \to \mathbf{R}^n, \quad t \in [0, \infty), \tag{1.2}$$

possesses some stability, and the integral term should be small enough not to disturb this stability. It should be noted that various results based on the semigroup equality lead to the same restriction. It is due to the following: The semigroup equality is not valid for equations possessing memory (see [1, p. 84]). Consequently, the semigroup approach is applicable only to the ordinary part and requires the smallness of the integral term. Therefore, all these methods cannot be used for stabilization, because usually a given ordinary differential system is unstable, and the aim of stabilization is to obtain a stable system introducing integral terms.

We now formulate some basic Floquet–Lyapunov results for a system of ordinary differential equations with reference to [4, 8, 33, 44]. Consider a linear system of ordinary differential equations

$$x'(t) = F(t)x(t), \quad x : [0, \infty) \to \mathbf{R}^n, \quad t \in [0, \infty), \tag{1.3}$$

where F is an ω-periodic piecewise continuous $n \times n$-matrix-valued function. The elements of the matrix $F(t)$ can be considered as complex-valued functions of t.

THEOREM 1.2 [44]. *The following assertions are true:*

(i) *Each fundamental matrix W of the ω-periodic system* (1.3) *can be represented as*

$$W(t) = \Phi(t)e^{\lambda t}, \quad W : [0, \infty) \to \mathbf{R}^n \times \mathbf{R}^n, \quad t \in [0, \infty), \tag{1.4}$$

where $\Phi(t)$ is an invertible matrix with absolutely continuous ω-periodic elements and λ is a constant $n \times n$-matrix.

(ii) *For system* (1.3) *there exists an invertible absolutely continuous ω-periodic matrix $\Phi(t)$ such that the transformation*

$$x(t) = \Phi(t)u(t), \quad u : [0, \infty) \to \mathbf{R}^n, \quad t \in [0, \infty) \tag{1.5}$$

reduces system (1.3) *to a system of ordinary differential equations with constant coefficients of the form*

$$u'(t) = \lambda u(t), \quad u : [0, \infty) \to \mathbf{R}^n, \quad t \in [0, \infty). \tag{1.6}$$

If there exists a transformation of type (1.5) *reducing system* (1.3) *to system* (1.6)*, then system* (1.3) *is ω-periodic and its fundamental matrix has the form* (1.4)*.*

REMARK 1.3. Theorem 1.2 reduces the stability research to the analysis of eigenvalues of the matrix λ that does not depend on the choice of the fundamental matrix W in (1.4) or on the matrix Φ in (1.5). If $W(0) = I_n$, where I_n is the $n \times n$-identity-matrix, then W is called the *matriciant* of system (1.3). In this case $W(\omega)$ is a monodromy matrix. It is known [44] that in (1.4) we can assume

$$\lambda = \frac{1}{\omega} \operatorname{Ln} W(\omega).$$

Eigenvalues α of the monodromy matrix are connected with the characteristic index γ by the equality $\gamma = e^{\alpha\omega}$. For the stability analysis one has to know either the matrix λ or the monodromy matrix $W(\omega)$. Various approximate methods for calculating and estimating characteristic indices of periodic systems were proposed in [8, 33, 44].

An analogue of the Floquet theory for functional differential equations can be found in [25]. For delay differential equations, this approach was developed in many publications. To name a few, the following recent papers should be mentioned: In [32] the Floquet multipliers were studied and in [42] an analytical approach was developed. It should be stressed that an infinite-dimensional fundamental system does not allow obtaining a full analogue of the Floquet theory (for example, the assertion about passing to systems with constant coefficients [25]).

Let us consider an integro-differential equation (1.1). If the kernel K is a square integrable function, then in a Hilbert space L_2 it can be represented in the form

$$K(t,s) = \sum_{i,j=1}^{\infty} c_{ij} A_i(t) A_j(s), \tag{1.7}$$

where $A_i(t)$, $i \in \mathbf{N}$, is an orthogonal basis in L_2 (see [29, 41]). In this paper we propose an analogue of the Floquet theory for integro-differential equations with kernels of the form

$$K(t,s) = \sum_{i=1}^{q} A_i(t) C_i(t,s) B_i(s), \tag{1.8}$$

where A_i and B_i are $n \times n$-matrices with continuous elements and C_i is the Cauchy matrix of an ordinary differential equation. We prove that, in contrast to the classical approach [25], the fundamental system of IDE with this type of kernels K is finite-dimensional. It is clear that the study of IDE with kernels of the form (1.8) is a natural basis for investigations of the case (1.7), and the first results in this direction are formulated in [10].

The basic idea is to reduce equation (1.1) to a corresponding system of ordinary differential equations of the order $r > n$. Our method is based on the use of the kernel K for the construction of the Cauchy matrix of an auxiliary system of ordinary differential equations. If C_i is the Cauchy matrix of an ordinary differential equation with ω-periodic coefficients, and A_i and B_i $(i = 1, \dots, q)$ are ω-periodic, then the new system can be periodic. In this case, the representation (1.4) of its solutions follows from the classical Floquet theory for ordinary differential systems.

In Section 2 we propose a reduction of integro-differential equations to systems of ordinary differential equations. Various examples, where sufficiently simple integro-differential equations can be reduced by elementary operations to ordinary differential equations, are well known (see, for example, the book [2]).

In Section 3 an analogue of the Floquet theorem and representations of solutions of integro-differential equations are proposed. Various results on exponential stability of integro-differential systems are obtained on this basis.

If C_i in (1.8) is the Cauchy matrix of an equation with constant coefficients (i.e., the elements of the matrix $C_i(t,s)$ are of the form $(t-s)^k e^{-\alpha(t-s)} \cdot \sin \beta(t-s)$, where $\alpha, \beta \in \mathbf{R}$, $k \in \mathbf{N}$) and A_i, B_i, $i = 1, \dots, q$, are constant $n \times n$-matrices, then the new system can be autonomous. In this case, a corresponding integro-differential equation can be called autonomous. In Section 4 stability results for such autonomous integro-differential equations are obtained. New results on the exponential stability obtained by our method are compared with the classical results obtained by the operator and Lyapunov functions methods. Note that our approach does not require stability of the ordinary part (1.2) of IDE (1.1) and can be applied even in case of its instability. When the difference between the corresponding coefficients in the autonomous and nonautonomous integro-differential equations is small enough, an extension of the stability results to nonautonomous equations is obtained.

In Section 5 a relationship between oscillation and asymptotic properties of solutions of integro-differential equations is established. The idea of this relationship for a second-order ordinary differential equation

$$x''(t) + p(t)x(t) = 0, \quad t \in [0, \infty), \quad \text{where} \quad p(t) = p(t+\omega) \geqq c > 0,$$

with ω-periodic coefficient appeared in Lyapunov's investigation of stability. The classical Lyapunov result claims that all solutions of this equation are bounded on a semiaxis if the period ω is less than the distance between two adjacent zeros (see, e.g., [45]). A classical estimate of the distance between two adjacent zeros

$$\int_0^\omega p(t)\,dt \leqq \frac{4}{\omega}$$

implies that all the solutions are bounded. Assertions connecting the distance between zeros and stability/instability of integro-differential equations are obtained in Section 5.

One of the classical problems in stability theory of nonlinear ordinary differential equations is the problem of linear approximation stability. As applied to the integro-differential equation

$$x'(t) = f\left(t, x(t), \int_0^t K_1(t,s)\varphi_1\big(x(s)\big)\,ds, \dots, \int_0^t K_m(t,s)\varphi_m\big(x(s)\big)\,ds\right),$$
$$t \in [0,\infty),$$

it is studied in Section 6. We consider

$$x'(t) = A(t)x(t) + \psi\big(t, x(t)\big), \quad x: [0,\infty) \to \mathbf{R}^n, \quad t \in [0,\infty), \tag{1.9}$$

where A is a continuous $n \times n$-matrix-valued function and $\psi(t,y) = O\big(\|y\|^2\big)$ for $t \in [0,\infty)$. The classical stability theory result for ordinary equations states that if the linear equation

$$x'(t) = A(t)x(t), \quad x: [0,\infty) \to \mathbf{R}^n, \quad t \in [0,\infty)$$

is exponentially stable, then the trivial solution of the nonlinear equation (1.9) is also exponentially stable. The essence of this result consists in the following: The nonlinear term in the vicinity of the trivial solution is smaller than the linear term. Because of this, here the linear term determines the behavior of solutions. However, the integral term in integro-differential equations violates this principle. The following question arises: How can a corresponding linear approximation be written for a nonlinear integro-differential equation? This problem is solved in Section 6, and an analogue of the linear approximation stability theorem is obtained.

2. Reduction of IDE to ODE systems

The aim of this section is to obtain an ordinary differential system such that $x(t)$ will be a part of a solution vector. We consider the equation

$$x'(t) = f\left(t, x(t), \int_0^t K(t,s)\varphi\big(s, x(s)\big)\,ds\right), \quad x: [0,\infty) \to \mathbf{R}^n, \quad t \in [0,\infty), \tag{2.1}$$

where

$$K(t,s) = \Phi(t)e^{\lambda(t-s)}M(s)$$

with an invertible differentiable $n \times n$-matrix $\Phi(t)$, λ is a constant $n \times n$-matrix, Φ and M are ω-periodic $n \times n$-matrices. The kernel $K(t,s)$ can be written in the form

$$K(t,s) = \Phi(t)e^{\lambda(t-s)}\Phi^{-1}(s)K(s,s).$$

Let $z(t) = \Phi(t)e^{\lambda t}$. It is clear that

$$Z(t,s) = \Phi(t)e^{\lambda(t-s)}\Phi^{-1}(s) = z(t)z^{-1}(s)$$

is the Cauchy function of a corresponding ordinary differential system of the form (1.3). To construct an augmented ordinary differential system, we obtain the matrix F in equation (1.3). It is clear that the $n \times n$-matrices z and F satisfy the equality

$$z'(t) = F(t)z(t), \quad t \in [0,\infty).$$

It follows from this equality that

$$F(t) = z'(t)z^{-1}(s) = \left\{ \Phi'(t) + \Phi(t)\lambda \right\} \Phi^{-1}(t). \tag{2.2}$$

Using the equality $(\Phi^{-1})' = -\Phi^{-1}\Phi'\Phi^{-1}$, we can obtain the representation

$$F(t) = -\Phi(t)(\Phi^{-1})'(t) + \Phi(t)\lambda\Phi^{-1}(t). \tag{2.3}$$

We assume

$$y(t) = \int_0^t \Phi(t)e^{\lambda(t-s)}\Phi^{-1}(s)K(s,s)\varphi\big(s,x(s)\big)\,ds.$$

Consider the system

$$x'(t) = f\big(t,x(t),y(t)\big), \quad y'(t) = F(t)y(t) + K(t,t)\varphi\big(t,x(t)\big), \tag{2.4}$$
$$x,y : [0,\infty) \to \mathbf{R}^n,$$

where $t \in [0,\infty)$, the initial condition $y(0) = 0$ is satisfied, and F is determined by either equality (2.2) or (2.3).

From the process of constructing system (2.4), the following relationship between the solution x of the n-dimensional system (2.1) and a solution (x,y) of the $2n$-dimensional ordinary system (2.4) is clear.

THEOREM 2.1. *Let (x,y) be a solution of the $2n$-dimensional ordinary system* (2.4). *Then the component x is a solution of the n-dimensional integro-differential system* (2.1).

3. Floquet theorem for integro-differential equations

We consider

$$x'(t) = A(t)x(t) + B(t)\int_0^t K(t,s)C(s)x(s)\,ds, \tag{3.1}$$

$$x : [0,\infty) \to \mathbf{R}^n, \quad t \in [0,\infty),$$

where

$$K(t,s) = \Phi(t)e^{\lambda(t-s)}M(s)$$

with an invertible differentiable $n \times n$-matrix Φ, λ is a constant $n \times n$-matrix, A, B, C, and M are continuous ω-periodic $n \times n$-matrices, and Φ is an ω-periodic $n \times n$-matrix.

THEOREM 3.1 (Floquet's theorem for integro-differential equations). *There exist an ω-periodic $2n \times 2n$-matrix-valued function G and a constant $2n \times 2n$-matrix Q such that each solution of the n-dimensional system* (3.1) *can be represented in the form*

$$x(t) = LG(t)e^{Qt}w_0, \tag{3.2}$$

where $w_0 = \operatorname{col}(x_0, 0)$, $x_0 \in \mathbf{R}^n$ is a vector of initial conditions of system (3.1)*, $0 \in \mathbf{R}^n$ is the zero vector, $L = (I_n, O)$, where I_n is the $n \times n$-identity matrix, and O is the $n \times n$-zero-matrix.*

REMARK 3.2. In order to describe the matrices G and Q, we consider the system

$$w'(t) = D(t)w(t), \quad w : [0,\infty) \to \mathbf{R}^{2n}, \quad t \in [0,\infty), \tag{3.3}$$

where D is a $2n \times 2n$-matrix-valued function of the form

$$D(t) = \begin{pmatrix} A(t) & B(t) \\ \Phi(t)M(t)C(t) & \big(\Phi'(t) + \Phi(t)\lambda\big)\,\Phi^{-1}(t) \end{pmatrix}. \tag{3.4}$$

We can set

$$G(t) = W(t)e^{-Qt} \quad \text{and} \quad Q = \frac{1}{\omega}\operatorname{Ln} W(\omega), \tag{3.5}$$

where W is a matriciant of system (3.3), $W(0) = I_{2n}$. Then $W(\omega)$ is called a monodromy matrix. In this case each solution of system (3.1) can be written in the form

$$x(t) = LW(t)w_0. \tag{3.6}$$

REMARK 3.3. Theorem 3.1 and Remark 3.2 reduce the problem of stability of IDE (3.1) to the study of stability of the corresponding ordinary differential system (3.3).

PROOF OF THEOREM 3.1. Using the idea of reduction described in Section 2, we set

$$y(t) = \int_0^t \Phi(t)e^{\lambda(t-s)}M(s)x(s)\,ds. \tag{3.7}$$

Equation (3.1) can be reduced to system (3.3), where the $2n \times 2n$-matrix D is determined by equality (3.4) and $w = \operatorname{col}(x, y)$. Each solution x of IDE (3.1) generates a solution w of system (3.3), in which the component y for this x is determined by (3.7). On the contrary, if w is a solution of system (3.3) satisfying the condition $y(0) = 0$, then $x = Lw$ is a solution of equation (3.1).

We apply Theorem 1.2 to system (3.3) denoting by W the matriciant of the $2n$-dimensional system (3.3) and determining G and Q by the formulae (3.5). It is clear that $W(t) = G(t)e^{Qt}$, where $G(t+\omega) = G(t)$, and each solution of system (3.3) is of the form

$$w(t) = G(t)e^{Qt}w_0. \tag{3.8}$$

The equality (3.2) follows from (3.8) and the equality $x = Lw$. □

The following assertion follows from equality (3.6).

THEOREM 3.4. *If all roots of the characteristic equation*

$$\det\big(W(\omega) - \lambda I_{2n}\big) = 0$$

are located within the unit circle of the complex plane, then (3.1) *is exponentially stable.*

REMARK 3.5. It is noteworthy that various approximate methods for calculating and estimating spectra were proposed in [8, 33, 44] when considering periodic systems.

In order to demonstrate the possibilities of our approach for the autonomous case, one can consider the assertions of Section 4. In the nonautonomous case we can formulate several results on the exponential stability for the vector equation

(3.9)
$$x'(t) + A(t)x(t) + B(t)\int_0^t \exp\Big\{-\int_s^t r(\theta)\,d\theta\Big\}C(s)x(s)\,ds = 0, \quad t \in [0,\infty),$$

where A, B, C, and r are $n\times n$-matrix-valued functions with ω-periodic coefficients a_{ij}, b_{ij}, c_{ij}, and r_{ij}, respectively. Equation (3.9) can be reduced to the system

$$\begin{cases} x'(t)+A(t)x(t)+B(t)u(t)=0, \\ u'(t)+r(t)u(t)-C(t)x(t)=0, \end{cases} \qquad t\in[0,\infty). \tag{3.10}$$

Using known results for the ordinary differential system (3.10), we obtain several assertions on the exponential stability of the integro-differential equation (3.9).

THEOREM 3.6. *If there exists* $\varepsilon>0$ *such that*

$$a_{ii}(t)>\sum_{j=1,\ j\neq i}^{n}\left|a_{ij}(t)\right|+\sum_{j=1}^{n}\left|b_{ij}(t)\right|+\varepsilon,$$

$$r_{ii}(t)>\sum_{j=1,\ j\neq i}^{n}\left|r_{ij}(t)\right|+\sum_{j=1}^{n}\left|c_{ij}(t)\right|+\varepsilon$$

at $t\in[0,\omega]$, $i=1,\dots,n$, *then the system* (3.9) *is exponentially stable.*

After the reduction of IDE (3.9) to system (3.10), the proof of Theorem 3.6 follows from the result given in [1, p. 97].

REMARK 3.7. Note that Theorem 3.6 is only applicable in the case of diagonally dominated matrices. It has been pointed out by a referee that Theorem 3.6 can be easily improved to the case of slowly varying matrices by using the so-called freezing method, see e.g., [3, 43].

For the scalar case $n=1$, two convenient coefficient features are suggested.

THEOREM 3.8. *Suppose* $n=1$, $A>0$, $r>0$, *and assume that there exists* $\varepsilon>0$ *such that* $A(t)r(t)>\left|B(t)C(t)\right|+\varepsilon$. *Then equation* (3.9) *is exponentially stable.*

THEOREM 3.9. *Let* $n=1$ *and one of the following two conditions be satisfied:*

(i) $r\geqq 0$, $C>0$, $B>0$, *and there exists* $\varepsilon>0$ *with* $A>\varepsilon+r+B+C$;

(ii) $r>0$, $C<0$, $B<0$, $A\geqq 0$, *and there exists* $\varepsilon>0$ *with* $r>A-C-B+\varepsilon$.

Then equation (3.9) *is exponentially stable.*

After the reduction of IDE (3.9) to system (3.10), the proofs of Theorems 3.8 and 3.9 follow from [9, Theorem 4.2 and Corollary 1 (pp. 1255–1256)], respectively.

We consider the scalar equation

$$x'(t) + A(t)x(t) + B(t) \int_0^t \exp\big(-r(t-s)\big) C(s)x(s)\,ds = 0, \quad t \in [0,\infty). \tag{3.11}$$

This equation can be reduced to a system (3.10) and then, if A and B are absolutely continuous and $B(t)$ is nonzero at $t \in [0,\infty)$, to a second-order ODE

$$x''(t) + a(t)x'(t) + b(t)x(t) = 0, \quad t \in [0,\infty), \tag{3.12}$$

where

$$a(t) = A(t) + r - \frac{B'(t)}{B(t)}, \quad b(t) = A'(t) + B(t)C(t) + rA(t) - \frac{B'(t)}{B(t)}A(t). \tag{3.13}$$

After a standard substitution

$$x(t) = y(t)\exp\left(-\frac{1}{2}\int_0^t a(s)\,ds\right),$$

we obtain the equation

$$y''(t) + p(t)y(t) = 0, \quad t \in [0,\infty),$$

where

$$p(t) = b(t) - \frac{1}{4}a^2(t) - \frac{1}{2}a'(t).$$

THEOREM 3.10. *Suppose at least one of the following four conditions is satisfied:*

(i) *There exists a positive constant c such that*

$$\frac{1}{c\omega}\int_0^\omega \left|p(t) - c^2\right| dt < \int_0^\omega \left(A(t) + r - \frac{B'(t)}{B(t)}\right) dt;$$

(ii) *the inequalities $p(t) \leqq m^2$ and*

$$2|m| < \int_0^\omega \left(A(t) + r - \frac{B'(t)}{B(t)}\right) dt,$$

where $P_a := \frac{1}{\omega}\int_0^\omega p(t)\,dt = -m^2$, *are satisfied;*

(iii) *the inequality* $2k < \int_0^\omega \left(A(t) + r - \frac{B'(t)}{B(t)}\right) dt$ *is satisfied, where*

$$k = \begin{cases} \dfrac{P_m - P_a}{2\sqrt{P_m}} & \text{if} \quad P_m + P_a > 0, \\ \sqrt{-P_a} & \text{if} \quad P_m + P_a \leqq 0, \end{cases}$$

and $P_m = \max_{t\in[0,\omega]} p(t)$*;*

(iv) *if* p *is a positive differentiable function,* t_i^+, $i = 1, \dots, m$, *are the points of maxima and* t_j^-, $j = 1, \dots, m$, *are the points of minima of* p *in the interval* $[0, \omega)$, *then the inequality*

$$\frac{2}{\omega} \ln \frac{p(t_1^+) \dots p(t_m^+)}{p(t_1^-) \dots p(t_m^-)} < \int_0^\omega \left(A(t) + r - \frac{B'(t)}{B(t)}\right) dt$$

is satisfied.

Then the integro-differential equation (3.11) *is exponentially stable.*

The proof of Theorem 3.10 is based on estimates of characteristic indices. After reduction of IDE (3.11) to the second-order equation (3.12) with the coefficients determined by formulae (3.13), our assertion follows from [44, Theorem 1 (p. 125–127)].

4. Use of autonomous integro-differential equations

The conditions of an integro-differential equation to be autonomous are pointed out in Section 1. Now it is clear that equation (3.9) can be called autonomous if system (3.10) is autonomous. Let us demonstrate the possibilities of our approach considering several autonomous integro-differential equations and then extend the results to stability in the nonautonomous case.

LEMMA 4.1. *The condition*

$$p + \gamma > 0 \quad \text{and} \quad p\gamma + b > 0 \tag{4.1}$$

is sufficient for exponential stability of the equation

$$x'(t) + px(t) + \int_0^t be^{-\gamma(t-s)}x(s)\,ds = 0, \quad x : [0, \infty) \to \mathbf{R}, \quad t \in [0, \infty) \tag{4.2}$$

with constant coefficients p, b, *and* γ.

REMARK 4.2. Notice that p may be even negative. Note also that γ may be negative, too, and instead of a small kernel there is a very big one in equation (4.2).

PROOF OF LEMMA 4.1. An analogue of system (3.3) can be written for equation (4.2) as

$$x' + px + bv = 0, \quad v' + \gamma v - x = 0, \quad v(0) = 0, \quad x, v : [0, \infty) \to \mathbf{R}, \quad t \in [0, \infty).$$

Its characteristic equation is $\lambda^2 + (p + \gamma)\lambda + p\gamma + b = 0$. Condition (4.1) is necessary and sufficient for the real parts of its roots to be negative. $\square$

Let us consider the system

$$\begin{cases} x_1'(t) + b \int_0^t e^{-\alpha(t-s)} x_1(s)\, ds + p_{12} x_2(t) = 0, \\ x_2'(t) + p_{21} x_1(t) + p_{22} x_2(t) = 0, \end{cases} \qquad t \in [0, \infty). \tag{4.3}$$

Here p_{ij}, b, and α are constants, and $x_1(t), x_2(t) \in \mathbf{R}$.

LEMMA 4.3. *The condition*

$$\begin{cases} p_{22} + \alpha > 0, \quad p_{22}\alpha + b - p_{12}p_{21} > 0, \quad p_{22}b - p_{12}p_{21}\alpha > 0, \\ p_{22}^2\alpha - p_{12}p_{21}p_{22} + p_{22}\alpha^2 + b\alpha > 0 \end{cases} \tag{4.4}$$

is sufficient for exponential stability of system (4.3).

PROOF. An analogue of system (3.3) can be written as

$$\begin{cases} x_1'(t) + p_{12} x_2(t) + b x_3(t) = 0, \\ x_2'(t) + p_{21} x_1(t) + p_{22} x_2(t) = 0, \\ x_3'(t) - x_1(t) + \alpha x_3(t) = 0, \end{cases} \qquad t \in [0, \infty). \tag{4.5}$$

Condition (4.4) is necessary and sufficient for the real parts of the solutions of the characteristic equation for system (4.5) to be negative [35]. $\square$

COROLLARY 4.4. *If* $p_{22} = 0$*, then the condition*

$$\alpha > 0, \quad b > 0, \quad p_{12}p_{21} < 0$$

is sufficient for exponential stability of system (4.3).

REMARK 4.5. Note that in the case $p_{22} = 0$, system (4.3) is not exponentially stable without the integral terms.

REMARK 4.6. In the following two lemmas, second-order equations are considered. Note that without integral terms these equations are not exponentially stable.

Consider the equation

$$x''(t) + px(t) - \gamma \int_0^t e^{-\beta(t-s)} x(s)\, ds = 0, \quad x : [0,\infty) \to \mathbf{R}, \quad t \in [0,\infty) \tag{4.6}$$

with constant coefficients p, β, and γ.

LEMMA 4.7. *Equation* (4.6) *is exponentially stable if* $\beta > 0$, $p > 0$, $\gamma > 0$, *and* $\beta p > \gamma$.

PROOF. An analogue of system (3.3) for equation (4.6) can be written as

$$x''(t) + px(t) - \gamma v(t) = 0, \quad v'(t) + \beta v(t) - x(t) = 0, \quad t \in [0,\infty).$$

Its characteristic equation is of the form $\lambda^3 + \beta\lambda^2 + p\lambda + \beta p - \gamma = 0$. All roots of this equation have negative real parts if and only if $\beta > 0$, $p > 0$, $\gamma > 0$, and $\beta p > \gamma$ [35]. □

Consider the equation

$$x''(t) + px(t) - \gamma \int_0^t e^{-\beta(t-s)} \sin r(t-s) x(s)\, ds = 0, \tag{4.7}$$
$$x : [0,\infty) \to \mathbf{R}, \quad t \in [0,\infty).$$

LEMMA 4.8. *Equation* (4.7) *is exponentially stable if* $\beta > 0$, $p > 0$, $(\beta^2 + r^2)p > r\gamma > 0$.

PROOF. Let us construct an analogue of system (3.3) for equation (4.7). Its characteristic equation is of the form $\lambda^4 + 2\beta\lambda^3 + (\beta^2 + r^2 + p)\lambda^2 + 2\beta p\lambda + (\beta^2 + r^2)p - r\gamma = 0$. All roots of this equation have negative real parts if and only if $\beta > 0$, $p > 0$, and $(\beta^2 + r^2)p > r\gamma > 0$ [35]. □

Let us examine the equation

$$x'(t) + A^0(t)x(t) + B^0(t) \int_0^t \exp\left(- \int_s^t r^0(\theta)\, d\theta \right) C^0(s)x(s)\, ds = 0, \tag{4.8}$$
$$t \in [0,\infty),$$

where $A^0(t) = \left\{ a^0_{ij}(t) \right\}_{i,j=1}^n$, $B^0(t) = \left\{ b^0_{ij}(t) \right\}_{i,j=1}^n$, $C^0(t) = \left\{ c^0_{ij}(t) \right\}_{i,j=1}^n$, and $r^0(t) = \left\{ r^0_{ij}(t) \right\}_{i,j=1}^n$ are $n \times n$-matrices with sufficiently smooth coefficients.

Denote by $W(t,s)$ the Cauchy matrix of the $2n$-dimensional system

$$\begin{cases} x'(t) + A^0(t)x(t) + B^0(t)u(t) = 0, \\ u'(t) + r^0(t)u(t) - C^0(t)x(t) = 0, \end{cases} \qquad t \in [0,\infty).$$

THEOREM 4.9. *Suppose there exists ε such that*

$$\begin{cases} \left| a_{ij}(t) - a^0_{ij}(t) \right| < \varepsilon, \quad \left| b_{ij}(t) - b^0_{ij}(t) \right| < \varepsilon, \\ \left| c_{ij}(t) - c^0_{ij}(t) \right| < \varepsilon, \quad \left| r_{ij}(t) - r^0_{ij}(t) \right| < \varepsilon \end{cases} \tag{4.9}$$

for $i,j = 1,\dots,n$ and

$$\sup_{t \in [0,\infty)} \left\| \int_0^\infty \left| W(t,s) \right| ds \right\| < \frac{1}{2n\varepsilon}. \tag{4.10}$$

Then system (3.9) *is exponentially stable.*

PROOF. The integro-differential system (3.9) can be reduced to a $2n$-dimensional ordinary differential system (3.10). Now we can use the W-method of Azbelev described in his book [1, pp. 96–99]. Substituting $w(t) = \int_0^t W(t,s)z(s)\,ds$, where $w = \operatorname{col}(x,u)$ and z is an element of the $2n$-dimensional space of measurable bounded functions, into system (3.10), we obtain that the function z satisfies $z = Kz$ with a corresponding compact operator K. Inequality (4.10) guarantees that the norm of the operator K is less than a unity, and this implies [1] the exponential stability of system (3.9). $\square$

COROLLARY 4.10. *Let system* (4.8) *be exponentially stable. If for a small enough $\varepsilon > 0$ the inequalities* (4.9) *hold, then system* (3.9) *is also exponentially stable.*

REMARK 4.11. The equations in each of Lemmas 4.1, 4.3, 4.7, and 4.8 can be taken as a model equation (4.8). Various coefficient features of exponential stability of equations with coefficients close to (4.8) can be obtained according to Theorem 4.9 on the basis of each of the exponentially stable equations.

5. Distance between zeros and stability of first-order IDEs

We examine a scalar IDE

$$x'(t) + A(t)x(t) + B(t)\int_0^t \exp\big(-r(t-s)\big)C(s)x(s)\,ds = 0, \quad t \in [0,\infty) \tag{5.1}$$

with ω-periodic coefficients A, B, and C. It is known [1] that this equation possesses a one-dimensional fundamental system.

If we compare the integro-differential equations (5.1) and (1.1), it is clear that

$$K(t,s) = B(t)e^{-r(t-s)}C(s).$$

Using our approach, equation (5.1) can be reduced to the system

$$x'(t) + A(t)x(t) + B(t)u(t) = 0, \quad u'(t) + ru(t) - C(t)x(t) = 0, \quad t \in [0,\infty),$$

and then, if A and B are absolutely continuous and $B(t)$ is nonzero at $t \in [0,\infty)$, to a second-order equation

$$\begin{aligned} x''(t) &+ \left(A(t) + r - \frac{B'(t)}{B(t)}\right)x'(t) \\ &+ \left(A'(t) + B(t)C(t) + rA(t) - \frac{B'(t)}{B(t)}A(t)\right)x(t) = 0, \quad t \in [0,\infty). \end{aligned} \tag{5.2}$$

For this equation there exist solutions satisfying the condition

$$x(t+\omega) = \lambda x(t).$$

Using Floquet theory for ODE, λ satisfies the equation

$$\lambda^2 - \big(x_1(\omega) + x_2'(\omega)\big)\lambda + W(\omega) = 0, \tag{5.3}$$

where x_1 and x_2 are two solutions of the ODE (5.2) such that $x_1(0) = 1$, $x_1'(0) = 0$, $x_2(0) = 0$, $x_2'(0) = 1$, and

$$W(t) = \det\begin{pmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{pmatrix}$$

is the Wronskian of the fundamental system of (5.2). It is clear that $W(0) = 1$.

If λ_1 is a real root of equation (5.3), a corresponding solution has the representation

$$y(t) = g(t) \exp\left(\frac{\ln|\lambda_1|}{\omega} t\right), \tag{5.4}$$

where g is an ω-periodic function if $\lambda_1 > 0$ and a 2ω-periodic function if $\lambda_1 < 0$. If equation (5.3) has two complex roots $\lambda_1 = |\lambda_1| \exp(i\theta)$ and $\lambda_2 = |\lambda_1| \exp(-i\theta)$, the corresponding solutions are of the forms

$$y_1(t) = \left(g_1(t) \cos\frac{\theta t}{\omega} - g_2(t) \sin\frac{\theta t}{\omega}\right) \exp\left(\frac{\ln|\lambda_1|}{\omega} t\right) \tag{5.5}$$

and

$$y_2(t) = \left(g_2(t) \cos\frac{\theta t}{\omega} + g_1(t) \sin\frac{\theta t}{\omega}\right) \exp\left(\frac{\ln|\lambda_1|}{\omega} t\right), \tag{5.6}$$

where g_1 and g_2 are ω-periodic functions.

REMARK 5.1. Using approximate methods, one can obtain the coefficients $x_1(\omega) + x_2'(\omega)$ and $W(\omega)$ in equation (5.3), and, solving this equation, get λ_1 in the representations of the solutions (5.4), (5.5), and (5.6).

The following assertion connects oscillation properties of the second-order equation (5.2) and asymptotic properties of solutions to the integro-differential equation (5.1).

THEOREM 5.2. *Suppose*

$$\int_0^\omega \left(A(t) + r - \frac{B'(t)}{B(t)}\right) dt > 0 \tag{5.7}$$

and assume that (5.2) *is oscillatory and the distance between zeros of the solutions of equation* (5.2) *is different from* 2ω. *Then equation* (5.1) *is exponentially stable.*

REMARK 5.3. Inequality (5.7) is weaker than the corresponding inequalities in Theorem 3.10.

REMARK 5.4. Condition (5.7), according to the classical formula of Ostrogradskii, guarantees that $W(\omega) < W(0) = 1$. It follows from equation (5.3) that the ODE (5.2) has a solution that exponentially tends to zero for $t \to \infty$. Can we conclude that the first-order integro-differential equation (5.1) possesses a solution tending to zero? The following example demonstrates that the condition concerning the distance between zeros of the solutions of ODE (5.2) is essential.

EXAMPLE 5.5. Let us fix a constant r in equation (5.1) and set

$$A(t)=\frac{2\sin t+\cos t}{1+\sin t\cos t}-r,\ \ C(t)=\frac{\sin^2 t-\sin t\cos t}{1+\sin t\cos t}-A'(t)-rA(t),\ \ B(t)=1.$$

We obtain the equation

(5.8)

$$x'(t)+\left(\frac{2\sin t+\cos t}{1+\sin t\cos t}-r\right)x(t)+\int_0^t \exp\big(-r(t-s)\big)\left\{\frac{\sin^2 s-\sin s\cos s}{1+\sin s\cos s}\right.$$

$$\left.-\left(\frac{2\sin s+\cos s}{1+\sin s\cos s}\right)'-r\frac{2\sin t+\cos t}{1+\sin t\cos t}+r^2\right\}x(s)\,ds=0,\quad t\in[0,\infty).$$

It is clear that 2π is a period of the coefficients. The function $x(t)=e^{-t}\cos t+\sin t$ is a fundamental solution of the integro-differential equation (5.8), which does not tend to zero at $t\to\infty$. Inequality (5.7) has the form

$$\int_0^{2\pi}\frac{2\sin t+\cos t}{1+\sin t\cos t}\,dt>0,$$

and it is satisfied. But there exist solutions of equation (5.2) with the distance between zeros equal to 4π. In fact, the functions $x_1(t)=e^{-t}\cos t$ and $x_2(t)=\sin t$ represent a fundamental system of the second-order equation (5.2).

REMARK 5.6. It is clear from Example 5.5 that it is not correct to require that the distance between zeros of the solution of the integro-differential equation (5.1) is different from 2ω instead of imposing a condition on the solutions of the second-order equation (5.2). Denote by t_i $(t_{i+1}>t_i)$, $i\in\mathbf{N}$, all the zeros of an oscillating solution of equation (5.1). Naturally, it seems that the following assertion is true: If inequality (5.7) holds, and if for each $k\in\mathbf{Z}$ the limit $\lim_{i\to\infty}(t_{i+k}-t_i)$ is different from 2ω, then each solution of equation (5.1) satisfies an exponential estimate. Such an assertion could reduce, in a certain sense, the research of the exponential stability to estimates of the distance between zeros of integro-differential equation (5.1).

THEOREM 5.7. *If the inequality*

$$\int_0^{\omega}\left(A(t)+r-\frac{B'(t)}{B(t)}\right)dt<0 \tag{5.9}$$

holds and if equation (5.2) *is oscillatory and the distance between zeros of solutions of equation* (5.2) *is different from* 2ω, *then the fundamental solutions of equation* (5.1) *are of the form* (5.5), *where* $|\lambda_1|>1$.

THEOREM 5.8. *If the equality*

$$\int_0^\omega \left(A(t) + r - \frac{B'(t)}{B(t)} \right) dt = 0 \tag{5.10}$$

holds, equation (5.2) *is oscillatory, and the distance between zeros of solutions of equation* (5.2) *is different from* 2ω, *then the fundamental solutions of equation* (5.1) *are bounded.*

PROOFS OF THEOREMS 5.2, 5.7, AND 5.8. It follows from the classical formula of Ostrogradskii that condition (5.7) implies the inequality $W(\omega) < 1$, condition (5.9) implies the inequality $W(\omega) > 1$, and condition (5.10) implies that $W(\omega) = 1$. The condition that the distance between zeros of solutions of equation (5.2) is different from 2ω excludes the existence of real roots of equation (5.3). In this case the inequality $W(\omega) < 1$ implies that $|\lambda_1| < 1$, the equality $W(\omega) = 1$ implies that $|\lambda_1| = 1$, and the inequality $W(\omega) > 1$ implies that $|\lambda_1| > 1$. Now the representation of solutions (5.5) completes the proofs. □

REMARK 5.9. Estimates of the distance between two adjacent zeros (nonoscillation interval) represent one of the classical topics in the theory of ordinary differential equations. Various connections between nonoscillation and many other important properties of n-th order ordinary differential equations were described in [31].

Using known estimates of nonoscillation intervals and Theorems 5.2, 5.7, and 5.8, we can suggest assertions about stability of integro-differential equations.

Consider equation (5.1) in the case of $B(t) \equiv B$ and $A(t) \equiv -r$:

$$x'(t) - rx(t) + \int_0^t \exp\big(-r(t-s)\big) C(s)x(s)\,ds = 0, \quad t \in [0,\infty). \tag{5.11}$$

Equation (5.2) can be written as

$$x''(t) + \big[C(t) - r^2\big]\,x(t) = 0. \tag{5.12}$$

The conditions of the following theorem imply oscillation of solutions and exclude the existence of real roots of equation (5.3).

THEOREM 5.10. *If* $C(t) \geqq r^2$, $0 < \int_0^\omega \big[C(t) - r^2\big]\,dt$, *and the distance between zeros of solutions of equation* (5.12) *is different from* 2ω, *then the fundamental solutions of equation* (5.11) *are bounded.*

Using the classical Lyapunov estimate of nonoscillation intervals from below we obtain the following assertion.

THEOREM 5.11. *If* $C(t) \geqq r^2$ *and*

$$0 < \int_0^\omega \left[C(t) - r^2 \right] dt \leqq \frac{4}{\omega},$$

then the fundamental solutions of equation (5.11) *are bounded.*

We introduce the notation

$$P = \min_{t \in [0,\omega]} \left[C(t) - r^2 \right] \quad \text{and} \quad R = \max_{t \in [0,\omega]} \left[C(t) - r^2 \right].$$

Based on the lower and upper estimates of the distance between zeros [31], the following assertion is obtained.

THEOREM 5.12. *Suppose* $C(t) \geqq r^2$, $0 < \int_0^\omega \left[C(t) - r^2 \right] dt$, *and for* $k \in \mathbf{N}$,

$$\frac{k-1}{k} < \frac{\sqrt{P}}{\sqrt{R}}.$$

Moreover, assume

$$\omega \in \left(0, \frac{\pi}{2\sqrt{R}} \right] \cup \left(\frac{1}{2} \frac{\pi}{\sqrt{P}}, \frac{\pi}{\sqrt{R}} \right) \cup \cdots \cup \left(\frac{k-1}{2} \frac{\pi}{\sqrt{P}}, \frac{\pi}{2\sqrt{R}} k \right).$$

Then the fundamental solutions of equation (5.11) *are bounded.*

6. Stability of nonlinear IDE

We examine the system

$$x'(t) = f\left(t, x(t), \int_0^t K(t,s) \varphi\big(s, x(s)\big)\, ds \right), \quad t \in [0, \infty). \tag{6.1}$$

Assume that the following conditions are satisfied:

(a) The vector-valued functions $f(t,x,y)$ and $\varphi(t,x)$ are continuous with respect to all variables and satisfy a Lipschitz condition in x and in y;

(b) the kernel $K(t,s)$ has the representation

$$K(t,s) = \Phi(t) e^{\lambda(t-s)} M(s),$$

where $\Phi(t)$ is an invertible differentiable $n \times n$-matrix, λ is a constant $n \times n$-matrix, and M is a continuous $n \times n$-matrix.

Similar systems of integro-differential equations were considered, for example, in [2, 6, 7, 20, 28, 34].

The method described in Section 2 reduces equation (6.1) to an ordinary differential system (2.4) (see Theorem 2.1). It is clear that in this case for system (6.1) the conditions of the theorem on existence and uniqueness of the solution of the Cauchy problem are satisfied.

Equation (6.1) can be reduced to system (2.4). Let us write this system in the form

$$\begin{cases} x'(t) = A(t)x(t) + p\big(t, x(t), y(t)\big) + B(t)y(t), \\ y'(t) = E(t)x(t) + h\big(t, x(t)\big) + F(t)y(t), \end{cases} \qquad t \in [0, \infty), \tag{6.2}$$

where F is determined by formula (2.3).

We introduce the following conditions:

(c) $p(t, 0, 0) = h(t, 0) = 0$, $\|p(t, x, y)\| = O\big(\|(x, y)\|^2\big)$, $\|h(t, x)\| = O\big(\|x\|^2\big)$;

(d) f (for each x and y), φ (for each y), Φ, M, A, B, and E are ω-periodic in t.

Let us formulate an analogue of the classical linear approximation theorem.

THEOREM 6.1. *Let the conditions* (a)–(d) *be satisfied. If the* $2n$*-dimensional ordinary system*

$$\begin{cases} x'(t) = A(t)x(t) + B(t)y(t), \\ y'(t) = E(t)x(t) + F(t)y(t), \end{cases} \qquad t \in [0, \infty) \tag{6.3}$$

is exponentially stable, then the trivial solution of the nonlinear system (6.1) *is also exponentially stable.*

PROOF. The proof follows from Theorem 2.1. □

Assertions about exponential stability of the linear IDE were obtained in Sections 2 and 3 by reducing the IDE to a corresponding $2n$-dimensional system of ordinary differential equations. The exponential stability of the linear IDE follows from the exponential stability of this $2n$-dimensional system. Using Theorem 6.1, one can obtain results on exponential stability of the trivial solution.

THEOREM 6.2. *Let the conditions* (a)–(d) *be satisfied. If for system* (6.3) *the conditions of at least one of Theorems* 3.4, 3.6, 3.8, 3.9, 3.10, 4.9, *Lemmas* 4.1, 4.3, 4.7, 4.8 *are satisfied, then the trivial solution of the nonlinear system* (6.1) *is also exponentially stable.*

Let us consider the following particular case of equation (6.1), where the integral term is linearly included:

(6.4)

$$x'(t) + A(t)x(t) + g\big(t, x(t)\big) + B(t)\int_0^t K(t,s)\Big(C(s)x(s) + h\big(s, x(s)\big)\Big)\,ds = 0,$$

$t \in [0, \infty)$. For this equation a stronger result connecting x-stability of system (6.3) with stability of IDE (6.2) can be obtained.

Definition 6.3 [5, 40]. The trivial solution $w := \mathrm{col}\,(x, y) = \mathrm{col}\,(0, 0)$ is called *asymptotically stable* if there exists $\varepsilon > 0$ such that $\big\|x(t)\big\| \to 0$ as $t \to \infty$ for each solution w with initial condition satisfying the inequality $\big\|w(0)\big\| < \varepsilon$.

Definition 6.4 [5, 40]. The trivial solution $w := \mathrm{col}\,(x, y) = \mathrm{col}\,(0, 0)$ is called *exponentially x-stable* if there exist positive constants N, ε, and θ such that

$$\big\|x(t)\big\| \leqq N\big\|w(0)\big\|e^{-\theta t}, \quad t \in [0, \infty)$$

for each solution w with initial condition satisfying $\big\|w(0)\big\| < \varepsilon$.

Let us assume that for the nonlinearities $g(t, x)$ and $h(t, x)$ the following condition is satisfied:

(e) $g(t, 0) = h(t, 0) = 0$, $\big\|g(t, x)\big\| = \big\|h(t, x)\big\| = O\big(\|x\|^2\big)$, g and h are continuous.

In this case the following is clear:

(f) A, B, and C are continuous $n \times n$-matrix-valued functions.

Assume also that the following conditions of periodicity are satisfied:

(g) f (for each x and y), φ (for each y), Φ, M, A, B, and C are ω-periodic in t.

Theorem 6.5. *Let the conditions* (b) *and* (e)–(g) *be satisfied. If the* $2n$*-dimensional ordinary system* (3.3) *is asymptotically x-stable, then the trivial solution of the nonlinear system* (6.4) *is exponentially stable for sufficiently small perturbations of the initial condition* $x(0)$*.*

Proof. From the asymptotic x-stability and Lyapunov's reducibility of system (3.3) given by Theorem 1.2, it follows that system (3.3) is exponentially x-stable. According to Corduneanu's theorem [5, 40] for system (3.3), there exists a function $V(t, w)$ such that

$$\|x\| \leqq V(t, w) \leqq N\|w\|, \quad V_t'(t, w) \leqq -\theta V(t, w)$$

and

$$\big|V(t, w_1) - V(t, w_2)\big| \leqq N\|w_1 - w_2\|, \quad t \in [0, \infty).$$

Reducing system (6.4), we obtain the $2n$-dimensional system

$$\begin{cases} x'(t) + A(t)x(t) + B(t)y(t) + g\big(t, x(t)\big)\,, \\ y'(t) + K(t,t)C(t)x(t) + F(t)y(t) + h\big(t, x(t)\big)\,, \end{cases} \qquad t \in [0, \infty). \tag{6.5}$$

Let us construct a scalar equation for system (6.5):

$$u' = -\theta u + Nr(u), \tag{6.6}$$

where

$$r\big(\|x\|\big) = O\big(\|x\|^2\big) \geqq \big\| g(t,x)\big\| + \big\| h(t,x)\big\| .$$

It is clear that the trivial solution $u = 0$ of the scalar equation (6.6) is exponentially stable. According to Corduneanu's assertion (see [40, Theorem 35.1]), the trivial solution of system (6.5) is exponentially x-stable. Now the exponential stability of the trivial solution of IDE (6.4) follows from Lemma 6.6 below. □

LEMMA 6.6. *If the trivial solution of system* (6.5) $w := \mathrm{col}\,(x, y) = \mathrm{col}\,(0, 0)$ *is asymptotically x-stable (or exponentially x-stable, or else x-unstable), then the trivial solution $x = 0$ of the integro-differential equation* (6.4) *is asymptotically stable (or, respectively, exponentially stable, or else unstable) for sufficiently small perturbations of the initial condition $x(0)$.*

PROOF. The proof follows from Theorem 2.1. □

EXAMPLE 6.7. Let us consider the scalar equation

$$x'(t) = -\mu(1 - 2\sin t)x(t) + \sum_{i=2}^{n} p_i(t)x^i(t) \tag{6.7}$$

$$+ \mu^2 \int_0^t \exp\big(-(t-s)\big)\left\{x(s) + \sum_{j=2}^{m} q_j(s)x^j(s)\right\} ds, \quad \mu > 0, \quad t \in [0, \infty),$$

where p_i, $i \in \{2, \ldots, n\}$ and q_j, $j \in \{2, \ldots, m\}$ are continuous bounded functions.

The linear approximation system can be written as

$$x'(t) = \mu(-1 + 2\sin t)x(t) + \mu v(t), \quad v'(t) = \mu x(t) - v(t), \quad t \in [0, \infty).$$

Using an estimate of the monodromy matrix spectrum (see [44, pp. 124–125]), we conclude that this system is exponentially stable and, consequently, equation (6.7) is exponentially stable if $\mu < 2/3$.

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