Half-Linear Dynamic Equations

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Abstract. We survey half-linear dynamic equations on time scales. These contain the well-known half-linear differential and half-linear difference equations as special cases, but also other kinds of half-linear equations. Special cases of half-linear equations are the well-studied linear equations of second order. We discuss existence and uniqueness of solutions of corresponding initial value problems and, using a Picone identity, derive a Reid roundabout theorem that gives conditions equivalent to disconjugacy of half-linear dynamic equations, among them solvability of an associated Riccati equation and positive definiteness of an associated functional. We also develop a corresponding Sturmian theory and discuss methods of oscillation theory, which we use to present oscillation as well as nonoscillation criteria for half-linear dynamic equations.

Keywords. Dynamic equations, time scales, half-linear equations, Sturmian theory, oscillation, Reid roundabout theorem, Picone identity.

1 Introduction

In this contribution we would like to present some aspects of the qualitative theory of the second-order half-linear dynamic equation

$$[r(t)\Phi(y^\Delta)]^\Delta + p(t)\Phi(y^\sigma) = 0 \qquad (\mathrm{HL}^\Delta\mathrm{E})$$

on an arbitrary time scale $\mathbb{T}$, where r and p are real rd-continuous functions on $\mathbb{T}$ with $r(t) \neq 0$, and $\Phi(x) = |x|^{\alpha-1}\operatorname{sgn} x$ with $\alpha > 1$. This survey can be understood as summarizing all known results which have been done for (HL$^\Delta$E) so far. In particular, we show that the IVP involving (HL$^\Delta$E) is (globally) uniquely solvable and the basic results known from the oscillation theory of the Sturm–Liouville differential equation

$$(r(t)y')' + p(t)y = 0, \qquad (\mathrm{LDE})$$

$r(t) > 0$, can be extended to (HL$^\Delta$E).

The terminology *half-linear* is justified by the fact that the space of all solutions of (HL$^\Delta$E) is homogeneous, but not generally additive. Thus, it has just "half of the properties" of a linear space. However, surprisingly, some sophisticated methods known from the theory of linear equations may be used to investigate half-linear equations. Equation (HL$^\Delta$E) covers the half-linear differential equation (when $\mathbb{T} = \mathbb{R}$)

$$[r(t)\Phi(y')]' + p(t)\Phi(y) = 0 \tag{HLDE}$$

as well as the half-linear difference equation (when $\mathbb{T} = \mathbb{Z}$)

$$\Delta[r_k\Phi(\Delta y_k)] + p_k\Phi(y_{k+1}) = 0. \tag{HLΔE}$$

Furthermore, (LDE) is a special case of (HLDE) (when $\alpha = 2$), and if $\Phi = \mathrm{id}$ (i.e., $\alpha = 2$), then (HLΔE) reduces to the Sturm–Liouville difference equation

$$\Delta(r_k\Delta y_k) + p_k y_{k+1} = 0. \tag{LΔE}$$

Finally, the linear dynamic equation

$$(r(t)y^\Delta)^\Delta + p(t)y^\sigma = 0, \tag{L$^\Delta$E}$$

which covers (LDE) and (LΔE) when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, respectively, is a special case of (HL$^\Delta$E) (when $\alpha = 2$).

The most important oscillatory properties are described by the so-called Reid roundabout theorem, for the first time proved for (LDE), see e.g., [37] (for the discrete case, see [5]). This theorem shows the connection among such concepts like disconjugacy of (LDE) on $[a, b]$, positive definiteness of the quadratic functional

$$\int_a^b \Big[r(t)\xi'^2(t) - p(t)\xi^2(t)\Big]\, dt$$

and solvability of the Riccati equation $w' + p(t) + r^{-1}(t)w^2 = 0$ on $[a, b]$. This theorem is of basic importance in oscillation theory; as its consequence we easily get the Sturm separation theorem and the Sturm comparison theorem. Thanks to the separation theorem, (LDE) can be classified as oscillatory or nonoscillatory. In the first case, one, and hereby every, solution vanishes at an infinite number of isolated points (the so-called zeros of a solution) in $[a, \infty)$; in the second case each solution has only a finite number of zeros in $[a, \infty)$. Reid's theorem also provides two powerful tools for the investigation of oscillatory properties, namely the Riccati technique and the variational principle. For more information concerning (LDE) see e.g., [22, 36, 46].

Later it was shown that Reid's theorem and many other results may be extended to (LΔE), where the concept of a zero of a solution is replaced by the concept of a generalized zero of a solution, see e.g., [1, 4, 23, 29]. The theories for these two equations can be unified and extended to (L$^\Delta$E), see e.g., [2, 8, 18]. Mirzov [34] in 1976 and Elbert [16] in 1979 showed independently that the Sturmian theory works for more general, namely half-linear equations (HLDE). For more information concerning (HLDE) see e.g., [13, 14, 17, 25, 30] and the references given therein. Finally, very recently, in [39, 40, 41, 42, 44] the half-linear discrete version of Reid's theorem was established and many other results for (HLΔE) were presented.

The above mentioned facts suggest an idea to develop an oscillation theory of the half-linear dynamic equation (HL$^\Delta$E) which unifies and extends the theories of all other mentioned equations, and also explains some discrepancies between them. (Note that some of the results for this equation were already presented in [43, 45].) This approach brings some interesting problems, especially of the following two types:

1) The first type is related to the aspect of unification. The problems arise in particular from the fact that not every "continuous" result has a "direct discrete" counterpart and vice versa. So, if we unify continuous and discrete theory, we come to the problem like, for example, the absence of a natural extension of the chain rule for computing the derivative of a composite function, and many others.

2) The second type of problems is related to the extension to the half-linear case. There exist certain limitations in the use of the linear approach to the investigation of half-linear equations (on the other hand, we can still use very sophisticated methods). These limitations are first of all the absence of a transformation theory similar to that for linear equations or the impossibility of the extension of the Wronskian identity to the half-linear case, which are very useful tools in the oscillation theory.

These problems complicate, or quite change the techniques that we use.

This survey is organized as follows. In the next section we give some basic information concerning the time scales calculus and also recall Schauder's fixed point theorem which is needed later. In Section 3, we discuss global existence and uniqueness of solutions of initial value problems involving half-linear dynamic equations (HL$^\Delta$E). The main tool in our proof of Reid's roundabout theorem in Section 5 is Picone's identity, which is presented in Section 4. The roundabout theorem gives statements that are all equivalent to disconjugacy of our equation (HL$^\Delta$E), among those positive definiteness of a related α-degree functional and solvability of a related Riccati equation. Mainly using Reid's roundabout theorem, we develop in Section 6 a Sturmian theory for half-linear dynamic equations, containing a Sturm comparison theorem and a Sturm separation theorem. The remainder of this survey is devoted to methods of oscillation theory (Section 7), including the variational principle and a thorough discussion of the Riccati technique, oscillation criteria (Section 8) and nonoscillation criteria (Section 9), and comparison results (Section 10). We conclude with a final section offering some examples that illustrate our theory.

2 Preliminaries about Time Scales

We start by introducing the following concepts related to the notion of time scales. A *time scale* is a special case of a *measure chain*. The calculus of measure chains was introduced by S. Hilger in his Ph.D. dissertation in 1988, see [24], in order to unify continuous and discrete analysis. A time scale $\mathbb{T}$ is defined as a nonempty closed subset of $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. Define the *forward jump operator* $\sigma(t)$ for $t \in \mathbb{T}$ by

$$\sigma(t) := \inf\{\tau > t : \tau \in \mathbb{T}\},$$

and the *backward jump* operator $\rho(t)$ at t, for $t \in \mathbb{T}$, by

$$\rho(t) := \sup\{\tau < t : \tau \in \mathbb{T}\}.$$

If $\sigma(t) > t$, we say t is *right-scattered*, while if $\rho(t) < t$, we say t is *left-scattered*. If $\sigma(t) = t$, we say t is *right-dense*, while if $\rho(t) = t$, we say t is *left-dense*. We shall also use the notation $\mu(t) := \sigma(t) - t$ which is called the *graininess function*. If $\mathbb{T}$ has a left-scattered maximum $t_{\max}$, then we define $\mathbb{T}^\kappa = \mathbb{T} \setminus \{t_{\max}\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

Let $\mathcal{X}$ be a real Banach space. The function $f : \mathbb{T} \to \mathcal{X}$ is called *(delta) differentiable at* $t \in \mathbb{T}^\kappa$ with *(delta) derivative* $f^\Delta(t) \in \mathcal{X}$, if for any $\varepsilon > 0$ there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\big|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\big| \le \varepsilon|\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

The function f is *differentiable on* $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. We will use the notation $f^\sigma(t) = f(\sigma(t))$ for $t \in \mathbb{T}^\kappa$, i.e., $f^\sigma = f \circ \sigma$. The following lemma shows some important properties of f^Δ.

Lemma 2.1 ([8, 24]). *Let $f, g : \mathbb{T} \to \mathcal{X}$ be two functions, and let $t \in \mathbb{T}^\kappa$. Then we have*

(i) *If* $f^\Delta(t)$ *and* $g^\Delta(t)$ *exist, then* $Af+Bg$ *is differentiable at* t *with* $(Af+Bg)^\Delta(t) = Af^\Delta(t) + Bg^\Delta(t)$ *for any constants* A, B.

(ii) *If* $f^\Delta(t)$ *exists, then* f *is continuous at* t.

(iii) *If* t *is right-scattered and* f *is continuous at* t*, then* $f^\Delta(t)$ *exists and*

$$f^\Delta(t) = \frac{f^\sigma(t) - f(t)}{\mu(t)}.$$

(iv) *If* $f^\Delta(t)$ *exists, then* $f^\sigma(t) = f(t) + \mu(t)f^\Delta(t)$.

(v) *If* $f^\Delta(t)$ *and* $g^\Delta(t)$ *exist, then* fg *is differentiable at* t *with*

$$(fg)^\Delta(t) = f^\sigma(t)g^\Delta(t) + f^\Delta(t)g(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t).$$

(vi) *Let* $f, g : \mathbb{T} \to \mathbb{R}$ *be such that* $g(t)g^\sigma(t) \neq 0$ *and* $f^\Delta(t), g^\Delta(t)$ *exist. Then* f/g *is differentiable at* t *with*

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g^\sigma(t)}.$$

(vii) *Assume that* $g : \mathbb{R} \to \mathbb{R}$ *is continuous,* $g : \mathbb{T} \to \mathbb{R}$ *is differentiable on* $\mathbb{T}^\kappa$*, and* $f : \mathbb{R} \to \mathbb{R}$ *is continuously differentiable (in* $\mathbb{R}$*). Then* $(f \circ g)^\Delta(t) = f'(g(\xi))g^\Delta(t)$*, where* $\xi \in [t, \sigma(t)]$.

A function $f : \mathbb{T} \to \mathcal{X}$ is called *rd-continuous* provided it is continuous at each right-dense point and has a left-sided limit at each point, which is at the same time right-scattered and left-dense, we write $f \in C_{\mathrm{rd}}(\mathbb{T}) = C_{\mathrm{rd}}(\mathbb{T}, \mathcal{X})$. If f is differentiable on $\mathbb{T}^\kappa$ with $f^\Delta \in C_{\mathrm{rd}}(\mathbb{T}^\kappa)$, we write $f \in C^1_{\mathrm{rd}}(\mathbb{T})$. If f is piecewise rd-continuously differentiable, we abbreviate this by writing $f \in C^1_{\mathrm{p}}(\mathbb{T})$. One can show, see [24], that if $f : \mathbb{T} \to \mathcal{X}$ is an rd-continuous function, then there exists a unique function (*antiderivative*) $F : \mathbb{T} \to \mathcal{X}$ with the properties $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$ and $F(\tau) = \eta$, where $\tau \in \mathbb{T}$ and $\eta \in \mathcal{X}$. Then we define the *Cauchy integral* of f by $\int_a^b f(t)\,\Delta t = F(b) - F(a)$, where $a, b \in \mathbb{T}$. In the following lemma we present some properties of the integral that we will need later. A so-called *right-sequence for* t is a strictly decreasing sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq \mathbb{T}$ with $\lim_{n\to\infty} t_n = t$. Note that due to the definition of σ there always exist right-sequences for any right-dense $t \in \mathbb{T}$ (except if $t = \max \mathbb{T}$ in case $\mathbb{T}$ is bounded above).

Lemma 2.2 ([8, 24]). *Let* $f, g \in C_{\mathrm{rd}}(\mathbb{T})$ *and* $a, b, c \in \mathbb{T}$*. Then*

(i) $\int_a^b [Af(t)+Bg(t)]\,\Delta t = A\int_a^b f(t)\,\Delta t + B\int_a^b g(t)\,\Delta t$, *where* A, B *are any constants.*

(ii) $\int_t^{\sigma(t)} f(s)\,\Delta s = \mu(t)f(t)$ *for* $t \in \mathbb{T}^\kappa$.

(iii) $\int_a^b f(t)\,\Delta t = \int_a^c f(t)\,\Delta t + \int_c^b f(t)\,\Delta t$.

(iv) $\int_a^b f(t)g^\Delta(t)\,\Delta t = [f(t)g(t)]_a^b - \int_a^b f^\Delta(t)g^\sigma(t)\,\Delta t$ *(integration by parts).*

(v) *If* a *is right-dense and* $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{T}$ *is a right-sequence for* a, *then*

$$\lim_{n\to\infty}\int_{a_n}^b f(t)\,\Delta t = \int_a^b f(t)\,\Delta t.$$

Let f be an rd-continuous function on $\mathbb{T}$ satisfying $1+\mu(t)f(t) \neq 0$ for all $t \in \mathbb{T}$ (such a function f is then called *regressive*). The *generalized exponential function* e_f is defined as the unique solution $x(t) = e_f(t,a)$ of the initial value problem $x^\Delta = f(t)x,\ x(a) = 1$. In fact, there is an explicit formula for $e_f(t,a)$, using a certain cylinder transformation. We refer to [8] for additional details concerning the calculus on time scales.

The notations $[a,b], [a,b), [a,\infty)$, and so on, will denote time scales intervals, i.e., for example, $[a,b) = \{t \in \mathbb{T} : a \leq t < b\}$, where $a, b \in \mathbb{T}$. Moreover, $\mathcal{I} := [a,b]$ with $a < \rho(b)$ and $\mathcal{I}_a := [a,\infty)$ in case $\sup\mathbb{T} = \infty$.

To prove our results we will need the following auxiliary statements.

Proposition 2.1 (Gronwall Inequality [8, Theorem 6.4]). *Assume that* $f, g \in C_{\rm rd}(\mathcal{I},\mathbb{R})$ *with* $f(t) \geq 0, g(t) \geq 0$ *on* $\mathcal{I}$ *and let* K, L *be nonnegative real numbers such that*

$$f(t) \leq K + L\int_a^t f(s)g(s)\,\Delta s$$

for $t \in \mathcal{I}$. *Then*

$$f(t) \leq Ke_{Lg}(t,a)$$

holds for $t \in \mathcal{I}$.

The following local existence result corresponds to the classical Peano theorem.

Proposition 2.2 (Peano Theorem [27, Theorem 2.1.1]). *Let* $F : \mathcal{I}^\kappa \times \mathbb{R}^n \to \mathbb{R}^n$ *be rd-continuous in* $\mathcal{I}^\kappa \times \{x \in \mathbb{R}^n : |x - x_0| \leq \delta\}$, *where* $\delta > 0$. *Then the initial value problem*

$$x^\Delta = F(t,x), \quad x(a) = x_0$$

has at least one solution x *in some right neighborhood of* a.

The following theorem is also a generalization of a standard result from the theory of differential and difference equations.

Proposition 2.3 (Existence/Uniqueness [24, Theorem 5.7]). *Let $t_0 \in \mathcal{I}^\kappa$. Suppose that the right-hand side $F : \mathcal{I}^\kappa \times \mathcal{X} \to \mathcal{X}$ of the initial value problem*

$$x^\Delta = F(t, x), \quad x(t_0) = x_0 \tag{2.1}$$

satisfies the following conditions:

(i) *F is rd-continuous with respect to the first argument in $\mathcal{I}^\kappa$.*

(ii) *For each $t \in \mathcal{I}^\kappa$ there exists a compact neighborhood U_t such that F in U_{t^κ} satisfies a Lipschitz condition with respect to the second argument, i.e.,*

$$|F(s, x) - F(s, y)| \le L_t |x - y|$$

for $(s, x), (s, y) \in U_{t^\kappa} \times \mathcal{X}$, L_t being some constant.

(iii) *At each $t < t_0$, F is regressive, i.e., the function $\mathrm{id} + \mu(t) F(t, \cdot) : \mathcal{X} \to \mathcal{X}$ is invertible.*

Then the IVP (2.1) *has a unique solution on $\mathcal{I}$.*

The following result is referred to as Young's inequality.

Lemma 2.3 ([21]). *If $\alpha > 1$ and $\beta > 1$ are mutually conjugate numbers, i.e., $1/\alpha + 1/\beta = 1$, then*

$$|uv| \le \frac{|u|^\alpha}{\alpha} + \frac{|v|^\beta}{\beta} \tag{2.2}$$

holds for any $u, v \in \mathbb{R}$.

Now we prove a generalization of the well-known Hölder inequality that we will need in the proof of Lemma 3.1. This generalization was first proved in [7, Lemma 2.2 (iv)], and it can be found also in [8, Theorem 6.13].

Lemma 2.4 (Hölder Inequality). *Let $f, g \in C_{\mathrm{rd}}(\mathcal{I}, \mathcal{X})$ and β be the conjugate number of α. Then*

$$\int_a^b |f(t) g(t)| \, \Delta t \le \left(\int_a^b |f(t)|^\alpha \, \Delta t \right)^{\frac{1}{\alpha}} \left(\int_a^b |g(t)|^\beta \, \Delta t \right)^{\frac{1}{\beta}}. \tag{2.3}$$

Proof. Inequality (2.3) is homogeneous. This means that if (2.3) holds for some f, g, then it is fulfilled also for $\lambda f, \nu g$, where λ, ν are arbitrary real numbers. Hence it is sufficient to verify (2.3) only in the case where

$$\int_a^b |f(t)|^\alpha \, \Delta t = \int_a^b |g(t)|^\beta \, \Delta t = 1. \tag{2.4}$$

Thus, let (2.4) hold. We prove that

$$\int_a^b |f(t)g(t)| \, \Delta t \le 1.$$

From the Young inequality (2.2) we have

$$\begin{aligned} \int_a^b |f(t)g(t)| \, \Delta t &\le \frac{1}{\alpha} \int_a^b |f(t)|^\alpha \, \Delta t + \frac{1}{\beta} \int_a^b |g(t)|^\beta \, \Delta t \\ &= \frac{1}{\alpha} + \frac{1}{\beta} = 1, \end{aligned}$$

and the lemma is proved. □

The following statement can be viewed as a time scales version of the second mean value theorem of integral calculus. We will use it in the proof of our oscillation criteria.

Lemma 2.5. *Let $f \in C^1_{\mathrm{rd}}(\mathcal{I}, \mathbb{R})$ be monotonic on $\mathcal{I}$, i.e., $f^\Delta(t)$ does not change its sign for $t \in \mathcal{I}^\kappa$. Then for any function $g \in C_{\mathrm{rd}}(\mathcal{I}, \mathbb{R})$ there exist $c, d \in \mathcal{I}^\kappa$ such that*

$$\int_a^b f^\sigma(t) g(t) \, \Delta t \le f(a) \int_a^c g(t) \, \Delta t + f(b) \int_c^b g(t) \, \Delta t$$

and

$$\int_a^b f^\sigma(t) g(t) \, \Delta t \ge f(a) \int_a^d g(t) \, \Delta t + f(b) \int_d^b g(t) \, \Delta t.$$

Proof. Suppose that f is nondecreasing, i.e., $f^\Delta(t) \ge 0$ for all $t \in \mathcal{I}^\kappa$ (if $f^\Delta(t) \le 0$, we proceed in the same way). Let $c, d \in \mathcal{I}^\kappa$ be such that

$$\int_a^c g(s) \, \Delta s \le \int_a^t g(s) \, \Delta s \le \int_a^d g(s) \, \Delta s$$

for $t \in \mathcal{I}^\kappa$. Using integration by parts (Lemma 2.2 (iv)), we have

$$\int_a^b f^\Delta(t) \left(\int_a^t g(s) \, \Delta s \right) \Delta t = \left[f(t) \int_a^t g(s) \, \Delta s \right]_a^b - \int_a^b f^\sigma(t) g(t) \, \Delta t$$

$$= \quad f(b)\int_a^b g(t)\,\Delta t - \int_a^b f^\sigma(t)g(t)\,\Delta t,$$

and hence

$$\begin{aligned} f(b)\int_a^c g(t)\,\Delta t - f(a)\int_a^c g(t)\,\Delta t &= (f(b)-f(a))\int_a^c g(t)\,\Delta t \\ = \int_a^b f^\Delta(t)\,\Delta t\int_a^c g(t)\,\Delta t \ &\le \ \int_a^b f^\Delta(t)\left(\int_a^t g(s)\,\Delta s\right)\Delta t \\ = f(b)\int_a^b g(t)\,\Delta t - \int_a^b f^\sigma(t)g(t)\,\Delta t \ &\le \ \int_a^d g(t)\,\Delta t\int_a^b f^\Delta(t)\,\Delta t \\ = f(b)\int_a^d g(t)\,\Delta t - f(a)\int_a^d g(t)\,\Delta t, \end{aligned}$$

which implies the statement. □

In the last part of this section we give some background for application of the Schauder fixed point theorem. It will be used in the proof of Theorem 10.1. We start by recalling the Schauder theorem that is applicable for our setting in dynamic equations. See [20, Theorem 6.44] for a proof.

Proposition 2.4 (Schauder's Fixed Point Theorem). *Let $\mathcal{N}$ be a normed space and X be a nonempty, closed, convex subset of $\mathcal{N}$. If $\mathcal{T}$ is a continuous mapping such that $\mathcal{T}(X) \subseteq X$ (i.e., a mapping from X into itself) and $\mathcal{T}(X)$ is relatively compact, then $\mathcal{T}$ has a fixed point in X.*

Denote by $C_{TS}^B(\mathcal{I}_a)$ the linear space of all continuous functions $f : \mathcal{I}_a \to \mathbb{R}$ such that $\sup_{t\in\mathcal{I}_a} |f(t)| < \infty$. Define this supremum to be the norm $\|f\| = \sup_{t\in\mathcal{I}_a} |f(t)|$.

The following statement can be understood as a time scales version of the Arzelà–Ascoli theorem. For the "discrete analogue" of this well-known theorem see [11, Theorem 3.3]. Note that we get $C_{TS}^B = \ell^\infty$ for $\mathbb{T} = \mathbb{N}$, and then condition (ii) in the next lemma holds trivially.

Lemma 2.6. *Let X be a subset of $C_{TS}^B(\mathcal{I}_a)$ having the following properties:*

(i) *X is bounded.*

(ii) *On every compact subinterval J of $[a,\infty)$ we have: For any $\varepsilon > 0$ there exists $\delta > 0$ such that $t_1, t_2 \in J$, $|t_1 - t_2| < \delta$ implies $|f(t_1) - f(t_2)| < \varepsilon$ for all $f \in X$ (i.e., the functions in X are locally equicontinuous).*

(iii) *For every $\varepsilon > 0$ there exists $b \in \mathcal{I}_a$ such that $t_1, t_2 \in [b,\infty)$ implies $|f(t_1) - f(t_2)| < \varepsilon$ for all $f \in X$ (in the "discrete terminology", X is said to be uniformly Cauchy).*

Then X is relatively compact.

Proof. By [20, Theorem 6.33] it is sufficient to construct a finite ε-net for any $\varepsilon > 0$. Since the proof is more or less obvious, we mention just some of its important points and omit details. In view of the properties (i)–(iii), it is possible to construct a two-dimensional grid where the vertical values are the elements $y_1, \dots, y_m \in \mathbb{R}$, $-K = y_1 < y_2 < \cdots < y_m = K$, K being such that $\|f\| \le K$ for all $f \in X$, and sufficiently close to neighbors, i.e., $y_{i+1} - y_i$ is a sufficiently small number depending on ε. The horizontal values $x_1, \dots, x_m \in \mathbb{T}$, $a = x_1 < x_2 < \cdots < x_n = b$, are sufficiently close to their neighbors in the sense that if they are close to dense points, the differences of the values of $f \in X$ at these points are small – depend on ε (this is possible thanks to the local equicontinuity), or they are isolated and sufficiently far from from each other; $b \ge a$ being such that $|f(t_1) - f(t_2)|$ is sufficiently small (depends on ε) whenever $t_1, t_2 \in [b, \infty)$ for all $f \in X$. Such b exists because of the property (iii). Now, having such a grid, for any $f \in X$ we can construct a linear fractional function g which approximate f (in fact, $\|f - g\| < \varepsilon$). The number of functions g constructed in this way is finite and thus the set of such functions forms a finite ε-net for X. □

3 Global Existence and Uniqueness of Solutions of IVPs

Consider the second-order half-linear dynamic equation

$$[r(t)\Phi(y^\Delta)]^\Delta + p(t)\Phi(y^\sigma) = 0, \tag{3.1}$$

where $\Phi(x) = |x|^{\alpha-1}\operatorname{sgn} x$, $\alpha > 1$. We will assume that $p, r \in C_{\mathrm{rd}}(\mathcal{I}, \mathbb{R})$ [resp. $p, r \in C_{\mathrm{rd}}(\mathcal{I}_a, \mathbb{R})$] with $r(t) \neq 0$, if it is not said otherwise. We call $y \in C^1_{\mathrm{rd}}(\mathcal{I}, \mathbb{R})$ $\big[$resp. $y \in C^1_{\mathrm{rd}}(\mathcal{I}_a, \mathbb{R})\big]$ a solution of (3.1) provided

$$\left\{(r\Phi(y^\Delta))^\Delta + p\Phi(y^\sigma)\right\}(t) = 0 \;\text{ holds for all } t \in (\mathcal{I}^\kappa)^\kappa \;\; [\text{resp. } t \in \mathcal{I}_a].$$

Notice that any solution y of (3.1) satisfies $r\Phi(y^\Delta) \in C^1_{\mathrm{rd}}(\mathcal{I}, \mathbb{R})$ $\big[$resp. $r\Phi(y^\Delta) \in C^1_{\mathrm{rd}}(\mathcal{I}_a, \mathbb{R})\big]$.

In this section we will suppose that $\mathbb{T}$ is unbounded above. Let us consider the initial value problem (IVP)

$$\left.\begin{array}{c}(r(t)\Phi(y^\Delta))^\Delta + p(t)\Phi(y^\sigma) = 0,\\ y(t_0) = A, \quad y^\Delta(t_0) = B\end{array}\right\} \tag{3.2}$$

on $\mathcal{I}_a$, where $A, B \in \mathbb{R}$, $t_0 \in \mathcal{I}_a$. Our aim is to examine, whether (and under what conditions) the IVP (3.2) has exactly one solution on the whole interval $\mathcal{I}_a$. The answer to this question is given in Theorem 3.1, the proof of which is based on the following lemmas. Sometimes it is convenient to deal with a system equivalent to the original problem (3.2), namely

$$Y^{\Delta} = F(t, Y), \quad Y(t_0) = Y_0, \tag{3.3}$$

where Y denotes $(x, u)^T$, $F : \mathcal{I}_a \times \mathbb{R}^2 \to \mathbb{R}^2$ is the vector function having the components

$$\begin{aligned} f_1(t, x, u) &= \Phi^{-1}\left(1/r(t)\right)\Phi^{-1}(u), \\ f_2(t, x, u) &= -p(t)\Phi\left[x + \mu(t)\Phi^{-1}\left(1/r(t)\right)\Phi^{-1}(u)\right], \end{aligned}$$

and $Y_0 = (A, r(t_0)\Phi(B))^T$.

In the following auxiliary lemmas we are interested in a right neighborhood of the point t_0. In fact, there we put $t_0 = a$, and the situation in a left neighborhood will be described in Remark 3.2.

Lemma 3.1 (Global Existence). *Let $p, r \in C_{\mathrm{rd}}(\mathcal{I}_a, \mathbb{R})$. Then the initial value problem* (3.2) *with $t_0 = a$ has at least one solution defined on the whole interval $\mathcal{I}_a$.*

Proof. The local existence follows from the result corresponding to the classical Peano theorem, Proposition 2.2. We will show that this solution can be extended to the entire interval $\mathcal{I}_a$. Integrating the system $Y^{\Delta} = F(t, Y)$ over an interval $[a, c]$, $c \in \mathcal{I}_a$, we obtain

$$\begin{aligned} x(t) &= x(a) + \int_a^t \Phi^{-1}\left(\frac{u(s)}{r(s)}\right)\Delta s, \\ u(t) &= u(a) - \int_a^t \left\{p\Phi\left[x + \mu\Phi^{-1}\left(\frac{u}{r}\right)\right]\right\}(s)\,\Delta s \end{aligned}$$

for $t \in [a, c]$. Using Hölder's inequality on time scales (Lemma 2.4), we have

$$|x(t) - x(a)|^{\alpha} \le \left(\int_a^t |1/r(s)|^{\beta/(\alpha-1)}\,\Delta s\right)^{\alpha/\beta} \int_a^t |u(s)|^{\beta}\,\Delta s$$

and

$$|u(t) - u(a)|^{\beta} \le \left(\int_a^t |p(s)|^{\alpha}\,\Delta s\right)^{\beta/\alpha} \int_a^t \left\{\left|x + \mu\Phi^{-1}\left(\frac{u}{r}\right)\right|^{\alpha}\right\}(s)\,\Delta s.$$

Taking into account that

$$|\lambda+\nu|^\alpha \le 2^{\alpha-1}\left(|\lambda|^\alpha+|\nu|^\alpha\right)$$

for $\lambda, \nu \in \mathbb{R}$, we get

$$\begin{aligned}
|x(t)|^\alpha + |u(t)|^\beta &\le 2^{\alpha-1}|x(a)|^\alpha + 2^{\beta-1}|u(a)|^\beta \\
&\quad + 2^{\alpha-1}\left(\int_a^t |1/r(s)|^{\beta/(\alpha-1)}\,\Delta s\right)^{\alpha/\beta}\int_a^t |u(s)|^\beta\,\Delta s \\
&\quad + 2^{\beta-1}\left(\int_a^t |p(s)|^\alpha\,\Delta s\right)^{\beta/\alpha}\int_a^t \left\{\left|x+\mu\Phi^{-1}\left(\frac{u}{r}\right)\right|^\alpha\right\}(s)\,\Delta s \\
&\le \; 2^{\alpha-1}|x(a)|^\alpha + 2^\alpha\left(\int_a^t |1/r(s)|^{\beta/(\alpha-1)}\,\Delta s\right)^{\alpha/\beta}\int_a^t \frac{1}{2}|u(s)|^\beta\,\Delta s \\
&\quad + 2^{\beta-1}|u(a)|^\beta + 2^{\alpha-1+\beta-1}\left(\int_a^t |p(s)|^\alpha\,\Delta s\right)^{\beta/\alpha}\int_a^t |x(s)|^\alpha\,\Delta s \\
&\quad + 2^{\alpha-1+\beta-1}\left(\int_a^t |p(s)|^\alpha\,\Delta s\right)^{\beta/\alpha}\int_a^t \left\{\frac{1}{2}\mu^\alpha\left|\frac{u}{r}\right|\right\}\,\Delta s \\
&\le \; K + H(c)\int_a^t \left(|x(s)|^\alpha + |u(s)|^\beta\right)(s)\,\Delta s
\end{aligned}$$

for $t \in [a,c]$, where

$$K = 2^{\alpha-1}|x(a)|^\alpha + 2^{\beta-1}|u(a)|^\beta \tag{3.4}$$

and

$$\begin{aligned}
H(c) = \max_{t\in[a,c]}\Bigg\{&2^{\alpha-1}\left(\int_a^t |p(s)|^\alpha\,\Delta s\right)^{\beta/\alpha}, \\
&2^\alpha\left(\int_a^t |1/r(s)|^{\beta/(\alpha-1)}\,\Delta s\right)^{\alpha/\beta}, \\
&(2\mu(t))^\alpha\left(\int_a^t |p(s)|^\alpha\,\Delta s\right)^{\beta/\alpha}|1/r(t)|^\beta\Bigg\}.
\end{aligned}$$

Using the Gronwall type inequality on time scales (Proposition 2.1), we have from the above estimate

$$|x(t)|^\alpha + |u(t)|^\beta \le K e_{H(c)}(t,a)$$

for $t \in [a,c]$, and thus the solution can be extended to $[a,c]$. Since $c \in \mathcal{I}_a$ is arbitrary, the assertion of the lemma follows. □

Lemma 3.2. *Let $p, r \in C_{\rm rd}(\mathcal{I}_a, \mathbb{R})$. Then* (3.1) *possesses only the trivial solution satisfying the initial conditions $y(a) = y^\Delta(a) = 0$ on $\mathcal{I}_a$.*

Proof. The statement follows from the proof of the previous lemma since the zero initial conditions imply $K = 0$, where K is defined by (3.4), and so $|x(t)|^\alpha + |u(t)|^\beta \le 0$ for all $t \in \mathcal{I}_a$. □

Lemma 3.3. *Let $p, r \in C_{\rm rd}(\mathcal{I}, \mathbb{R})$. Suppose that either (i) $A \neq 0, B \neq 0$, or (ii) $A = 0, B \neq 0$ and $\alpha \ge 2$, or (iii) $A \neq 0, B = 0$ and $\alpha \le 2$. Then the IVP* (3.2) *with $t_0 = a$ has a unique solution in a small right neighborhood of the point a.*

Of course, it makes sense to speak about points of $\mathbb{T}$ in a small right neighborhood of a only in the case when a is a right-dense point, but if a is right-scattered, then the continuation to the right is always possible in a unique manner. See the proof below.

Proof. It is easy to see that the vector function F satisfies a local Lipschitz condition on the set $\mathcal{D}$ defined in one of the following ways:

$$\begin{aligned}
\mathcal{D} &= \mathcal{I}_a \times (\mathbb{R} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\}), \\
\mathcal{D} &= \mathcal{I}_a \times \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \ \text{ and } \ \alpha \ge 2, \\
\mathcal{D} &= \mathcal{I}_a \times (\mathbb{R} \setminus \{0\}) \times \mathbb{R} \ \text{ and } \ \alpha \le 2.
\end{aligned}$$

Hence, the function F is Lipschitz in some right neighborhood of a. Note that in general the Lipschitz condition is trivially fulfilled in some right neighborhood of any right-scattered point. So the assumptions of Proposition 2.3 hold in $[a, c]$, where $c > a$ is sufficiently close to the (right-dense) point a, and the statement follows. □

Lemma 3.4. *Let $p, r \in C_{\rm rd}(\mathcal{I}, \mathbb{R})$. Suppose that $A = 0$, $B \neq 0$ and $\alpha < 2$. Then the IVP* (3.2) *with $t_0 = a$ has a unique solution in a small right neighborhood of the point a.*

Proof. Let y_1 and y_2 be two (local) solutions of (3.2) with $t_0 = a$, $A = 0$ and $B \neq 0$ (they exist by Proposition 2.2). We can suppose that a is right-dense since for right-scattered a, uniqueness is guaranteed (in some right neighborhood). Integrating (3.1) with $y = y_i$ twice from a to $t \in \mathrm{dom}(y_1) \cap \mathrm{dom}(y_2) \cap \mathcal{I}$ we get

$$y_i(t) = \int_a^t \frac{1}{r(s)} \Phi^{-1}\left[\tilde{B} - \int_a^s p(\tau)\Phi(y_i^\sigma(\tau))\,\Delta\tau\right] \Delta s, \quad i \in \{1, 2\},$$

where $\tilde{B} = r(a)\Phi(B)$. Hence

$$y_1(t) - y_2(t) = \int_a^t \frac{1}{r(s)} \left[\Phi^{-1}(\tilde{B} - I_1(s)) - \Phi^{-1}(\tilde{B} - I_2(s))\right] \Delta s,$$

where

$$I_i(t) = \int_a^t p(s)\Phi(y_i^\sigma(s))\,\Delta s, \quad i \in \{1,2\},$$

and, by the mean value theorem,

$$y_1(t) - y_2(t) = (\beta - 1)\int_a^t \frac{1}{r(s)}|\eta(s)|^{\beta-2}(I_2(s) - I_1(s))\,\Delta s,$$

where $\eta(t)$ lies between $\tilde{B} - I_1(t)$ and $\tilde{B} - I_2(t)$. Since $\tilde{B} - I_i(t) \to \tilde{B}$ as $t \to a$, $i \in \{1,2\}$, one can find $\delta > 0$ such that $|\eta(t)| \le 2|\tilde{B}|$ for $t \in [a, a+\delta] \cap \mathbb{T}$ (here we mean a real interval by $[a, a+\delta]$). Noting that $\beta - 2 > 0$ and using integration by parts (Lemma 2.2 (iv)), we obtain

$$\begin{aligned}
\frac{|y_1(t) - y_2(t)|}{(\beta-1)(2|\tilde{B}|)^{\beta-2}} &\le \int_a^t \frac{1}{|r(s)|}|I_1(s) - I_2(s)|\,\Delta s \\
&\le \int_a^t \frac{\int_a^s |p(\tau)||\Phi(y_1^\sigma(\tau)) - \Phi(y_2^\sigma(\tau))|\,\Delta\tau}{|r(s)|}\Delta s \\
&= \left[\int_a^s |p(\tau)||\Phi(y_1^\sigma(\tau)) - \Phi(y_2^\sigma(\tau))|\,\Delta\tau \int_a^s \frac{1}{|r(\tau)|}\,\Delta\tau\right]_a^t \\
&\quad - \int_a^t \left(\int_a^{\sigma(s)} \frac{1}{|r(\tau)|}\,\Delta\tau\right) |p(s)||\Phi(y_1^\sigma(s)) - \Phi(y_2^\sigma(s))|\,\Delta s \\
&= \int_a^t \left(\int_{\sigma(s)}^t \frac{1}{|r(\tau)|}\,\Delta\tau\right) |p(s)||\Phi(y_1^\sigma(s)) - \Phi(y_2^\sigma(s))|\,\Delta s
\end{aligned}$$

for $t \in [a, a+\delta] \cap \mathbb{T}$. Define the rd-continuous function

$$\tilde{y}_i(t) = \begin{cases} y_i(t)\left\{\int_a^t \frac{1}{|r(s)|}\Delta s\right\}^{-1} & \text{for } t \in (a, a+\delta] \cap \mathbb{T}, \\ B|r(a)| & \text{for } t = a, \end{cases}$$

$i \in \{1,2\}$. Then, using the above estimates, $|\tilde{y}_1(t) - \tilde{y}_2(t)|$ is less than or equal to

$$(\beta-1)(2|\tilde{B}|)^{\beta-2}\int_a^t \left(\int_a^{\sigma(s)} \frac{1}{|r(\tau)|}\,\Delta\tau\right)^{\alpha-1} \{|p||\Phi(\tilde{y}_1^\sigma) - \Phi(\tilde{y}_2^\sigma)|\}\,(s)\,\Delta s$$

for $t \in [a, a+\delta] \cap \mathbb{T}$. By the mean value theorem

$$|\Phi(\tilde{y}_1(t)) - \Phi(\tilde{y}_2(t))| \le (\alpha - 1)|\xi(t)|^{\alpha-2}|\tilde{y}_1(t) - \tilde{y}_2(t)|$$

for $t \in [a, a+\delta] \cap \mathbb{T}$, where $\xi(t)$ lies between $\tilde{y}_1(t)$ and $\tilde{y}_2(t)$. Since $\tilde{y}_i(t) \to B|r(a)|$ as $t \to a$, $i \in \{1, 2\}$, $\xi(t)$ can be made to satisfy $|Br(a)|/2 \le |\xi(t)|$, if $t \in \mathcal{I}$ is taken sufficiently close to a. Hence there exists $0 < \omega < \delta$ such that

$$|\Phi(\tilde{y}_1(t)) - \Phi(\tilde{y}_2(t))| \le (\alpha - 1)\left(|Br(a)|/2\right)^{\alpha-2}|\tilde{y}_1(t) - \tilde{y}_2(t)|$$

for $t \in [a, a+\omega] \cap \mathbb{T}$. Now, using the above and the fact that

$$(\beta - 1)(\alpha - 1)(2|\tilde{B}|)^{\beta-2}\left(|Br(a)|/2\right)^{\alpha-2} = 2^{\beta-\alpha}|r(a)|^{\alpha+\beta-4},$$

we find that $|\tilde{y}_1^\sigma(t) - \tilde{y}_2^\sigma(t)|$ is less than or equal to

$$2^{\beta-\alpha}|r(a)|^{\alpha+\beta-4}\int_a^{\sigma(t)}\left(\int_a^{\sigma(s)}\frac{1}{|r(\tau)|}\,\Delta\tau\right)^{\alpha-1}|p(s)||\tilde{y}_1^\sigma(s) - \tilde{y}_2^\sigma(s)|\,\Delta s$$

for $t \in ([a, a+\omega] \cap \mathbb{T})^\kappa$. Applying the Gronwall type inequality (Proposition 2.1) we conclude that $\tilde{y}_1^\sigma(t) \equiv \tilde{y}_2^\sigma(t)$ on $([a, a+\omega] \cap \mathbb{T})^\kappa$, which implies that $y_1(t)$ and $y_2(t)$ coincide in a small right neighborhood of the (right-dense) point a. □

Lemma 3.5. *Let $p, r \in C_{\mathrm{rd}}(\mathcal{I}, \mathbb{R})$. Suppose that $A \neq 0$, $B = 0$ and $\alpha > 2$. Then the IVP (3.2) with $t_0 = a$ has a unique solution in a small right neighborhood of the point a.*

Proof. For $p(a) \neq 0$, where a is right-dense, the statement can be proved in a similar way as the previous lemma. See also Remark 3.1 after this proof.

We have to show that the statement is valid also for the case where $p(a) = 0$ with right-dense a. Let y_1 and y_2 be two (local) solutions of (3.2) with $t_0 = a$, $A \neq 0$ and $B = 0$. Then there exists $\delta > 0$ such that $y_i(t)y_i^\sigma(t) > 0$ on $[a, a+\delta] \cap \mathbb{T}$, $i \in \{1, 2\}$. Then, as it is shown in Section 5, $w_i = r(t)\Phi(y_i^\Delta(t)/y_i(t))$, $i \in \{1, 2\}$, solve the generalized Riccati dynamic equation (5.1) on $[a, a+\delta] \cap \mathbb{T}$. Further, $w_1(a) = 0 = w_2(a)$. The initial value problem (5.1), $w(a) = w_0$ with $w_0 \in \mathbb{R}$, which corresponds to a nonzero solution y of (3.2) on $[a, a+\delta] \cap \mathbb{T}$, has exactly one solution on $[a, a+\delta] \cap \mathbb{T}$ by Proposition 2.3. Indeed, we will show that the right-hand side of $w^\Delta(t) = -p(t) - \mathcal{S}[w, r](t)$ is Lipschitz with respect to w. Observe that $\Phi(\Phi^{-1}(r) + \mu\Phi^{-1}(w)) = r\Phi(y^\sigma/y) \neq 0$ at each $t \in [a, a+\delta] \cap \mathbb{T}$. Differentiating $\mathcal{S}$ with respect to w we obtain

$$\frac{\partial \mathcal{S}[w, r]}{\partial w} = \lim_{\lambda \to \mu}\frac{|\Phi^{-1}(r) + \lambda\Phi^{-1}(w)|^\alpha - |r|^\beta}{\lambda|\Phi^{-1}(r) + \lambda\Phi^{-1}(w)|^\alpha},$$

see also the proof of the first part of Lemma 7.1, and so the existence and boundedness of this derivative implies that the assumption (ii) of Proposition 2.3 is satisfied. Hence $w_1(t) = w_2(t)$ and thus $y_1^\Delta(t)/y_1(t) = y_2^\Delta(t)/y_2(t)$, which implies $y_1(t) = Cy_2(t)$ for $t \in [a, a+\delta] \cap \mathbb{T}$, where C is any real constant. Since $y_1(a) = A = y_2(a)$, we get $C = 1$, and the statement follows. □

Remark 3.1. In the second part of the last proof it is not necessary to assume $p(a) = 0$, and the statement can be proved using the Riccati technique also for the case $p(a) \neq 0$. If σ and Δ commute (i.e., $f^{\sigma\Delta} = f^{\Delta\sigma}$), then we have another possibility how to prove it in the case $p(a) \neq 0$, namely the method based on the reciprocal equation

$$[(p(t))^{1-\beta}\Phi^{-1}(u^\Delta)]^\Delta + (r^\sigma(t))^{1-\beta}\Phi^{-1}(u^\sigma) = 0. \tag{3.5}$$

This equation is related to (3.1) by the substitution $u(t) = r(t)\Phi(y^\Delta(t))$, where y is a solution of (3.1) (which exists by Proposition 2.2). Then u is a solution of (3.5) (satisfying the initial conditions $u(a) = 0,\ u^\Delta(a) \neq 0$ in case $y(a) \neq 0,\ y^\Delta(a) = 0$). Moreover, we have $\beta < 2$ (since $\alpha > 2$), and so Lemma 3.4 can be applied.

Note that the existence and uniqueness of IVPs for (HLDE) is proved in [16] using the generalized Prüfer transformation. However, this tool is not available in the half-linear time scales case so far.

Remark 3.2. It is not difficult to see that similar statements as above can be proved also in the case, where we examine the problem of the existence and uniqueness of the IVP with $t_0 \in \mathcal{I}_a \setminus \{a\}$ in a left neighborhood of t_0. However, in addition, we must show that the vector function F is regressive for $t < t_0$ so that condition (iii) from Proposition 2.3 is fulfilled. This is important for the continuation of a solution of (3.1) in backward time direction in a unique manner on discrete time scales. Thus, suppose for this moment that $\mathbb{T}$ is a discrete time scale. The regressivity of F in t, in other words, means the possibility to compute the value of $Y = (x, u)^T$ at t in a unique manner knowing its value at $\sigma(t)$. But for $\mu(t) > 0$ the system $Y^\Delta = F(t, Y)$ is equivalent to the system

$$\begin{aligned} x &= x^\sigma - \mu(t)\Phi^{-1}(1/r(t))\,\Phi^{-1}(u^\sigma + \mu(t)p(t)\Phi(x^\sigma)), \\ u &= u^\sigma + \mu(t)p(t)\Phi(x^\sigma), \end{aligned}$$

and hence F is regressive. Actually, every equation in “self-adjoint” form (like (3.1), but also of higher order) with a nonzero leading coefficient can be rewritten as a regressive system.

Theorem 3.1 (Existence and Uniqueness). *Let $p, r \in C_{\rm rd}(\mathcal{I}_a, \mathbb{R})$. Then the IVP* (3.2) *has exactly one solution on $\mathcal{I}_a$.*

Proof. The statement follows from Lemmas 3.1–3.5 and Remark 3.2. □

4 Picone's Identity

Consider the operators

$$\begin{aligned} \mathcal{L}_{r,p}[y] &:= (r(t)\Phi(y^\Delta))^\Delta + p(t)\Phi(y^\sigma), \\ \mathcal{L}_{R,P}[z] &:= (R(t)\Phi(z^\Delta))^\Delta + P(t)\Phi(z^\sigma) \end{aligned}$$

for $t \in (\mathcal{I}^\kappa)^\kappa$, where $p, P, r, R \in C_{\rm rd}(\mathcal{I}, \mathbb{R})$ with $r(t) \neq 0 \neq R(t)$. Denote by $D_{r,p}(\mathcal{I})$ and $D_{R,P}(\mathcal{I})$ the domains of the operators $\mathcal{L}_{r,p}$ and $\mathcal{L}_{R,P}$, respectively, that are defined as sets of all rd-continuous real functions y and z defined on $\mathcal{I}$ such that $r(t)\Phi(y^\Delta)$ and $R(t)\Phi(z^\Delta)$ belong to $C^1_{\rm rd}(\mathcal{I}^\kappa, \mathbb{R})$. For convenience, in this and the next two sections we assume, without loss of generality, that $\max \mathbb{T} = b$. We need to avoid the situation where $b < \sigma(b) \notin \mathcal{I} = \mathcal{I}^\kappa$ (this is in connection with the concept of a generalized zero).

Next we derive a generalized Picone identity involving the two above defined operators, which plays a crucial rôle in the proof of our extension of the Roundabout theorem.

Lemma 4.1 (Picone Identity). *Let $y \in D_{r,p}(\mathcal{I})$, $z \in D_{R,P}(\mathcal{I})$ and $z(t) \neq 0$ for $t \in \mathcal{I}^\kappa$. Then the equality*

$$\begin{aligned} \left\{ \frac{y}{\Phi(z)}[\Phi(z) r \Phi(y^\Delta) - \Phi(y) R \Phi(z^\Delta)] \right\}^\Delta &= (P - p)|y^\sigma|^\alpha \\ &+ (r - R)|y^\Delta|^\alpha + \frac{y^\sigma}{\Phi(z^\sigma)} \left\{ \mathcal{L}_{r,p}[y]\Phi(z^\sigma) - \mathcal{L}_{R,P}[z]\Phi(y^\sigma) \right\} + G(y,z), \end{aligned}$$

where

$$G(y,z) = R|y^\Delta|^\alpha - \frac{R\Phi(z^\Delta)}{\Phi(z)}(|y|^\alpha)^\Delta + \frac{R\Phi(z^\Delta)(\Phi(z))^\Delta}{\Phi(z)\Phi(z^\sigma)}|y^\sigma|^\alpha,$$

holds on $(\mathcal{I}^\kappa)^\kappa$.

Proof. On $(\mathcal{I}^\kappa)^\kappa$ we have

$$(yr\Phi(y^\Delta))^\Delta = y^\sigma (r\Phi(y^\Delta))^\Delta + y^\Delta r \Phi(y^\Delta)$$

$$= \quad y^\sigma \mathcal{L}_{r,p}[y] - p|y^\sigma|^\alpha + r|y^\Delta|^\alpha.$$

Further, again on $(\mathcal{I}^\kappa)^\kappa$,

$$\begin{aligned}
&\left(\frac{y}{\Phi(z)}\Phi(y)R\Phi(z^\Delta)\right)^\Delta = \left(|y|^\alpha \frac{R\Phi(z^\Delta)}{\Phi(z)}\right)^\Delta \\
&= \quad (|y|^\alpha)^\Delta \frac{R\Phi(z^\Delta)}{\Phi(z)} + |y^\sigma|^\alpha \frac{(R\Phi(z^\Delta))^\Delta \Phi(z)}{\Phi(z)\Phi(z^\sigma)} - |y^\sigma|^\alpha \frac{R\Phi(z^\Delta)(\Phi(z))^\Delta}{\Phi(z)\Phi(z^\sigma)} \\
&= \quad \frac{y^\sigma \Phi(y^\sigma)(R\Phi(z^\Delta))^\Delta}{\Phi(z^\sigma)} + \frac{y^\sigma \Phi(y^\sigma) P \Phi(z^\sigma)}{\Phi(z^\sigma)} - R|y^\Delta|^\alpha \\
&\qquad + (|y|^\alpha)^\Delta \frac{R\Phi(z^\Delta)}{\Phi(z)} - |y^\sigma|^\alpha \frac{R\Phi(z^\Delta)(\Phi(z))^\Delta}{\Phi(z)\Phi(z^\sigma)} - P|y^\sigma|^\alpha + R|y^\Delta|^\alpha \\
&= \quad \frac{y^\sigma}{\Phi(z^\sigma)} \mathcal{L}_{R,P}[z]\Phi(y^\sigma) - G(y,z) - P|y^\sigma|^\alpha + R|y^\Delta|^\alpha.
\end{aligned}$$

Combining these two equalities, we get the desired result. □

Lemma 4.2. *In addition to the assumptions of the previous lemma let z be such that $(Rzz^\sigma)(t) > 0$ for $t \in \mathcal{I}^\kappa$. Then $G(y,z)(t) \geq 0$ for $t \in \mathcal{I}^\kappa$, where the equality holds if and only if $y^\Delta = (z^\Delta/z)y$.*

Proof. We distinguish the following two cases:

I. Suppose that $t \in \mathcal{I}^\kappa$ is right-scattered. Then, using Lemma 2.1, we have at t

$$\begin{aligned}
G(y,z) &= \left(\frac{Rz}{z^\sigma}\right)\left\{\frac{z^\sigma}{z}|y^\Delta|^\alpha - \frac{\Phi(z^\Delta)}{\mu z|z^\sigma|^{\alpha-2}}|y^\sigma|^\alpha + \frac{z^\sigma \Phi(z^\Delta)}{\mu|z|^\alpha}|y|^\alpha\right\} \\
&= \left(\frac{Rz}{\mu^\alpha z^\sigma}\right)\left\{\frac{z^\sigma}{z}|y^\sigma - y|^\alpha - \frac{\Phi|z^\sigma - z|}{z|z^\sigma|^{\alpha-2}}|y^\sigma|^\alpha + \frac{z^\sigma}{z}\Phi\left(\frac{z^\sigma - z}{z}\right)|y|^\sigma\right\}.
\end{aligned}$$

The nonnegativity of the expression in the brackets follows from [41, Lemma 2].

II. Suppose that $t \in \mathcal{I}^\kappa$ is right-dense. Then, using Lemma 2.1, we obtain

$$G(y,z)(t) = R\left\{|y^\Delta|^\alpha - (|y|^\alpha)^\Delta \frac{\Phi(z^\Delta)}{\Phi(z)} + \frac{\Phi(z^\Delta)(\Phi(z))^\Delta}{\Phi(z)\Phi(z)}\right\}(t)$$

$$= R\left\{|y^\Delta|^\alpha - \alpha\Phi\left(\frac{yz^\Delta}{z}\right)y^\Delta + (\alpha-1)\left|\frac{yz^\Delta}{z}\right|^\alpha\right\}(t) \geq 0$$

by the Young inequality (where we put $u = y^\Delta$ and $v = \Phi(yz^\Delta/z)$), see Lemma 2.3, since $(Rz^2)(t) = (Rzz^\sigma)(t) > 0$ implies $R(t) > 0$. □

5 Reid Roundabout Theorem

Now we are in a position to formulate the central statement of the oscillation theory for (3.1), namely a half-linear version of the Roundabout theorem on time scales. First we introduce some necessary concepts.

Consider the equation (3.1), where $p, r \in C_{\mathrm{rd}}(\mathcal{I}, \mathbb{R})$ with $r(t) \neq 0$. Along with (3.1) consider the generalized Riccati dynamic equation

$$\mathcal{R}[w] := w^{\Delta} + p(t) + \mathcal{S}[w, r](t) = 0, \tag{5.1}$$

where

$$\mathcal{S}[w, r] = \lim_{\lambda \to \mu} \frac{w}{\lambda} \left(1 - \frac{r}{\Phi(\Phi^{-1}(r) + \lambda \Phi^{-1}(w))} \right).$$

In fact, (5.1) is related to the original equation by the Riccati type substitution $w(t) = r(t)\Phi(y^{\Delta}(t))/\Phi(y(t))$. Observe that

$$\mathcal{S}[w, r](t) = \begin{cases} \left\{ \frac{\alpha - 1}{\Phi^{-1}(r)} |w|^{\beta} \right\}(t) & \text{for right-dense } t, \\ \left\{ \frac{w}{\mu} \left(1 - \frac{r}{\Phi(\Phi^{-1}(r) + \mu \Phi^{-1}(w))} \right) \right\}(t) & \text{for right-scattered } t, \end{cases}$$

where l'Hôpital's rule is used in the first case. Note that using the Lagrange mean value theorem, one can find the following form of the operator $\mathcal{S}$, namely

$$\mathcal{S}[w, r] = \frac{(\alpha - 1)|\eta|^{\alpha - 2}|w|^{\beta}}{\Phi(\Phi^{-1}(r) + \mu \Phi^{-1}(w))}, \tag{5.2}$$

where η is between $\Phi^{-1}(r)$ and $\Phi^{-1}(r) + \mu\Phi^{-1}(w)$, and this form will be sometimes useful.

Definition 5.1. (i) We say that a solution y of (3.1) has a *generalized zero* at t in case $y(t) = 0$. We say y has a *generalized zero* in $(t, \sigma(t))$ in case $r(t)y(t)y(\sigma(t)) < 0$. We say that (3.1) is *disconjugate* on the interval $\mathcal{I}$, if there is no nontrivial solution of (3.1) with two (or more) generalized zeros in $\mathcal{I}$. (For further discussion of the concept of a generalized zero see also Example 11.1 in Section 11 and Remark 6.1 in the next section.)

(ii) Define a class $U = U(a, b)$ of so-called *admissible functions* by

$$U(a, b) = \{\xi \in C^1_{\mathrm{p}}(\mathcal{I}, \mathbb{R}) : \ \xi(a) = \xi(b) = 0\}.$$

Define an *"α-degree" functional* $\mathcal{F}$ on U by

$$\mathcal{F}(\xi; a, b) = \int_a^b \left\{ r|\xi^{\Delta}|^{\alpha} - p|\xi^{\sigma}|^{\alpha} \right\}(t)\, \Delta t.$$

We say that $\mathcal{F}$ is *positive definite on U* provided $\mathcal{F}(\xi) \geq 0$ for all $\xi \in U$ and $\mathcal{F}(\xi) = 0$ if and only if $\xi = 0$.

Theorem 5.1 (Roundabout Theorem). *The following statements are equivalent:*

(i) *Equation* (3.1) *is disconjugate on* $\mathcal{I}$.

(ii) *Equation* (3.1) *has a solution* y *such that* $(ryy^\sigma)(t) > 0$ *for* $t \in \mathcal{I}^\kappa$ *(i.e., a solution having no generalized zeros on* $\mathcal{I}$*).*

(iii) *Equation* (5.1) *has a solution* w *with*

$$\left\{\Phi^{-1}(r) + \mu\Phi^{-1}(w)\right\}(t) > 0 \tag{5.3}$$

for $t \in \mathcal{I}^\kappa$.

(iv) $\mathcal{F}$ *is positive definite on* U.

Proof. We show the four following implications:

(i)⇒(ii): Let $\bar{y}$ be the solution of (3.1) given by the initial conditions $\bar{y}(a) = 0$, $\bar{y}^\Delta(a) = 1$. From (i) it follows that $(ryy^\sigma)(t) > 0$ for $t \in \mathcal{I}^\kappa \setminus \{a\}$. Consider a solution y_ε satisfying the initial conditions

$$y_\varepsilon(a) = \varepsilon r(a), \quad y_\varepsilon^\Delta(a) = \mu^\dagger(a)(1 - \varepsilon r(a) - \mu(a)) + 1,$$

where we put $\mu^\dagger = 0$ if $\mu = 0$ and $\mu^\dagger = \mu^{-1}$ if $\mu > 0$. Then $y_\varepsilon \to y$ as $\varepsilon \to 0$. Hence, if we choose $\varepsilon > 0$ sufficiently small, then $y \equiv y_\varepsilon$ satisfies $(ryy^\sigma)(t) > 0$ for $t \in \mathcal{I}^\kappa \setminus \{a\}$. Moreover, for right-scattered a we have $(ryy^\sigma)(a) = \varepsilon r^2(a) > 0$ since

$$\left(\frac{y^\sigma - y}{\mu}\right)(a) = \frac{1 - \varepsilon r(a)}{\mu(a)}$$

by Lemma 2.1. In the case when a is right-dense we get

$$(ryy^\sigma)(a) = (ry^2)(a) = r(a)(\varepsilon r(a))^2,$$

and this is positive if and only if $r(a) > 0$. Suppose, for the contrary, that $r(a) < 0$. Consider the solution $\tilde{y}$ of (3.1) satisfying the initial conditions $\tilde{y}(c) = 0$, $\tilde{y}^\Delta(c) = 1$, where $c \in \mathcal{I}^\kappa \setminus \{a\}$. The disconjugacy of (3.1) implies that $(r\tilde{y}\tilde{y}^\sigma)(a) > 0$. Since a is right-dense, we have $r(a) > 0$, a contradiction. Altogether, y is the solution of (3.1) with $(ryy^\sigma)(t) > 0$ for $t \in \mathcal{I}^\kappa$, and hence (ii) holds.

(ii)⇒(iii): Assume that y is a solution of (3.1) with $(ryy^\sigma)(t) > 0$ on $\mathcal{I}^\kappa$. Use the Riccati type substitution

$$w = \frac{r(t)\Phi(y^\Delta)}{\Phi(y)}.$$

Then we have

$$w^{\Delta}(t) = -p(t) - \left\{ \frac{w(\Phi(y))^{\Delta}}{\Phi(y^{\sigma})} \right\}(t)$$

for $t \in \mathcal{I}^{\kappa}$. Observe that

$$\left\{ r\Phi\left(\frac{y^{\sigma}}{y}\right) \right\}(t) = \Phi(\Phi^{-1}(r) + \mu\Phi^{-1}(w))(t)$$

for $t \in \mathcal{I}^{\kappa}$, and hence (5.3) holds. Further, at each right-scattered t we have

$$\frac{w(\Phi(y))^{\Delta}}{\Phi(y^{\sigma})} = \frac{w}{\mu}\left(1 - \Phi\left(\frac{y}{y^{\sigma}}\right)\right) = \frac{w}{\mu}\left(1 - \frac{r}{\Phi(\Phi^{-1}(r) + \mu\Phi^{-1}(w))}\right)$$

by Lemma 2.1, while at right-dense t we obtain

$$\frac{w(\Phi(y))^{\Delta}}{\Phi(y^{\sigma})} = \frac{(\alpha-1)w|y|^{\alpha-2}y^{\Delta}}{\Phi(y)} = (\alpha-1)r^{1-\beta}|w|^{\beta},$$

again by Lemma 2.1.

(iii)⇒(iv): Assume that w is a solution of (5.1) with the property (5.3) for $t \in \mathcal{I}^{\kappa}$. It is not difficult to see that the solution z of

$$z^{\Delta} = \Phi^{-1}\left(\frac{w(t)}{r(t)}\right)z, \quad z(t_0) = A \neq 0, \quad t_0 \in \mathcal{I}, \tag{5.4}$$

is a solution of (3.1) satisfying $(rzz^{\sigma})(t) > 0$ for $t \in \mathcal{I}^{\kappa}$. Note that, since

$$\left\{1 + \mu\Phi^{-1}\left(\frac{w}{r}\right)\right\}(t) = \left\{\frac{1}{\Phi^{-1}(r)}(\Phi^{-1}(r) + \mu\Phi^{-1}(w))\right\}(t) \neq 0$$

on $\mathcal{I}^{\kappa}$, the mapping $\Phi^{-1}(w/r)$ is regressive, so the initial value problem (5.4) has only one solution z. Pick any $\xi \in U$. From the Picone identity applied to the case $p = P$, $r = R$, $y = \xi$ and $w = r\Phi(z^{\Delta}/z)$, we obtain

$$\left\{\xi r\Phi(\xi^{\Delta}) - w|\xi|^{\alpha}\right\}^{\Delta} = \xi^{\sigma}(r\Phi(\xi^{\Delta}))^{\Delta} + p|\xi^{\sigma}|^{\alpha} + \tilde{G}(\xi, w),$$

where

$$\tilde{G}(\xi, w) = r|\xi^{\Delta}|^{\alpha} - w\left(|\xi|^{\alpha}\right)^{\Delta} + \mathcal{S}[w, r]|\xi^{\sigma}|^{\alpha}.$$

Hence

$$r|\xi^{\Delta}|^{\alpha} - p|\xi^{\sigma}|^{\alpha} = \{w|\xi|^{\alpha}\}^{\Delta} + \tilde{G}(\xi, w).$$

The integration from a to b yields

$$\mathcal{F}(\xi; a, b) = \left[\{w|\xi|^{\alpha}\}(t)\right]_a^b + \int_a^b \tilde{G}(\xi, w)(t)\,\Delta t.$$

Therefore, $\mathcal{F}(\xi;a,b) \geq 0$ because of Lemma 4.2. If, however, $\mathcal{F}(\xi;a,b) = 0$, then, again by Lemma 4.2, $\xi^\Delta = (z^\Delta/z)(t)\xi = \Phi^{-1}(w/r)(t)\xi$ on $\mathcal{I}^\kappa$. Since $\xi(a) = 0$, the initial value problem admits only one, namely the trivial solution. Consequently, $\mathcal{F}(\xi;a,b) > 0$ for all nontrivial admissible ξ.

(iv)⇒(i): Suppose, by contradiction, that (3.1) is not disconjugate on $\mathcal{I}$. Then there is a solution y of (3.1) and $c, d \in \mathbb{T}$ for which exactly one of the following cases may happen:

I. $\sigma(a) \leq \sigma(c) < d \leq b$ such that $y(t) \neq 0$ for $t \in (c,d)$ and $y(c) = 0 = y(d)$,

II. $\sigma(a) \leq \sigma(c) < d \leq \rho(b)$ such that $y(t) \neq 0$ for $t \in (c, \sigma(d))$, $y(c) = 0$ and $(ryy^\sigma)(d) < 0$,

III. $\sigma(a) \leq \sigma(c) < d \leq b$ such that $y(t) \neq 0$ for $t \in [c,d)$, $(ryy^\sigma)(c) < 0$ and $y(d) = 0$,

IV. $\sigma(a) \leq \sigma(c) < d \leq \rho(b)$ such that $y(t) \neq 0$ for $t \in [c,d]$, $(ryy^\sigma)(c) < 0$ and $(ryy^\sigma)(d) < 0$.

We examine for example the cases I and II since the remaining ones can be treated in a similar way. We can investigate I and II simultaneously except for the case II with right-dense d; we will deal with this special case later. Thus, define

$$\xi(t) = \begin{cases} 0 & \text{for } t \in [a,c), \\ y(t) & \text{for } t \in [c,d], \\ 0 & \text{for } t \in (d,b], \end{cases}$$

which yields $\xi(a) = \xi(b) = 0$, $\xi(t) \neq 0$ for $t \in (c,d)$ and $\xi \in C^1_{\mathrm{p}}(\mathcal{I})$. Using integration by parts, we have

$$\begin{aligned}
\mathcal{F}(\xi) &= \int_a^b \left\{ r|\xi^\Delta|^\alpha - p|\xi^\sigma|^\alpha \right\}(t)\,\Delta t \\
&= \int_c^d \left\{ r|\xi^\Delta|^\alpha - p|\xi^\sigma|^\alpha \right\}(t)\,\Delta t + \int_d^{\sigma(d)} \left\{ r|\xi^\Delta|^\alpha - p|\xi^\sigma|^\alpha \right\}(t)\,\Delta t \\
&= \left[\left\{ yr\Phi(y^\Delta) \right\}(t) \right]_c^d - \int_c^d \left\{ \xi^\sigma \mathcal{L}_{r,p}[\xi] \right\}(t)\,\Delta t \\
&\quad + \left\{ r|\xi^\Delta|^\alpha - p|\xi^\sigma|^\alpha \right\}(d)\mu(d) \\
&= \left\{ yr\Phi(y^\Delta) \right\}(d) + \left\{ r|\xi^\Delta|^\alpha \right\}(d)\mu(d),
\end{aligned}$$

where the operator $\mathcal{L}_{r,p}$ is defined at the beginning of Section 4. Now, if d is right-dense, then the only possibility is $y(d) = 0$ and, moreover, $\mu(d) = 0$.

Therefore $\mathcal{F} \not> 0$ and we will proceed with right-scattered d so that $\sigma(d) > d$ and $\mu(d) > 0$. Then we have

$$\begin{aligned}\mathcal{F}(\xi) &= \left\{yr\Phi(y^\Delta)\right\}(d) + \left\{r|\xi^\sigma - \xi|^\alpha \mu^{1-\alpha}\right\}(d) \\ &= \left\{yr\Phi(y^\Delta)\right\}(d) + \left\{r|y|^\alpha \mu^{1-\alpha}\right\}(d).\end{aligned}$$

We have to verify the inequality

$$\left\{\mu^{\alpha-1} yr\Phi(y^\Delta)\right\}(d) \le -\left\{yr\Phi(y)\right\}(d),$$

i.e.,

$$\left\{yr\Phi(y^\sigma - y)\right\}(d) \le -\left\{ry\Phi(y)\right\}(d).$$

But this inequality holds since it is equivalent to

$$\left\{\Phi^{-1}(yr)y^\sigma\right\}(d) - \left\{\Phi^{-1}(yr)y\right\}(d) \le -\left\{\Phi^{-1}(yr)y\right\}(d)$$

and we suppose $(ryy^\sigma)(d) \le 0$, a contradiction.

It remains to discuss the case II with right-dense d. This implies $r(d) < 0$. First we suppose that d is left-scattered. Then $d \ne b$. Let $\{t_m\}_{m\in\mathbb{N}} \subseteq \mathcal{I}$ be a right-sequence for d. For $m \in \mathbb{N}$ put

$$\xi_m(t) = \begin{cases} \frac{t_m - t}{(t_m-d)^{1/\alpha}} & \text{for } t \in [d, t_m], \\ 0 & \text{otherwise.}\end{cases}$$

Then $\xi_m(a) = \xi_m(b) = 0$, $\xi_m(d) = (t_m - d)^{1/\beta} \ne 0$ and $\xi_m \in C^1_{\mathrm{p}}(\mathcal{I})$. Hence

$$\begin{aligned}\mathcal{F}(\xi_m) &= \int_d^{t_m} \left\{r|\xi_m^\Delta|^\alpha - p|\xi_m^\sigma|^\alpha\right\}(t)\,\Delta t \\ &= \int_d^{t_m} \frac{r(t)}{t_m - d}\,\Delta t - \int_d^{t_m} \frac{(\sigma(t)-t)^\alpha p(t)}{t_m - d}\,\Delta t \to r(d) < 0\end{aligned}$$

as $m \to \infty$, a contradiction. The remaining case of left-dense d can be treated in a similar way using a suitable left-sequence (it is defined in a dual way) for $d = b$ and left- and right-sequences otherwise. The theorem is proved. □

Remark 5.1. The Picone identity could be also used to show that absence of generalized zeros of the solution of (3.1) implies positive definiteness of the functional $\mathcal{F}$ (the implication (ii)⇒(iv)). But we included the generalized Riccati equation into the Roundabout theorem since such an equivalence is important for applications of our theory.

Next we state the version of Reid's roundabout theorem involving a different type of boundary conditions than those in Theorem 5.1. More precisely, we consider a functional with one free endpoint. Later we use this theorem in the proofs of a Leighton type comparison theorem and a comparison theorem for generalized Riccati dynamic equations.

Theorem 5.2 (Roundabout Theorem). *The following statements are equivalent:*

(i) *The solution y of* (3.1) *given by $[r\Phi(y^\Delta/y)](a) = A$ satisfies $(ryy^\sigma)(t) > 0$ for $t \in \mathcal{I}^\kappa$.*

(ii) *Equation* (5.1) *has a solution w on $\mathcal{I}$ such that $w(a) = A$ and* (5.3) *holds for $t \in \mathcal{I}^\kappa$.*

(iii) *The functional*

$$\mathcal{F}_A(\xi; a, b) = A|\xi(a)|^\alpha + \mathcal{F}(\xi; a, b)$$

is positive definite on $U_A(a, b)$ defined by

$$U_A(a, b) = \{\xi \in C_p^1(\mathcal{I}, \mathbb{R}) :\ \xi(b) = 0\}.$$

Proof. We combine the technique of the last proof with the idea of [33, Theorem 1]. Hence we can omit details. Note just that in (i)⇒(iii) we use the Picone identity, and in (iii)⇒(i), a function ξ from U_A is defined by

$$\xi(t) = \begin{cases} y(t) & \text{for } t \in [a, d], \\ 0 & \text{for } t \in (d, b], \end{cases}$$

where d is similar as in the last proof. Finally note that the proof of (i)⇔(ii) is very easy. □

6 Sturmian Theory

This section is devoted to Sturmian theory. Consider two equations $\mathcal{L}_{r,p}[y] = 0$ and $\mathcal{L}_{R,P}[z] = 0$ (the operators $\mathcal{L}_{r,p}, \mathcal{L}_{R,P}$ are defined at the beginning of Section 4). Denote

$$\mathcal{F}_{R,P}(\xi; a, b) := \int_a^b \left\{R|\xi^\Delta|^\alpha - P|\xi^\sigma|^\alpha\right\}(t)\,\Delta t.$$

Then we have the following extension of Sturmian theorems for half-linear dynamic equations.

Theorem 6.1 (Sturm Comparison Theorem). *Suppose that we have* $R(t) \geq r(t)$ *and* $p(t) \geq P(t)$ *for* $t \in \mathcal{I}^\kappa$. *If* $\mathcal{L}_{r,p}[y] = 0$ *is disconjugate on* $\mathcal{I}$, *then* $\mathcal{L}_{R,P}[z] = 0$ *is also disconjugate on* $\mathcal{I}$.

Proof. Suppose that equation $\mathcal{L}_{r,p}[y] = 0$ is disconjugate on $\mathcal{I}$. Then Theorem 5.1 yields $\mathcal{F}(\xi; a, b) > 0$ for all (nontrivial) admissible sequences ξ. For such a ξ we also have

$$\mathcal{F}_{R,P}(\xi; a, b) \geq \mathcal{F}(\xi; a, b) > 0.$$

Hence $\mathcal{F}_{R,P}(\xi; a, b) > 0$, and thus $\mathcal{L}_{R,P}[z] = 0$ is disconjugate on $\mathcal{I}$ by Theorem 5.1. □

As far as the separation result is concerned, note that the implication (ii)⇒(i) from Theorem 5.1 is in fact a Sturm type separation theorem. Hence we have the following statement.

Theorem 6.2 (Sturm Separation Theorem). *Two nontrivial solutions* x *and* y *of* (3.1)*, which are not proportional, cannot have a common zero. If there are* $c_1, c_2 \in \mathbb{T}$, $c_1 < c_2$, *such that* $(rxx^\sigma)(c_1) \leq 0$ *and* $(rxx^\sigma)(c_2) \leq 0$ *(we exclude the case where* $\sigma(c_1) = c_2$ *and* $y(c_2) = 0$*), then there is* $d \in [c_1, c_2]$ *such that* $(ryy^\sigma)(d) \leq 0$.

Remark 6.1. There exist examples, e.g., in linear discrete oscillation theory, see [4], showing that two (nontrivial) nonproportional solutions of (LΔE) (with $r(t) > 0$) may have a common generalized zero (not just zero). This is possible because of the discrete nature of the domain. However, in Example 11.1 (Section 11) we present equation (11.2) possessing two nonproportional solutions that have common generalized zeros at each point regardless whether they are right-scattered or not. This is possible because we allow $r(t)$ to be also negative and we use a specific definition of generalized zeros. Since such an example is not known in the usual oscillation theory, we also discuss the problem of generalized zeros (more precisely, the problem of their definition) at right-dense points in that section.

Proof of Theorem 6.2. It is sufficient to prove the part concerning the common zero of nonproportional solutions since the remaining part follows from Theorem 5.1. Suppose, by contradiction, that $x(a) = 0 = y(a)$. Let z be the solution of (3.1) such that $z(a) = 0$, $z^\Delta(a) = 1$. Then $x = Az$ and $y = Bz$, where A, B are suitable nonzero constants, are also nontrivial solutions of (3.1) satisfying

$$x(a) = 0, \quad x^\Delta(a) = A \quad \text{and} \quad y(a) = 0, \quad y^\Delta(a) = B,$$

respectively. Hence $x = Cy$, where $C = A/B$, a contradiction. □

7 Methods of Oscillation Theory

In the following four sections we will consider equation (3.1) on $\mathcal{I}_a$ (where $\sup \mathbb{T} = \infty$) with $r(t) \neq 0$. Before we present our methods, let us define the concept of (non)oscillation of (3.1).

Definition 7.1. Equation (3.1) is said to be *nonoscillatory* if there exists $c \in \mathcal{I}_a$ such that (3.1) is disconjugate on $[c, d]$ for every $d > c$, $d \in \mathbb{T}$. In the opposite case, (3.1) is said to be *oscillatory*. Oscillation of (3.1) may be equivalently defined as follows. A nontrivial solution y of (3.1) is called *oscillatory* if it has infinitely many (isolated) generalized zeros. In view of the fact that the Sturm type separation theorem extends to (3.1), we have the following equivalence: One solution of (3.1) is oscillatory if and only if every solution of (3.1) is oscillatory. Hence we can speak about *oscillation* or *nonoscillation of equation* (3.1).

7.1 Riccati Technique

First we give an auxiliary result in which a behavior of the operator $\mathcal{S}$ with respect to its arguments is described by the properties of the function

$$S(x, y, \alpha) = \lim_{\lambda \to \mu} \frac{x}{\lambda} \left(1 - \frac{y}{\Phi(\Phi^{-1}(y) + \lambda \Phi^{-1}(x))} \right).$$

Note that the function S can be understood as a "half-linear generalization" of the function $x^2/(y + \mu x)$ that corresponds to the operator appearing in the Riccati dynamic equation associated to linear dynamic equation ($\mathrm{L}^{\Delta}\mathrm{E}$), and hence a similar behavior of these functions can be expected in a certain sense.

Lemma 7.1. *The function S has the following properties:*

(i) *Let $y > 0$. Then $x \frac{\partial S}{\partial x}(x, y, \alpha) \geq 0$ for $\Phi^{-1}(y) + \mu \Phi^{-1}(x) > 0$, where $\frac{\partial S}{\partial x}(x, y, \alpha) = 0$ if and only if $x = 0$.*

(ii) *$S(x, y, \alpha) \geq 0$ for $\Phi^{-1}(y) + \mu \Phi^{-1}(x) > 0$, where the equality holds if and only if $x = 0$.*

(iii) *If $x > 0$, $y > 0$ and*

$$\gamma := \lim_{\lambda \to \mu} \frac{(1 + \lambda z) \ln(1 + \lambda z) - \lambda z \ln z}{\lambda} \geq 0,$$

where $z := (x/y)^{\frac{1}{\alpha - 1}}$, then $\frac{\partial S}{\partial \alpha}(x, y, \alpha) \geq 0$.

(iv) *If* $\Phi^{-1}(y)+\mu\Phi^{-1}(x)>0$, *then* $\frac{\partial S}{\partial y}(x,y,\alpha)\le 0$.

Proof. (i) It is not difficult to compute that

$$\frac{\partial S(x,y,\alpha)}{\partial x}=\lim_{\lambda\to\mu}\frac{|\Phi^{-1}(y)+\lambda\Phi^{-1}(x)|^{\alpha}-|r|^{\beta}}{\lambda|\Phi^{-1}(y)+\lambda\Phi^{-1}(x)|^{\alpha}}.$$

Observe that if $\mu=0$ (which corresponds to a right-dense t in (5.1)), using the L'Hôpital rule we get

$$\frac{\partial S(x,y)}{\partial x}=\frac{\alpha\Phi^{-1}(x)}{\Phi^{-1}(y)}\frac{\partial}{\partial x}\left(\frac{(\alpha-1)|x|^{\beta}}{\Phi^{-1}(y)}\right),$$

which means

$$\lim_{\lambda\to 0}\frac{\partial}{\partial x}\left[\frac{x}{\lambda}\left(1-\frac{y}{\Phi(\Phi^{-1}(y)+\lambda\Phi^{-1}(x))}\right)\right]=\frac{\partial}{\partial x}\left(\frac{(\alpha-1)|x|^{\beta}}{\Phi^{-1}(y)}\right).$$

Since $y>0$, the statement is obvious for the case when $\mu=0$. If $\mu>0$, then we can write

$$\frac{\partial S(x,y,\alpha)}{\partial x}=\frac{\left(y^{\beta-1}+\mu\Phi^{-1}(x)\right)^{\alpha}-\left(y^{\beta-1}\right)^{\alpha}}{\mu\left(y^{\beta-1}+\mu\Phi^{-1}(x)\right)^{\alpha}}.$$

Suppose that $y>0$ and $x\ge 0$. Then obviously $S_x(x,y,\alpha)\ge 0$. If we suppose $y>0$ and $x\le 0$, then we have

$$\left(y^{\beta-1}-\mu|x|^{\beta-1}\right)^{\alpha}\le\alpha\left(y^{\beta-1}-\mu|x|^{\beta-1}\right)^{\alpha-1}\left(-\mu|x|^{\beta-1}\right)\le 0$$

and hence $\frac{\partial S}{\partial x}(x,y,\alpha)\le 0$.

(ii) The statement is obvious if $\mu=0$. Hence, suppose $\mu>0$. For the case $y>0$, (ii) follows from (i) and from the fact that $S(0,y,\alpha)=0$. One can observe that the function $S(x,y,\alpha)$ with arbitrary fixed $y<0$ and $\Phi^{-1}(y)+\mu\Phi^{-1}(x)>0$ is increasing with respect to the first variable for $x>(2/\mu)^{\alpha-1}|y|$, decreasing for $|y|<x<(2/\mu)^{\alpha-1}|y|$ and $S((2/\mu)^{\alpha-1}|y|,y,\alpha)>0$. The statement now follows from the continuity of S.

To prove the property (iii) first note that for $\mu>0$ the function S can be rewritten as

$$S(x,y,\alpha)=\frac{x}{\mu}\left[1-\left(1+\mu\left(\frac{x}{y}\right)^{\frac{1}{\alpha-1}}\right)^{1-\alpha}\right],$$

while for $\mu = 0$ it takes the the form $S(x, y, \alpha) = (\alpha - 1)x\,(x/y)^{\frac{1}{\alpha-1}}$. Differentiating S with respect to α, using the known rules, we get

$$\frac{\partial S}{\partial \alpha} = \frac{x}{\mu}(1 + \mu z)^{-\alpha}\,[(1 + \mu z)\ln(1 + \mu z) - \mu z \ln z]$$

in case when $\mu > 0$. If $\mu = 0$, then we obtain $\partial S/\partial\alpha = x(z - z \ln z)$ and, using L'Hôpital's rule, we have $\gamma = z - z \ln z$. In view of the assumptions of the lemma, Remark 7.1 (i) below, and the equality

$$\lim_{\lambda\to\mu} \frac{\partial}{\partial\alpha}\left\{\frac{x}{\lambda}\left[1 - \left(1 + \lambda\left(\frac{x}{y}\right)^{\frac{1}{\alpha-1}}\right)^{1-\alpha}\right]\right\} = \frac{\partial}{\partial\alpha}\left\{\lim_{\lambda\to\mu}\frac{x}{\lambda}\left[1 - \left(1 + \lambda\left(\frac{x}{y}\right)^{\frac{1}{\alpha-1}}\right)^{1-\alpha}\right]\right\},$$

we get the statement.

(iv) Similarly as in the first case we get

$$\frac{\partial S}{\partial y}(x, y, \alpha) = \frac{-|x|^{\beta}}{(\Phi^{-1}(y) + \mu\Phi^{-1}(x) > 0)^{\alpha}} \le 0,$$

and so the statement follows. □

Remark 7.1. (i) It is easy to see that if $\mu \ge 1$, then $\gamma \ge 0$, since

$$\gamma = \frac{\ln(1 + \mu z) + \mu z \ln \frac{1+\mu z}{z}}{\mu}.$$

On the other hand, if $\mu \in [0, 1)$, then z being small (more precisely, $z \le 1$, but in fact the right-hand side may be greater than 1, it depends on μ) is a sufficient condition for γ to be nonnegative. We notice how the graininess function plays the rôle in the monotone nature of S. Observe that S is not always nondecreasing with respect to α, even when $x, y > 0$.

(ii) In view of the last remark, if, for example, $w(t) > 0$, $r(t) > 0$, $\lim_{t\to\infty} w(t) = 0$ and $\liminf_{t\to\infty} r(t) > 0$, then $\frac{\partial S(w(t),r(t);\alpha)}{\partial\alpha} \ge 0$ for large t. It is clear that the last two conditions may be dropped when $\mu(t) \ge 1$ eventually.

The method of oscillation theory for (3.1), which uses the ideas of the following lemma (and also of Lemmas 7.4 and 7.5), is usually referred to as the *Riccati technique*. Important rôles in the proof are played by the Roundabout theorem and the Sturmian comparison theorem.

Lemma 7.2 (Riccati Technique). *a) The following statements are equivalent:*

(i) *Equation* (3.1) *is nonoscillatory.*

(ii) *There is* $a \in \mathbb{T}$ *and a function* $w : \mathcal{I}_a \to \mathbb{R}$ *such that* (5.3) *holds and*

$$\mathcal{R}[w](t) = 0 \quad \textit{for} \quad t \in \mathcal{I}_a.$$

(iii) *There is* $a \in \mathbb{T}$*, a constant* $C \in \mathbb{R}$ *and a function* $w : \mathcal{I}_a \to \mathbb{R}$ *such that* (5.3) *holds and*

$$w(t) = C - \int_a^t \{p + \mathcal{S}[w, r]\}(s)\,\Delta s \quad \textit{for} \quad t \in \mathcal{I}_a.$$

(iv) *There is* $a \in \mathbb{T}$ *and a function* $w : \mathcal{I}_a \to \mathbb{R}$ *such that* (5.3) *holds and*

$$\mathcal{R}[w](t) \leq 0 \quad \textit{for} \quad t \in \mathcal{I}_a.$$

(v) *There is* $a \in \mathbb{T}$ *and a function* $y : \mathcal{I}_a \to \mathbb{R}$

$$(ryy^\sigma)(t) > 0 \quad \textit{and} \quad \{y^\sigma \mathcal{L}_{r,p}[y]\}(t) \leq 0 \quad \textit{for} \quad t \in \mathcal{I}_a. \tag{7.1}$$

b) Suppose that there is $a \in \mathbb{T}$*, a constant* $C \in \mathbb{R}$ *and a function* $w : \mathcal{I}_a \to \mathbb{R}$ *with* (5.3) *such that* $r(t) > 0$ *and either*

$$w(t) \geq C - \int_a^t \{p + \mathcal{S}[w, r]\}(s)\,\Delta s \geq 0$$

or

$$w(t) \leq C - \int_a^t \{p + \mathcal{S}[w, r]\}(s)\,\Delta s \leq 0$$

for $t \in \mathcal{I}_a$*. Then (i)–(v) from part a) hold. If, in addition,* $\int_a^t p(s)\,\Delta s \to \infty$ *as* $t \to \infty$*, then the above condition is necessary for (i)–(v) (here we do not need to assume* $r(t) > 0$*).*

Proof. To prove part a) we show the following implications:

(i)⇒(ii): This implication follows from the Roundabout theorem since nonoscillation of (3.1) implies existence of $a \in \mathbb{T}$ such that (3.1) is disconjugate on $\mathcal{I}_a$.

(ii)⇒(iii): Trivial.

(iii)⇒(iv): Trivial.

(iv)⇒(v): Let w satisfy $\mathcal{R}[w] \le 0$ with (5.3) on $\mathcal{I}_a$. Further, $y(t) = e_{\Phi^{-1}(w/r)}(t,a),\ t \in \mathcal{I}_a$, is a solution of the initial value problem

$$y^\Delta = \Phi^{-1}\left(\frac{w(t)}{r(t)}\right) y, \quad y(a) = 1.$$

Note that this problem has exactly one solution. Clearly, $y(t) \neq 0$ since

$$\left\{1 + \mu\Phi^{-1}\left(\frac{w}{r}\right)\right\}(t) = \left\{\frac{1}{\Phi^{-1}(r)}(\Phi^{-1}(r) + \mu\Phi^{-1}(w))\right\}(t) \neq 0$$

and, moreover, $(ryy^\sigma)(t) > 0$ since

$$\left\{r\Phi\left(\frac{y^\sigma}{y}\right)\right\}(t) = \Phi(\Phi^{-1}(r) + \mu\Phi^{-1}(w))(t) > 0.$$

Further,

$$\begin{aligned} y^\sigma \mathcal{L}_{r,p}[y] &= y^\sigma[(r(t)\Phi(y^\Delta))^\Delta + p(t)\Phi(y^\sigma)] - \frac{|y^\sigma|^\alpha r(t)\Phi(y^\Delta)(\Phi(y))^\Delta}{\Phi(y)\Phi(y^\sigma)} \\ &\quad + \frac{|y^\sigma|^\alpha r(t)\Phi(y^\Delta)(\Phi(y))^\Delta}{\Phi(y)\Phi(y^\sigma)} \\ &= y^\sigma\Phi(y^\sigma)\frac{(r(t)\Phi(y^\Delta))^\Delta\Phi(y) - r(t)\Phi(y^\Delta)(\Phi(y))^\Delta}{\Phi(y)\Phi(y^\sigma)} \\ &\quad + |y^\sigma|^\alpha p(t) + |y^\sigma|^\alpha \frac{r(t)\Phi(y^\Delta)(\Phi(y))^\Delta}{\Phi(y)\Phi(y^\sigma)} \\ &= |y^\sigma|^\alpha \mathcal{R}[w] \le 0, \end{aligned}$$

hence (v) holds.

(v)⇒(i): Suppose that a function y satisfies (7.1) on $\mathcal{I}_a$. Then

$$\varphi(t) := -\left\{y^\sigma \mathcal{L}_{r,p}[y]\right\}(t)$$

is a nonnegative function on this interval. Set $\bar{r}(t) = r(t)$ and $\bar{p}(t) = p(t) - \varphi(t)/|y^\sigma|^\alpha$. Then $\bar{p} \ge p$ and

$$(\bar{r}(t)\Phi(y^\Delta))^\Delta + \bar{p}(t)\Phi(y^\sigma) = (r(t)\Phi(y^\Delta))^\Delta + \left(p(t) - \frac{\varphi(t)}{|y^\sigma|^\alpha}\right)\Phi(y^\sigma) = 0.$$

Thus the equation $(\bar{r}(t)\Phi(y^\Delta))^\Delta + \bar{p}(t)\Phi(y^\sigma) = 0$ is disconjugate on $\mathcal{I}_a$. Therefore, (3.1) is disconjugate on $\mathcal{I}_a$ as well by the Sturm type comparison theorem (Theorem 6.1) and hence nonoscillatory.

To prove part b) we show that the assumptions imply (iv). Let

$$v(t) = C - \int_a^t \{p + \mathcal{S}[w, r]\}(s)\,\Delta s.$$

Then $v^\Delta + p(t) + \mathcal{S}[w, r](t) = 0$. Obviously,

$$\{\Phi^{-1}(r) + \mu\Phi^{-1}(v)\}(t) \geq \{\Phi^{-1}(r) + \mu\min\{0, w\}\}(t) > 0.$$

We have $w \geq v \geq 0$ or $w \leq v \leq 0$ and hence $\mathcal{S}[w, r] \geq \mathcal{S}[v, r]$ by Lemma 7.1. Therefore we get $\mathcal{R}[v](t) \leq 0$ for $t \in \mathcal{I}_a$ and so (iv) holds. The part concerning necessity is obvious. □

In the next lemmas we describe other asymptotic properties of solutions of (5.1). First we show that, under certain assumptions, an eventually positive solution of a (nonoscillatory) equation (3.1) has an eventually positive delta derivative. Consequently, (5.1) has a positive solution.

Lemma 7.3. *Assume*

$$\liminf_{t\to\infty} \int_T^t p(s)\Delta s \geq 0 \quad \textit{and} \quad \not\equiv 0 \tag{7.2}$$

for all large T *and*

$$r(t) > 0, \quad \int_a^\infty r^{1-\beta}(s)\Delta s = \infty. \tag{7.3}$$

If y *is a solution of* (3.1) *such that* $y(t) > 0$ *for* $t \in [T, \infty)$*, then there exists* $S \in [T, \infty)$ *such that* $y^\Delta(t) > 0$ *for* $t \in [S, \infty)$*.*

Proof. The proof is by contradiction. We consider two cases:

Case I. Suppose that $y^\Delta(t) < 0$ for $t \in [T, \infty)$. Then also $[\Phi(y)]^\Delta(t) < 0$ for $t \in [T, \infty)$ since

$$[\Phi(y)]^\Delta(t) = \frac{d}{dy}\Phi[y(\xi)]y^\Delta(t) = (\alpha - 1)|y(\xi)|^{\alpha-2}y^\Delta(t) < 0$$

by Lemma 2.1 (vii), where $t \leq \xi \leq \sigma(t)$. Another argument for $[\Phi(y)]^\Delta(t) < 0$ is that if y is decreasing, then $\Phi(y)$ is decreasing as well because of the properties of the function Φ. Without loss of generality we may assume that T is such that $\int_T^t p(s)\Delta s \geq 0$, $t \in [T, \infty)$, reasoning as in [19, Proof of Lemma 13]. Define $Q(t, T) = \int_T^t p(s)\Delta s$. Integration by parts gives

$$\int_T^t p(s)\Phi(y^\sigma(s))\Delta s \quad = \quad \int_T^t Q^\Delta(s, T)\Phi(y^\sigma(s))\Delta s$$

$$= \quad Q(t,T)\Phi(y(t)) - \int_T^t Q(s,T)[\Phi(y(s))]^\Delta \Delta s \geq 0.$$

Integrating (3.1) we have, using the last estimate,

$$r(t)\Phi(y^\Delta(t)) - r(T)\Phi(y^\Delta(T)) = \int_T^t [r(s)\Phi(y^\Delta(s))]^\Delta \Delta s \leq 0.$$

Hence

$$y^\Delta(t) \leq \frac{r^{\beta-1}(T)y^\Delta(T)}{r^{\beta-1}(t)} \tag{7.4}$$

for $t \in [T,\infty)$. Integrating (7.4) for $t \geq T$ we see that $y(t) \to -\infty$ by (7.3), a contradiction. Therefore, $y^\Delta(t) < 0$ cannot hold for all large t.

Case II. Next, if $y^\Delta(t) \not> 0$ eventually, then for every (large) $T \in \mathcal{I}_a$ there exists $T_0 \in [T,\infty)$ such that $y^\Delta(T_0) \leq 0$ and we may suppose that $\liminf_{t\to\infty} \int_T^t p(s)\Delta s \geq 0$. Since $y(t) > 0$ for $t \in [T,\infty)$, the function $w(t) = r(t)\Phi\left[y^\Delta(t)/y(t)\right]$ satisfies the generalized Riccati equation (5.1) with (5.3) for $t \in [T,\infty)$. Integrating (5.1) from T_0 to t, $t \geq T_0$, gives

$$w(t) = w(T_0) - \int_{T_0}^t p(s)\Delta s - \int_{T_0}^t \mathcal{S}(w,r)(s)\Delta s.$$

Therefore, it follows that $\limsup_{t\to\infty} w(t) < 0$, using the facts $w(T_0) \leq 0$, $w(t)$ is eventually nontrivial, and (7.2). Indeed, there is $M > 0$ such that $\int_{T_0}^t \mathcal{S}(w,r)(s)\Delta s \geq M$ and $\int_{T_0}^t p(s)\Delta s \geq -M/2$ for all large t. Hence there exists $T_1 \in [T,\infty)$ such that $w(t) < 0$ for $t \in [T_1,\infty)$ and so $y^\Delta(t) < 0$ for $t \in [T_1,\infty)$, a contradiction to the first part. □

In the next lemma, a necessary condition for nonoscillation of (3.1) is given in terms of solvability of a generalized Riccati integral inequality involving improper integrals, under certain assumptions. Note that a closer examination of the proof of Theorem 10.1 shows that this condition is also sufficient.

Lemma 7.4. *Let the assumptions of Lemma 7.3 hold and assume further that*

$$\int_a^\infty p(s)\Delta s = \lim_{t\to\infty} \int_a^t p(s)\Delta s \quad \text{is convergent.} \tag{7.5}$$

Let y be a solution of (3.1) such that $y(t) > 0$ for $t \in [T,\infty)$. Then there exists $T_1 \in [T,\infty)$ such that

$$w(t) \geq \int_t^\infty p(s)\Delta s + \int_t^\infty \mathcal{S}(w,r)(s)\Delta s \tag{7.6}$$

for $t \in [T_1,\infty)$, where $w(t) = r(t)\Phi\left[y^\Delta(t)/y(t)\right] > 0$.

Proof. By Lemma 7.3 there exists $T_1 \in [T, \infty)$ such that $w(t) > 0$ for $t \in [T_1, \infty)$ and w satisfies (5.1) for $t \in [T, \infty)$ (clearly, with (5.3)). Integrating (5.1) from t to s, $s \geq t \geq T_1$, gives

$$w(s) - w(t) + \int_t^s p(\xi)\Delta\xi + \int_t^s \mathcal{S}(w, r)(\xi)\Delta\xi = 0. \tag{7.7}$$

Therefore,

$$0 < w(s) = w(t) - \int_t^s p(\xi)\Delta\xi - \int_t^s \mathcal{S}(w, r)(\xi)\Delta\xi,$$

and hence

$$w(t) \geq \int_t^s p(\xi)\Delta\xi + \int_t^s \mathcal{S}(w, r)(\xi)\Delta\xi$$

for $s \geq t \geq T_1$. Letting $s \to \infty$ we obtain (7.6). □

If we strengthen the assumptions of the previous lemma somewhat, then we may prove that there is a solution of (5.1) which is monotone, tending to zero and satisfies a certain integral equation involving improper integrals.

Lemma 7.5. *Suppose that* (7.3) *and* (7.5) *hold with* $p(t) \geq 0$ *(which is eventually nontrivial). Let* y *be a solution of* (3.1) *such that* $y(t) > 0$ *for* $t \in [T, \infty)$*. Then there exists* $T_1 \in [T, \infty)$ *such that the function* $w(t) = r(t)\Phi\left[y^\Delta(t)/y(t)\right] > 0$ *is positive, nonincreasing, tends to zero and satisfies*

$$w(t) = \int_t^\infty p(s)\Delta s + \int_t^\infty \mathcal{S}(w, r)(s)\Delta s \tag{7.8}$$

for $t \in [T_1, \infty)$*.*

Proof. From Lemma 7.3 there exists $T_1 \in [T, \infty)$ such that $w(t) > 0$ for $t \in [T_1, \infty)$. Note that this can actually be proved even much easier since $p(t)$ is nonnegative here. Further, w satisfies (5.1) with (5.3) on $[T, \infty)$. The fact that $w^\Delta(t) \leq 0$ for $t \in [T_1, \infty)$ follows from (5.1). Next we show that $w(t) \to 0$ as $t \to \infty$. Since y is positive and increasing, it either converges to a positive constant L or diverges to ∞. First suppose that $y(t) \to \infty$ as $t \to \infty$. Then, since $r(t)\Phi[y^\Delta(t)]$ is nonincreasing, we have

$$w(t) = \frac{r(t)\Phi[y^\Delta(t)]}{\Phi[y(t)]} \leq \frac{r(T_1)\Phi[y^\Delta(T_1)]}{\Phi[y(t)]} \to 0$$

as $t \to \infty$. Now, if $y(t) \to L$ as $t \to \infty$, then $r(t)\Phi[y^\Delta(t)] \to 0$ as $t \to \infty$ and, consequently, $w(t)$ tends to zero as $t \to \infty$. To see that $r(t)\Phi[y^\Delta(t)]$

converges to zero, note first that it converges since it is positive and nonincreasing. If, however, $r(t)\Phi[y^{\Delta}(t)]$ converges to a positive constant K, then we get

$$y(t) \geq y(T_1) + K^{\beta-1} \int_{T_1}^{t} r^{1-\beta}(s)\,\Delta s \to \infty$$

as $t \to \infty$, which contradicts the boundedness of y. Finally, the fact that y satisfies (7.8) on $[T_1, \infty)$ follows from (7.7). □

Remark 7.2. Clearly, solvability of (7.8) is also a sufficient condition for nonoscillation of (3.1).

7.2 Variational Principle

Another method, known from the oscillation theory of (LDE) and now extended also to the half-linear time scales case, is the so-called *variational principle*. It is based on the equivalence (i)⇔(iv) from the Roundabout theorem. More precisely, we use the following two lemmas.

Lemma 7.6. *Equation* (3.1) *is nonoscillatory if and only if there exists* $a \in \mathbb{T}$ *such that*

$$\mathcal{F}(\xi; a, \infty) = \int_{a}^{\infty} \left\{ r|\xi^{\Delta}|^{\alpha} - p|\xi^{\sigma}|^{\alpha} \right\}(t)\,\Delta t > 0$$

for every nontrivial $\xi \in U(a)$*, where*

$$U(a) := \{\xi \in C_p^1(\mathcal{I}_a, \mathbb{R}) : \ \exists b > a \text{ with } \xi(t) = 0 \text{ if } t \notin (a,b)\}.$$

Lemma 7.7. *Equation* (3.1) *is oscillatory if and only if for any* $a \in \mathbb{T}$ *there exists a (nontrivial) admissible function* $\xi \in U(a)$ *with* $\mathcal{F}(\xi; a, \infty) \leq 0$.

8 Oscillation Criteria

As an application of the above described methods we give oscillation (in this section) and nonoscillation (in the next section) criteria for (3.1). Our oscillation criteria unify and extend those presented in [12, 13, 14, 38, 41]. We start with an extension of the well-known Leighton–Wintner criterion (see also [6, 10]).

Theorem 8.1. *Suppose that* (7.3) *holds. If*

$$\int_{a}^{\infty} p(t)\,\Delta t = \infty, \tag{8.1}$$

then (3.1) *is oscillatory.*

Proof. According to Lemma 7.7 it is sufficient to find for any $c \in \mathcal{I}_a$ a nontrivial function ξ satisfying $\xi(t) = 0$ for $t \in [a, c] \cup [d, \infty)$, where $c < d$, $d \in \mathcal{I}_a$ (then ξ is admissible), such that

$$\mathcal{F}(\xi; c, d) = \int_c^d \left\{ r|\xi^\Delta|^\alpha - p|\xi^\sigma|^\alpha \right\}(t)\,\Delta t \le 0.$$

Let $t_1, t_2, t_3, t_4 \in \mathcal{I}_a$ be such that $a \le t_1 < t_2 < t_3 < t_4$. Define the function ξ by

$$\xi(t) = \begin{cases} 0 & \text{for } t \in [a, t_1], \\ \left(\int_{t_1}^t r^{1-\beta}(s)\,\Delta s\right)\left(\int_{t_1}^{t_2} r^{1-\beta}(s)\,\Delta s\right)^{-1} & \text{for } t \in [t_1, t_2], \\ 1 & \text{for } t \in [t_2, t_3], \\ \left(\int_t^{t_4} r^{1-\beta}(s)\,\Delta s\right)\left(\int_{t_3}^{t_4} r^{1-\beta}(s)\,\Delta s\right)^{-1} & \text{for } t \in [t_3, t_4], \\ 0 & \text{for } t \in [t_4, \infty), \end{cases}$$

which yields $\xi(t_1) = \xi(t_4) = 0$, $\xi(t) > 0$ for $t \in (t_1, t_4)$ and $\xi \in C^1_{\mathrm{p}}(\mathcal{I}_a)$. Using integration by parts, we have

$$\begin{aligned} \int_{t_1}^{t_4} r(t)|\xi^\Delta(t)|^\alpha \Delta\tau &= \int_{t_1}^{t_2} r(t)|\xi^\Delta(t)|^\alpha\,\Delta t + \int_{t_2}^{t_3} r(t)|\xi^\Delta(t)|^\alpha\,\Delta t \\ &\quad + \int_{t_3}^{t_4} r(t)|\xi^\Delta(t)|^\alpha\,\Delta t \\ &= \left[\left\{\xi r\Phi(\xi^\Delta)\right\}(t)\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \left\{\xi^\sigma (r\Phi(\xi^\Delta))^\Delta\right\}(t)\,\Delta t \\ &\quad + \left[\left\{\xi r\Phi(\xi^\Delta)\right\}(t)\right]_{t_3}^{t_4} - \int_{t_3}^{t_4} \left\{\xi^\sigma (r\Phi(\xi^\Delta))^\Delta\right\}(t)\,\Delta t \\ &= \xi(t_2)r(t_2)\Phi(\xi^\Delta(t_2)) - \xi(t_3)r(t_3)\Phi(\xi^\Delta(t_3)) \\ &= \left(\int_{t_1}^{t_2} r^{1-\beta}(t)\,\Delta t\right)^{1-\alpha} + \left(\int_{t_3}^{t_4} r^{1-\beta}(t)\,\Delta t\right)^{1-\alpha}. \end{aligned}$$

Further, $|\xi|^\alpha \in C^1_{\mathrm{rd}}$ is monotone on $[t_1, t_2]$ and on $[t_3, t_4]$ since $(|\xi|^\alpha)^\Delta$ is equal to

$$\alpha|\eta_1(t)|^{\alpha-1}\xi^\Delta(t) = \alpha|\eta_1(t)|^{\alpha-1} r^{1-\beta}(t)\left(\int_{t_1}^{t_2} r^{1-\beta}(s)\,\Delta s\right)^{-1} > 0$$

for $t \in [t_1, t_2]^\kappa$, and equal to

$$\alpha|\eta_2(t)|^{\alpha-1}\xi^\Delta(t) = -\alpha|\eta_1(t)|^{\alpha-1} r^{1-\beta}(t)\left(\int_{t_1}^{t_2} r^{1-\beta}(s)\,\Delta s\right)^{-1} < 0$$

for $t \in [t_3, t_4]^\kappa$, where $\eta_1(t), \eta_2(t)$ are between $\xi(t)$ and $\xi^\sigma(t)$. Hence, by Lemma 2.5, there exists $s_1 \in [t_1, t_2]^\kappa$ such that

$$\begin{aligned} \int_{t_1}^{t_2} p(t)|\xi^\sigma(t)|^\alpha \Delta t &\geq |\xi(t_1)|^\alpha \int_{t_1}^{s_1} p(s)\,\Delta t + |\xi(t_2)|^\alpha \int_{s_1}^{t_2} p(s)\,\Delta t \\ &= \int_{s_1}^{t_2} p(t)\,\Delta t \end{aligned}$$

and, similarly, there exists $s_2 \in [t_3, t_4]^\kappa$ for which

$$\begin{aligned} \int_{t_3}^{t_4} p(t)|\xi^\sigma(t)|^\alpha \Delta t &\geq |\xi(t_3)|^\alpha \int_{t_3}^{s_2} p(s)\,\Delta t + |\xi(t_4)|^\alpha \int_{s_2}^{t_4} p(s)\,\Delta t \\ &= \int_{t_3}^{s_2} p(t)\,\Delta t. \end{aligned}$$

Using these estimates we get

$$\mathcal{F}(\xi) \leq \left(\int_{t_1}^{t_2} r^{1-\beta}(t)\,\Delta t \right)^{1-\alpha} + \left(\int_{t_3}^{t_4} r^{1-\beta}(t)\,\Delta t \right)^{1-\alpha} - \int_{s_1}^{s_2} p(t)\,\Delta t.$$

Now, denote $M = \left(\int_{t_1}^{t_2} r^{1-\beta}(t)\,\Delta t \right)^{1-\alpha}$ and let $\varepsilon > 0$ be arbitrary. According to (8.1), the point t_3 can be chosen in such a way that $\int_{s_1}^{t} p(s)\,\Delta s \geq M + \varepsilon$ whenever $t \in [t_3, \infty)$. Since (7.3) holds, we have $\left(\int_{t_3}^{t_4} r^{1-\beta}(s)\,\Delta s \right)^{1-\alpha} \leq \varepsilon$, if t_4 is sufficiently large. Summarizing the above estimates, if t_3, t_4 are sufficiently large, then we have $\mathcal{F}(\xi; t_1, t_4) \leq M + \varepsilon - (M + \varepsilon) = 0$, which completes the proof. □

In the case when $\int_a^\infty p(t)\,\Delta t$ is convergent, we can use the following criterion, which can be understood as a generalization of the Hinton–Lewis criterion. Denote

$$\mathcal{A}(t) := \left(\int_a^t r^{1-\beta}(s)\,\Delta s \right)^{\alpha-1} \int_t^\infty p(s)\,\Delta s.$$

Theorem 8.2. *Suppose that* (7.3) *and* (7.5) *hold. If*

$$\lim_{t\to\infty} \mathcal{A}(t) > 1, \tag{8.2}$$

then (3.1) *is oscillatory.*

A closer examination of the proof shows that "lim" in (8.2) can be replaced by "lim inf".

Proof. Let the function ξ be the same as in the proof of the previous theorem. Hence we have

$$\mathcal{F}(\xi) \le \left(\int_{t_1}^{t_2} r^{1-\beta}(t)\,\Delta t\right)^{1-\alpha} + \left(\int_{t_3}^{t_4} r^{1-\beta}(t)\,\Delta t\right)^{1-\alpha} - \int_{s_1}^{s_2} p(t)\,\Delta t,$$

where $a \le t_1 < t_2 < t_3 < t_4$, $s_1 \in [t_1, t_2]^\kappa$ and $s_2 \in [t_3, t_4]^\kappa$. Now, let $\varepsilon > 0$ be such that the limit in (8.2) is greater than or equal to $1 + 4\varepsilon$. According to (8.2), t_1 may be chosen in such a way that $\mathcal{A}(t) \ge 1 + 3\varepsilon$ for $t \in [t_1, \infty)$. Obviously, there is $t_2 > t_1$ such that

$$\left(\int_a^{t_2} r^{1-\beta}(t)\,\Delta t\right)^{\alpha-1} \left(\int_{t_1}^{t_2} r^{1-\beta}(t)\,\Delta t\right)^{1-\alpha} \le 1 + \varepsilon.$$

In view of the fact that $\mathcal{A}(t) \ge 1 + 3\varepsilon$, there is $t_3 > t_2$ such that

$$\left(\int_a^{t} r^{1-\beta}(s)\,\Delta s\right)^{\alpha-1} \int_t^{\tau} p(s)\,\Delta s \ge 1 + 2\varepsilon$$

for $\tau \in [t_3, \infty)$. Finally, since (7.3) holds, we have

$$\left(\int_a^{t_2} r^{1-\beta}(t)\,\Delta t\right)^{\alpha-1} \left(\int_{t_3}^{t_4} r^{1-\beta}(t)\,\Delta t\right)^{1-\alpha} \le \varepsilon,$$

if t_4 is sufficiently large. Using these estimates, the fact that $\int_a^t r^{1-\beta}(s)\,\Delta s$ is positive and increasing with respect to $t \in \mathcal{I}_a$ (this means that we have $\int_a^{t_2} r^{1-\beta}(s)\,\Delta s \ge \int_a^{s_1} r^{1-\beta}(s)\,\Delta s$) and $\int_{s_1}^{s_2} p(s)\,\Delta s > 0$ if $s_1, s_2 \in \mathcal{I}_a$ are sufficiently large, we get

$$\mathcal{F}(\xi) \le \left(\int_a^{t_2} r^{1-\beta}(t)\,\Delta t\right)^{1-\alpha} (1 + \varepsilon + \varepsilon - 1 - 2\varepsilon) = 0,$$

which yields the desired result. □

In the next theorem we prove by using the Riccati technique that if some assumptions of Theorem 8.2 are strengthened, then the constant on the right-hand side of (8.2) can be lowered. We can also see how the application of different methods to the same problems gives different results.

Theorem 8.3. *Suppose that* (7.3) *and* (7.5) *hold with* $p(t) \ge 0$. *If there exists a constant* $M > 0$ *such that*

$$\mu(t) r^{1-\beta}(t) \le M \quad \textit{for all large } t \tag{8.3}$$

and

$$\liminf_{t\to\infty} \mathcal{A}(t) > \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}, \tag{8.4}$$

then (3.1) *is oscillatory.*

Proof. Suppose by contradiction that (3.1) is nonoscillatory. Then the assumptions of Lemma 7.5 are satisfied and so there exists a positive decreasing function w converging to zero, which satisfies (7.8) for large t. Multiplying this equation by $\left(\int_a^t r^{1-\beta}(s)\,\Delta s\right)^{\alpha-1}$, we get

$$w(t)\left(\int_a^t r^{1-\beta}(s)\,\Delta s\right)^{\alpha-1} = \frac{\int_t^\infty \mathcal{S}(w,r)(s)\,\Delta s}{\left(\int_a^t r^{1-\beta}(s)\,\Delta s\right)^{1-\alpha}} + \mathcal{A}(t). \tag{8.5}$$

Suppose that $\gamma := \liminf_{t\to\infty}\left(\int_a^t r^{1-\beta}(s)\,\Delta s\right)^{\alpha-1} w(t) < \infty$ (if $\gamma=\infty$, then a contradiction can be obtained even easier than for $\gamma<\infty$, as it can be seen from the text below). We claim that (8.3), (8.4) and (8.5) imply the existence of $\varepsilon>0$ such that

$$\gamma > \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} + \varepsilon + \gamma^\beta. \tag{8.6}$$

To see this, first note that the expression

$$\left(\int_t^\infty \mathcal{S}(w,r)(s)\,\Delta s\right)\left(\int_a^t r^{1-\beta}(s)\,\Delta s\right)^{\alpha-1}$$

is in fact of the type "0/0", and so L'Hôpital's rule on time scales (see [3, 8]) can be used in order to investigate its behavior in a neighborhood of ∞. Further, the identities

$$\frac{\left[\int_t^\infty \mathcal{S}(w,r)(s)\,\Delta s\right]^\Delta}{\left[\left(\int_a^t r^{1-\beta}(s)\,\Delta s\right)^{\alpha-1}\right]^\Delta} = \frac{-\mathcal{S}(w,r)(t)}{\theta^{-\alpha}(t)r^{1-\beta}(t)}$$

$$\left\{\frac{\eta^{\alpha-2}w^\beta\theta^\alpha r^{\beta-1}}{(r^{\beta-1}+\mu w^{\beta-1})^{\alpha-1}}\right\}(t) =: H(t)$$

hold, where $\int_a^t r^{1-\beta}(s)\,\Delta s \le \theta(t) \le \int_a^{\sigma(t)} r^{1-\beta}(s)\,\Delta s$ and $r^{\beta-1}(t) \le \eta(t) \le r^{\beta-1}(t)+\mu(t)w^{\beta-1}(t)$, by Lemma 2.1 (vii) and the alternative form of $\mathcal{S}$ given by (5.2). The function H can be estimated as

$$H(t) \;\ge\; w(t)\left(\int_a^t r^{1-\beta}(s)\,\Delta s\right)^{\alpha-1} r^{\beta-1}(t)\frac{(r^{\beta-1}+\delta_{\alpha-2}\mu w^{\beta-1})^{\alpha-2}(t)}{(r^{\beta-1}+\mu w^{\beta-1})^{\alpha-1}(t)}$$

$$= \quad w(t)\left(\int_a^t r^{1-\beta}(s)\,\Delta s\right)^{\alpha-1}\frac{(1+\delta_{\alpha-2}\mu(w/r)^{\beta-1})^{\alpha-2}(t)}{(1+\mu(w/r)^{\beta-1})^{\alpha-1}(t)},$$

where

$$\delta_{\alpha-2} = \begin{cases} 1 & \text{for } 1<\alpha\le 2, \\ 0 & \text{for } \alpha\ge 2. \end{cases}$$

Now it is easy to see that for any $\varepsilon_1 > 0$ one has $H(t) \ge \gamma^\beta(1-\varepsilon_1)$ for all sufficiently large t. Moreover, condition (8.4) implies the existence of $\varepsilon_2 > 0$ such that $\mathcal{A}(t) > \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} + \varepsilon_2$ for sufficiently large t. Clearly, ε_1 and ε_2 can be made to satisfy $\varepsilon_2 - \varepsilon_1\gamma^\beta =: \varepsilon > 0$, and so (8.6) follows. Since $|\lambda|^\beta - \lambda + \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} \ge 0$ for all $\lambda \in \mathbb{R}$, we get a contradiction. □

9 Nonoscillation Criteria

As a further application of the introduced methods we give two nonoscillation criteria that generalize those presented in [13, 15, 26, 38].

Theorem 9.1. *Suppose that* (7.3) *and* (7.5) *hold. Further assume that*

$$\lim_{t\to\infty}\frac{\mu(t)r^{1-\beta}(t)}{\int_a^t r^{1-\beta}(s)\,\Delta s} = 0. \tag{9.1}$$

If

$$-\frac{2\alpha-1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} < \liminf_{t\to\infty}\mathcal{A}(t) \le \limsup_{t\to\infty}\mathcal{A}(t) < \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}, \tag{9.2}$$

then (3.1) *is nonoscillatory.*

Proof. By Lemma 7.2 it is sufficient to show that the generalized Riccati dynamic inequality $\mathcal{R}[w](t) \le 0$ has a solution w with (5.3) in a neighborhood of infinity. Set

$$w(t) = C\left(\int_a^t r^{1-\beta}(s)\,\Delta s\right)^{1-\alpha} + \int_t^\infty p(s)\,\Delta s,$$

where C is a suitable constant which will be specified later. By Lemma 2.1 we have

$$\left[\left(\int_a^t r^{1-\beta}(s)\,\Delta s\right)^{1-\alpha}\right]^\Delta = (1-\alpha)r^{1-\beta}(t)\theta^{-\alpha}(t), \tag{9.3}$$

where $\int_a^t r^{1-\beta}(s)\,\Delta s \le \theta(t) \le \int_a^{\sigma(t)} r^{1-\beta}(s)\,\Delta s$. Further, we know that for the function η appearing in the operator $\mathcal{S}$ (in the form defined by (5.2)), η is between $\Phi^{-1}(r)$ and $\Phi^{-1}(r)+\mu\Phi^{-1}(w)$. Hence

$$r^{\beta-1} - \mu|w|^{\beta-1} \le \eta \le r^{\beta-1} + \mu|w|^{\beta-1}.$$

Let $C = ((\alpha-1)/\alpha)^{\alpha}$. Then

$$\begin{aligned}\left\{\frac{\mu|w|^{\beta-1}}{r^{\beta-1}}\right\}(t) &= \frac{\mu(t)}{r^{\beta-1}(t)}\left|C\left(\int_a^t r^{1-\beta}(s)\Delta s\right)^{1-\alpha} + \int_t^\infty p(s)\Delta s\right|^{\beta-1} \\ &= \frac{\mu(t)r^{1-\beta}(t)}{\int_a^t r^{1-\beta}(s)\,\Delta s}|\mathcal{A}(t)+C|^{\beta-1} \to 0\end{aligned}$$

as $t\to\infty$ according to (9.1) and therefore

$$r^{\beta-1} + \mu\Phi^{-1}(w) = r^{\beta-1}\left(1+\frac{\mu\Phi^{-1}(w)}{r^{\beta-1}}\right) > 0$$

for large $t\in\mathbb{T}$ and so (5.3) holds. Further, the assumption (9.2) implies the existence of $\bar{\varepsilon}>0$ such that

$$-\frac{2\alpha-1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} + 2\bar{\varepsilon} < \mathcal{A}(t) < \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} - 2\bar{\varepsilon}$$

for t sufficiently large, say $t\in[t_1,\infty)$. Therefore, $|C+\mathcal{A}(t)|+\bar{\varepsilon} < C^{\frac{1}{\beta}} - \bar{\varepsilon}$. This clearly implies the existence of $\varepsilon>0$ such that $|C+\mathcal{A}(t)|(1+\varepsilon)^{\frac{1}{\beta}} < C^{\frac{1}{\beta}}(1-\varepsilon)^{\frac{1}{\beta}}$ and hence

$$|C+\mathcal{A}(t)|^{\beta}(1+\varepsilon) < C(1-\varepsilon) \tag{9.4}$$

for $t\in[t_1,\infty)$. Now, for a given $\varepsilon>0$, there exists $t_2\in\mathbb{T}$ such that $r(t) > \mu(t)|w(t)|^{\beta-1}$ and

$$\left(\frac{\int_a^t r^{1-\beta}(s)\,\Delta s}{\theta(t)}\right)^{\alpha} \ge \left(\frac{\int_a^t r^{1-\beta}(s)\,\Delta s}{\int_a^{\sigma(t)} r^{1-\beta}(s)\,\Delta s}\right)^{\alpha} > 1-\varepsilon$$

for $t\in[t_2,\infty)$, since (9.1) implies

$$\lim_{t\to\infty}\frac{\int_a^{\sigma(t)} r^{1-\beta}(s)\,\Delta s}{\int_a^t r^{1-\beta}(s)\,\Delta s} = \lim_{t\to\infty}\frac{\int_a^t r^{1-\beta}(s)\,\Delta s + \int_t^{\sigma(t)} r^{1-\beta}(s)\,\Delta s}{\int_a^t r^{1-\beta}(s)\,\Delta s}$$

$$= \lim_{t\to\infty}\left(1+\frac{\mu(t)r^{1-\beta}(t)}{\int_a^t r^{1-\beta}(s)\,\Delta s}\right)=1.$$

Let $t_2 \in \mathbb{T}$ be at the same time so large that the following estimate holds for $t \in [t_2, \infty)$:

$$\begin{aligned}\frac{|\eta|^{\alpha-2}r^{\beta-1}}{(|r|^{\beta-1}+\mu\Phi^{-1}(w))^{\alpha-1}} &\le \frac{r^{\beta-1}[r^{\beta-1}+\operatorname{sgn}(\alpha-2)\mu|w|^{\beta-1}]^{\alpha-2}}{r(1+\mu\Phi^{-1}(w/r))^{\alpha-1}}\\ &= \frac{[1+\operatorname{sgn}(\alpha-2)\mu|w/r|^{\beta-1}]^{\alpha-2}}{(1+\mu\Phi^{-1}(w/r))^{\alpha-1}} < 1+\varepsilon.\end{aligned}$$

Multiplying (9.4) by $(\alpha-1)r^{1-\beta}\left(\int_a^t r^{1-\beta}(s)\,\Delta s\right)^{-\alpha}$ and using the above estimates, at each $t \in [\max\{t_1, t_2\}, \infty)$ we have

$$\begin{aligned}0 &> -(\alpha-1)Cr^{1-\beta}\left(\int_a^t r^{1-\beta}(s)\,\Delta s\right)^{-\alpha}(1+\varepsilon)\\ &\quad +(\alpha-1)|C+\mathcal{A}|^{\beta}r^{1-\beta}\left(\int_a^t r^{1-\beta}(s)\,\Delta s\right)^{-\alpha}(1+\varepsilon)\\ &> (1-\alpha)Cr^{1-\beta}\theta^{-\alpha}-p+p\\ &\quad +\frac{(\alpha-1)|C+\mathcal{A}|^{\beta}r^{1-\beta}\left(\int_a^t r^{1-\beta}(s)\,\Delta s\right)^{-\alpha}|\eta|^{\alpha-2}}{(\Phi^{-1}(r)+\mu\Phi^{-1}(w))^{\alpha-1}}\\ &= w^{\Delta}+p+\frac{(\alpha-1)|w|^{\beta}|\eta|^{\alpha-2}}{(\Phi^{-1}(r)+\mu\Phi^{-1}(w))^{\alpha-1}}\\ &= w^{\Delta}+p+\mathcal{S}[w,r],\end{aligned}$$

which completes the proof. ☐

The following theorem completes the previous statement and treats the "complementary" case $\int_a^\infty r^{1-\beta}(s)\,\Delta s < \infty$.

Theorem 9.2. *Suppose that $r(t) > 0$ for $t \in \mathcal{I}_a$, $\int_a^\infty r^{1-\beta}(s)\,\Delta s$ is convergent and*

$$\lim_{t\to\infty}\frac{\mu(t)r^{1-\beta}(t)}{\int_t^\infty r^{1-\beta}(s)\,\Delta s}=0. \tag{9.5}$$

If

$$\limsup_{t\to\infty}\left(\int_t^\infty r^{1-\beta}(s)\,\Delta s\right)^{\alpha-1}\int_a^t p(s)\,\Delta s < \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}$$

and

$$\liminf_{t\to\infty}\left(\int_t^\infty r^{1-\beta}(s)\,\Delta s\right)^{\alpha-1}\int_a^t p(s)\,\Delta s > -\frac{2\alpha-1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} \tag{9.6}$$

then (3.1) *is nonoscillatory.*

Proof. We set

$$w(t) = C\left(\int_t^\infty r^{1-\beta}(s)\,\Delta s\right)^{1-\alpha} + \int_a^t p(s)\,\Delta s,$$

and then the proof is similar to that of the previous theorem. □

Remark 9.1. We can see that conditions (9.1) and (9.5) (and also (8.3)) hold trivially in the continuous case. However, we needed these conditions in our proofs and it is an open problem whether they are really necessary or they are just needed because of the methods we used. A closer examination of the proofs reveals that the main reason why we needed these assumptions is the absence of the "natural" chain rule for computing the derivative of composed functions like in (9.3).

Let $\lambda \le 0$. Denote by $\omega_{\max}(\lambda)$ the greatest root of the equation $|x|^{1/\beta} + x + \lambda = 0$. If $\liminf_{t\to\infty}\mathcal{A}(t) > -\frac{2\alpha-1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}$ in (9.2) fails to hold, then we have the following criterion that completes Theorem 9.1.

Theorem 9.3. *Assume that* (7.3), (7.5) *and* (9.1) *hold. If*

$$\limsup_{t\to\infty}\mathcal{A}(t) < \left[\omega_{\max}\left(\liminf_{t\to\infty}\mathcal{A}(t)\right)\right]^{\frac{1}{\beta}} - \omega_{\max}\left(\liminf_{t\to\infty}\mathcal{A}(t)\right)$$

and

$$-\infty < \liminf_{t\to\infty}\mathcal{A}(t) \le -\frac{2\alpha-1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1},$$

then (3.1) *is nonoscillatory.*

Proof. The proof is similar to that of [26, Theorem 1.6]. □

Remark 9.2. If (9.6) fails to hold, then, similarly as above, we can state a complementary criterion to Theorem 9.2. The details are left to the reader.

Before the next statement, which is a generalization of [26, Theorem 1.9] and [38, Theorem 9], we introduce some notation. Set

$$\mathcal{B}(t) = \left(\int_a^t r^{1-\beta}(s)\,\Delta s\right)^{-1}\int_a^t\left(\int_a^s r^{1-\beta}(u)\,\Delta u\right)^{\alpha} p(s)\,\Delta s.$$

Denote by $\gamma_{\min}$ the smallest root of the equation

$$(\alpha-1)|x|^{\beta}+\alpha x+\frac{2\alpha-1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}=0.$$

Theorem 9.4. *Assume that* (7.3), (7.5) *and* (9.1) *hold. If*

$$\gamma_{\min}+\frac{2\alpha-1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}<\liminf_{t\to\infty}\mathcal{B}(t)\le\limsup\mathcal{B}(t)<\left(\frac{\alpha-1}{\alpha}\right)^{\alpha}, \tag{9.7}$$

then (3.1) *is nonoscillatory.*

Proof. In order to prove the statement, the technique of the proof of Theorem 9.1 is combined with that of [38, Theorem 9]. In view of this fact we may omit details. Notice only that the function

$$w(t)=\left(\int_a^t r^{1-\beta}(s)\,\Delta s\right)^{1-\alpha}\left[\frac{2\alpha-1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}-\mathcal{B}(t)\right]$$

is shown to satisfy $\mathcal{R}[w]\le 0$ with (5.3) in a neighborhood of infinity, and consequently the statement follows from Lemma 7.2. □

The following statement completes the previous one in the case that (7.3) fails to hold.

Theorem 9.5. *Suppose that $r(t)>0$ for $t\in\mathcal{I}_a$, $\int_a^\infty r^{1-\beta}(s)\,\Delta s<\infty$ and* (9.5) *hold. If $\limsup_{t\to\infty}\tilde{\mathcal{B}}(t)<\left(\frac{\alpha-1}{\alpha}\right)^{\alpha}$ and*

$$\liminf_{t\to\infty}\tilde{\mathcal{B}}(t)>\gamma_{\min}+\frac{2\alpha-1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}, \tag{9.8}$$

where

$$\tilde{\mathcal{B}}(t)=\left(\int_t^\infty r^{1-\beta}(s)\,\Delta s\right)^{-1}\int_t^\infty\left(\int_s^\infty r^{1-\beta}(u)\,\Delta u\right)^{\alpha}p(s)\,\Delta s,$$

then (3.1) *is nonoscillatory.*

Proof. We set

$$w(t)=-\left(\int_t^\infty r^{1-\beta}(s)\,\Delta s\right)^{1-\alpha}\left[\frac{2\alpha-1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}-\left(\int_t^\infty r^{1-\beta}(s)\,\Delta s\right)^{-1}\int_t^\infty\left(\int_s^\infty r^{1-\beta}(u)\,\Delta u\right)^{\alpha}p(s)\,\Delta s\right],$$

and then proceed as in the proof of the previous theorem. □

Let $\lambda_1 > -1$ and $0 < \lambda_2 < \left(\frac{\alpha-1}{\alpha}\right)^\alpha$. Denote by $\delta_{\max}(\lambda_1)$ and $\varphi_{\max}(\lambda_2)$ the greatest roots of the equations

$$(\alpha-1)|x|^\beta + \alpha x + \lambda_1 = 0 \quad \text{and} \quad (\alpha-1)|x|^\beta - (\alpha-1)x + \lambda_2 = 0,$$

respectively. If the first condition in (9.7) fails to hold, then we have the following complement to Theorem 9.4.

Theorem 9.6. *Assume that* (7.3)*,* (7.5) *and* (9.1) *hold. If*

$$\limsup_{t\to\infty} \mathcal{B}(t) < \liminf_{t\to\infty} \mathcal{B}(t) + \varphi_{\max}\left(\liminf_{t\to\infty} \mathcal{B}(t)\right) + \delta_{\max}\left[\liminf_{t\to\infty} \mathcal{B}(t) + \varphi_{\max}\left(\liminf_{t\to\infty} \mathcal{B}(t)\right)\right]$$

and

$$-\infty < \liminf_{t\to\infty} \mathcal{B}(t) \le \gamma_{\min} + \frac{2\alpha-1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1},$$

then (3.1) *is nonoscillatory.*

Proof. The proof is similar to that of [26, Theorem 1.9]. □

Remark 9.3. In a similar way as above, we can state a complementary criterion to Theorem 9.5 in case that (9.8) fails to hold.

10 Comparison Theorems

In Section 6 we proved an extension of a classical result — the Sturm type comparison theorem. In this section we present other types of comparison theorems for (3.1) and one comparison theorem for (5.1). The first one actually contains two statements. First, we give a condition in terms of an inequality between the integrals $\int_t^\infty p(s)\Delta s$ and $\int_t^\infty P(s)\Delta s$ (i.e., we compare the coefficients "on average"; note that in the classical Sturm type theorem the coefficients are compared pointwise). This statement unifies and generalizes [31, Theorem 2] and [44, Theorem 4], and for historical reasons it can be called of Hille–Wintner type. Note that in [31] (this paper concerns (HLDE)), the coefficient $p(t)$ is assumed to be nonnegative. Second, we assume a condition in terms of an inequality between the exponents of the power function Φ. This enables, among others, to compare a half-linear equation with a linear one. Note that, in this sense (that is, the relation between two equations with different nonlinearities), the statement is new even in the continuous case (i.e., when $\mathbb{T} = \mathbb{R}$). In the proof we combine the Riccati technique with using the Schauder fixed point theorem.

Along with (3.1) consider now the equation

$$[R(t)\Phi_{\bar{\alpha}}(x^{\Delta})]^{\Delta} + P(t)\Phi_{\bar{\alpha}}(x^{\sigma}) = 0, \tag{10.1}$$

where R and P satisfy the same assumptions as r and p, and $\Phi_{\bar{\alpha}}(x) = |x|^{\bar{\alpha}-1}\operatorname{sgn} x$ with $\bar{\alpha} > 1$.

Theorem 10.1. *Assume* $0 < R(t) \le r(t)$,

$$0 \le \int_t^{\infty} p(s)\Delta s \le \int_t^{\infty} P(s)\Delta s \tag{10.2}$$

for all large t *(in particular, these integrals exist as finite numbers and are eventually nontrivial with respect to* t*),*

$$\int_a^{\infty} R^{1-\bar{\beta}}(s)\Delta s = \infty$$

with $\bar{\beta}$ *being the conjugate number to* $\bar{\alpha}$*, and* $1 < \alpha \le \bar{\alpha}$*. Further suppose that* $\liminf_{t\to\infty} R(t) > 0$ *when* $\mu(t) \not\geq 1$ *eventually (if* $\mu(t) \ge 1$ *eventually, then this condition may be dropped – see also Remark 7.1). If* (10.1) *is nonoscillatory, then so is* (3.1).

Proof. By Lemma 7.4, the assumptions of the theorem imply the existence of a function z (actually, $z = R\Phi_{\bar{\alpha}}(x^{\Delta}/x)$, x being an eventually positive increasing solution of (10.1)) and $T \in \mathcal{I}_a$ such that

$$z(t) \ge \int_t^{\infty} P(s)\Delta s + \int_t^{\infty} \mathcal{S}(z(s), R(s); \bar{\alpha})\Delta s =: Z(t)$$

with $z(t) > 0$ for $t \ge T$. Without loss of generality, we may assume that (10.2) holds for $t \ge T$. Define the set

$$\Omega = \{w \in C_{TS}^{B}[T,\infty) :\ 0 \le w(t) \le Z(t) \text{ for } t \ge T\}$$

and the operator $\mathcal{T} : \Omega \to C_{TS}^{B}[T,\infty)$ by

$$\mathcal{T}(w)(t) = \int_t^{\infty} p(s)\Delta s + \int_t^{\infty} \mathcal{S}(w(s), R(s); \alpha)\Delta s$$

for $w \in \mathcal{T}$. In view of the assumptions of the theorem and the properties of $\mathcal{S}$, the operator $\mathcal{T}$ is well defined. It is very easy to see that Ω is closed and convex.

Let us show that $\mathcal{T}$ maps Ω into itself. Suppose that $w \in \Omega$ and define $v(t) = \mathcal{T}(w)(t)$, $t \ge T$. Obviously, $v(t) \ge 0$ for $t \ge T$. We prove that $v(t) \le$

$Z(t)$. First note that since $w \in \Omega$ is small for large t and $\liminf_{t\to\infty} R(t) > 0$ (provided $\mu(t) \not\geq 1$ eventually), we have $w(t)/R(t) \leq 1$ for large t (without loss of generality we may suppose that T is such that $Z(t)/R(t) \leq 1$ for $t \geq T$ in case $\mu(t) \not\geq 1$ eventually), and so the assumptions of Lemma 7.1 (iii) are satisfied (see also Remark 7.1). Now we get

$$\begin{aligned} v(t) &= \int_t^\infty p(s)\Delta s + \int_t^\infty \mathcal{S}(w(s), R(s); \alpha)\Delta s \\ &\leq \int_t^\infty P(s)\Delta s + \int_t^\infty \mathcal{S}(w(s), R(s); \bar{\alpha})\Delta s \leq Z(t) \end{aligned}$$

by the assumptions of the theorem and by Lemma 7.1. Hence $\mathcal{T}(\Omega) \subset \Omega$.

According to Lemma 2.6, to prove the relative compactness of $\mathcal{T}(\Omega)$, it is sufficient to verify that conditions (i)–(iii) hold for $\mathcal{T}(\Omega)$. Clearly, $\mathcal{T}(\Omega) \subset \Omega$ implies the boundedness of $\mathcal{T}(\Omega)$. In view of the definition of $\mathcal{T}$, for any $w \in \Omega$ we have

$$0 \leq -(\mathcal{T}(w))^\Delta(t) = p(t) + \mathcal{S}(w(t), R(t); \alpha) \leq p(t) + \mathcal{S}(z(t), R(t); \bar{\alpha}),$$

which proves equicontinuity of the elements of $\mathcal{T}(\Omega)$. Finally we verify that $\mathcal{T}(\Omega)$ is "uniformly Cauchy". Let $\varepsilon > 0$ be given. We have to show that there exists $t_0 \in [T, \infty)$ such that $t_1, t_2 \in [t_0, \infty)$ implies $|\mathcal{T}(w)(t_1) - \mathcal{T}(w)(t_2)| < \varepsilon$ for any $w \in \Omega$. Without loss of generality, suppose $t_1 < t_2$. Then we have

$$|\mathcal{T}(w)(t_1) - \mathcal{T}(w)(t_2)| \leq \left| \int_{t_1}^{t_2} p(s)\Delta s \right| + \int_{t_1}^{t_2} \mathcal{S}(w(s), R(s); \alpha)\Delta s. \quad (10.3)$$

Since the integrals in (10.3) are convergent, for any $\varepsilon > 0$ one can find $t_0 \in [T, \infty)$ such that

$$\left| \int_{t_1}^{t_2} p(s)\Delta s \right| < \frac{\varepsilon}{2} \quad \text{and} \quad \int_{t_1}^{t_2} \mathcal{S}(w(s), R(s); \alpha)\Delta s < \frac{\varepsilon}{2}$$

whenever $t_2 > t_1 \geq t_0$. From here and (10.3) we get the desired inequality. Hence $\mathcal{T}(\Omega)$ is relatively compact.

The last hypothesis, which has to be verified, is the continuity of $\mathcal{T}$ in Ω. Let $\{w_n\}_{n\in\mathbb{N}}$ be a sequence in Ω which converges uniformly on every compact subinterval of $[T, \infty)$ to $\bar{w} \in \Omega$. Because $\mathcal{T}(\Omega)$ is relatively compact, the sequence $\{\mathcal{T}(w_n)\}$ admits a subsequence $\{\mathcal{T}(w_{n_j})\}$ converging in the topology of $C^B_{TS}[T, \infty)$ to $\bar{v}$. The inequality $\mathcal{S}(w_{n_j}(t), R(t); \alpha) \leq \mathcal{S}(z(t), R(t); \bar{\alpha})$ implies the total convergence of $\int_t^\infty \mathcal{S}(w_{n_j}(s), R(s); \alpha)\Delta s$. Hence, by the Lebesgue dominated convergence theorem on time scales, see [35], the sequence $\{\mathcal{T}(w_{n_j})\}$ converges to $\mathcal{T}(\bar{w})$. In view of the uniqueness of the limit,

$\mathcal{T}(\bar{w}) = \bar{v}$ is the only cluster point of the sequence $\{\mathcal{T}(w_n)\}$. This proves the continuity of $\mathcal{T}$ in Ω.

Therefore, it follows from Proposition 2.4 that there exists an element $w \in \Omega$ such that $\mathcal{T}(w) = w$. In view of how the operator $\mathcal{T}$ is defined, this (positive) function w satisfies the equation

$$w(t) = \int_t^\infty p(s)\Delta s + \int_t^\infty \mathcal{S}(w(s), R(s); \alpha)\Delta s,$$

$t \geq T$, and hence also the equation $w^\Delta + p(t) + \mathcal{S}(w, R; \alpha)(t) = 0$, clearly, with $\Phi^{-1}(R) + \mu\Phi^{-1}(w) > 0$. Consequently, the function y given by

$$y(T) = A \neq 0 \quad \text{and} \quad y^\Delta = \left(\frac{w(t)}{R(t)}\right)^{\beta-1} y,$$

$t \geq T$, is a nonoscillatory solution of

$$[R(t)\Phi(y^\Delta)]^\Delta + p(t)\Phi(y^\sigma) = 0,$$

and hence this equation is nonoscillatory. The statement now follows from Theorem 6.1. □

Remark 10.1. (i) A closer examination of the proof shows that the necessary condition for nonoscillation of (3.1) in Lemma 7.4 is also sufficient.

(ii) It is not difficult to see that if the assumptions of Lemma 7.5 are satisfied, then the comparison theorem involving the condition in terms of the inequality between exponents of the nonlinearities can be proved immediately by the Riccati technique without using the Schauder fixed point theorem. Indeed, let w be a solution of (5.1) which is positive and tends to zero. Therefore

$$0 = w^\Delta(t) + p(t) + \mathcal{S}[w, r; \alpha] \geq w^\Delta(t) + p(t) + \mathcal{S}[w, r; \bar{\alpha}]$$

for large t provided $\alpha \geq \bar{\alpha} > 1$ and $\liminf_{t\to\infty} r(t) > 0$ (when $\mu(t) \not\geq 1$ eventually). The fact that

$$[r(t)\Phi_{\bar{\alpha}}(y^\Delta)]^\Delta + p(t)\Phi_{\bar{\alpha}}(y^\sigma) = 0$$

is nonoscillatory then follows from Lemma 7.2. Finally notice that if $\mu(t) \geq 1$ eventually, then this simple proof can be used even under the assumptions of the theorem since we do not need a solution of (5.1) to be close to zero.

Now we mention a few background details which serve to motivate our next result. Along with equation (3.1) consider the equation

$$[r(t)\Phi(y^{\Delta})]^{\Delta} + \lambda p(t)\Phi(y^{\sigma}) = 0, \tag{10.4}$$

where λ is a real constant and assume $r(t) > 0$. We claim that if (3.1) is nonoscillatory and $0 < \lambda \leq 1$, then (10.4) is also nonoscillatory. If $p(t) \geq 0$, then this statement follows immediately from the Sturm theorem. If $p(t)$ may change sign, then dividing (10.4) by λ yields an equivalent equation which is nonoscillatory by the Sturm theorem. This can be analogously done for oscillatory counterparts. If the constant λ is replaced by a function $q(t)$, then the situation is not so easy (when $p(t)$ may change sign; otherwise the Sturm theorem can be applied immediately). The following statements give the answer to the question: "What are the conditions which guarantee that (non)oscillation of (3.1) is preserved when multiplying the coefficient $p(t)$ by a function $q(t)$?". They generalize [19, Theorem 7 and Corollary 8]. Along with (3.1) consider the equation

$$[R(t)\Phi(x^{\Delta})]^{\Delta} + q(t)P(t)\Phi(x^{\sigma}) = 0, \tag{10.5}$$

where R and P satisfy the same assumptions as r and p.

Theorem 10.2. *Assume $q \in C^1_{rd}(\mathcal{I}_a)$, $r(t) \leq R(t)$, $P(t) \leq p(t)$, $0 < q(t) \leq 1$, $q^{\Delta}(t) \leq 0$. Further, let* (7.2) *and* (7.3) *hold. Then* (3.1) *is nonoscillatory implies* (10.5) *is nonoscillatory.*

Proof. The assumptions of the theorem imply that there exist a solution y of (3.1) and $T \in \mathcal{I}_a$ such that $y(t) > 0$ and $y^{\Delta}(t) > 0$ on $[T, \infty)$ by Lemma 7.3. Therefore, the function $w(t) := r(t)\Phi(y^{\Delta}(t)/y(t)) > 0$ satisfies (5.1) with (5.3) on $[T, \infty)$. We have

$$\begin{aligned} q\mathcal{S}[w,r] &= \lim_{\lambda \to \mu} \frac{wq}{\lambda}\left(1 - \frac{rq}{\Phi\left[\Phi^{-1}(q)\Phi^{-1}(r) + \lambda\Phi^{-1}(q)\Phi^{-1}(w)\right]}\right) \\ &= \mathcal{S}[qw, qr]. \end{aligned}$$

Now, multiplying (5.1) by $q(t)$, we get

$$\begin{aligned} 0 &= w^{\Delta}(t)q(t) + p(t)q(t) + \mathcal{S}[qw, qr](t) \\ &\geq w^{\Delta}(t)q(t) + P(t)q(t) + \mathcal{S}[qw, qr](t) \\ &\geq w^{\Delta}(t)q(t) + w^{\sigma}(t)q^{\Delta}(t) + P(t)q(t) + \mathcal{S}[qw, qr](t) \\ &= (wq)^{\Delta}(t) + P(t)q(t) + \mathcal{S}[qw, qr](t) \end{aligned}$$

for $t \in [T, \infty)$. Hence the function $v(t) = w(t)q(t)$ satisfies the generalized Riccati inequality

$$v^\Delta(t) + P(t)q(t) + \mathcal{S}[v, qr](t) \le 0$$

with

$$\left\{\Phi^{-1}(qr) + \mu\Phi^{-1}(v)\right\}(t) = \Phi^{-1}(q)\left\{\Phi^{-1}(r) + \mu\Phi^{-1}(w)\right\}(t) > 0$$

for $t \in [T, \infty)$. Therefore, the equation

$$[q(t)r(t)\Phi(x^\Delta)]^\Delta + q(t)P(t)\Phi(x^\sigma) = 0 \tag{10.6}$$

is nonoscillatory by Lemma 7.2, and so (10.5) is nonoscillatory by Theorem 6.1 since $q(t)r(t) \le r(t) \le R(t)$. ☐

Theorem 10.3. *Assume* $q \in C^1_{rd}(\mathcal{I}_a)$, $0 < R(t) \le r(t)$, $p(t) \le P(t)$, $q(t) \ge 1$, $q^\Delta(t) \ge 0$. *Further, let*

$$\liminf_{t\to\infty} \int_T^t q(s)P(s)\Delta s \ge 0 \quad \textit{and} \quad \not\equiv 0 \tag{10.7}$$

for all large T *and*

$$\int_a^\infty R^{1-\beta}(s)\Delta s = \infty.$$

Then (3.1) *is oscillatory implies* (10.5) *is oscillatory.*

Proof. Suppose by contradiction that (10.5) is nonoscillatory. Then there exist a solution x of (10.5) and $T \in \mathcal{I}_a$ such that $x(t) > 0$ and $x^\Delta(t) > 0$ on $[T, \infty)$ by Lemma 7.3. Therefore, the function $v(t) := R(t)\Phi(x^\Delta(t)/x(t)) > 0$ satisfies

$$v^\Delta(t) + q(t)P(t) + \mathcal{S}[v, R](t) = 0 \tag{10.8}$$

with $\left\{\Phi^{-1}(R) + \mu\Phi^{-1}(v)\right\}(t) > 0$ on $[T, \infty)$. We have

$$\frac{v^\Delta(t)}{q(t)} \ge \frac{v^\Delta(t)q(t)}{q^2(t)} - \frac{v(t)q^\Delta(t)}{q^2(t)} = \left(\frac{v(t)}{q(t)}\right)^\Delta$$

at right-dense t, while

$$\frac{v^\Delta(t)}{q(t)} = \frac{v^\sigma(t)}{\mu(t)q(t)} - \frac{v(t)}{\mu(t)q(t)} \ge \frac{v^\sigma(t)}{\mu(t)q^\sigma(t)} - \frac{v(t)}{\mu(t)q(t)} = \left(\frac{v(t)}{q(t)}\right)^\Delta$$

at right-scattered t. Dividing (10.8) by $q(t)$ and using the above estimates, we get

$$\begin{aligned} 0 &= \frac{v^{\Delta}(t)}{q(t)} + P(t) + \frac{1}{q(t)}\mathcal{S}[v,R](t) \\ &\geq \left(\frac{v(t)}{q(t)}\right)^{\Delta} + p(t) + \mathcal{S}\left[\frac{v}{q},\frac{R}{q}\right](t) \end{aligned}$$

for $t \in [T,\infty)$. Hence the function $w(t) = v(t)/q(t)$ satisfies the inequality $w^{\Delta}(t) + p(t) + \mathcal{S}\left[w, R/q\right](t) \leq 0$ with $\left\{\Phi^{-1}(R/q) + \Phi^{-1}(w)\right\} > 0$ for $t \in [T,\infty)$. Therefore the equation

$$\left[\frac{R(t)}{q(t)}\Phi(y^{\Delta})\right]^{\Delta} + p(t)\Phi(y^{\sigma}) = 0 \tag{10.9}$$

is nonoscillatory by Lemma 7.2. Now, since $R(t)/q(t) \leq R(t) \leq r(t)$, equation (3.1) is nonoscillatory by Theorem 6.1, a contradiction. □

Remark 10.2. A closer examination of the proofs shows that the last two theorems can be improved in the following way (assuming the same conditions):

- Theorem 10.2: If (3.1) is nonoscillatory, then (10.6) is as well.
- Theorem 10.3: If (10.9) is oscillatory, then (10.5) is as well.

Our theorems then follow from the above by virtue of the Sturm type comparison theorem.

The next comparison result is based on the Roundabout theorem that involves a functional with one free endpoint.

Theorem 10.4 (Leighton Comparison Theorem). *Let z be a solution of $\mathcal{L}_{R,P}[z] = 0$ such that $z(b) = 0 \neq z(a)$. Denote $B = [R\Phi(z^{\Delta}/z)](a)$. Let A be such that*

$$\mathcal{G}(z;a,b) := (A-B)|z(a)|^{\alpha} + \int_a^b \left\{(r-R)|z^{\Delta}|^{\alpha} - (p-P)|z^{\sigma}|^{\alpha}\right\}(s)\Delta s \leq 0.$$

Then the solution y of (3.1) *given by $[r\Phi(y^{\Delta}/y)](a) = A$ has a generalized zero in $\mathcal{I}$.*

Proof. Define the functional $\mathcal{F}_{BRP}(\xi;a,b) = B|\xi(a)|^\alpha + \mathcal{F}_{R,P}(\xi;a,b)$ and let z and B be as in the theorem. Integration by parts yields

$$\mathcal{F}_{BRP}(z;a,b) = B|z(a)|^\alpha + [z(t)R(t)\Phi(z^\Delta(t))]_a^b - \int_a^b \{z^\sigma \mathcal{L}_{R,P}[z]\}(s)\Delta s = 0.$$

Hence $\mathcal{F}_A(z;a,b) = \mathcal{F}_A(z;a,b) - \mathcal{F}_{BRP}(z;a,b) = \mathcal{G}(z;a,b) \le 0$. The statement now follows from Theorem 5.2. □

An immediate consequence is the following statement.

Corollary 10.1. *Let z and B be as in the previous theorem. Further assume that $A \le B$, $r(t) \le R(t)$ and $P(t) \le p(t)$ on $\mathcal{I}^\kappa$. Then the solution of* (3.1) *given by $[r\Phi(y^\Delta/y)](a) = A$ has a generalized zero in $\mathcal{I}$.*

The last theorem of this section deals with a comparison of generalized Riccati dynamic equations.

Theorem 10.5. *Suppose that $r(t) \le R(t)$ and $P(t) \le p(t)$ on $\mathcal{I}^\kappa$. Denote with w and v solutions of* (5.1) *and $v^\Delta + P(t) + \mathcal{S}[v,R](t) = 0$, respectively. If* (5.3) *holds on $\mathcal{I}^\kappa$ and $v(a) \ge w(a)$, then $v(t) \ge w(t)$ for $t \in \mathcal{I}$ with $\{\Phi^{-1}(R) + \mu\Phi^{-1}(v)\}(t) > 0$.*

Proof. Since (5.1) is solvable on $\mathcal{I}$ with (5.3), it follows from Theorem 5.2 that $\mathcal{F}_A(\xi;a,b) = w(a)|\xi(a)|^\alpha + \mathcal{F}(\xi;a,b) > 0$ for all nontrivial $\xi \in U_A(a,b)$. Then $\mathcal{F}_{BRP} = v(a)|\xi(a)|^\alpha + \mathcal{F}_{R,P}(\xi;a,b) > 0$. Hence, by Theorem 5.2,

$$\{\Phi^{-1}(R) + \mu\Phi^{-1}(v)\}(t) > 0$$

on $\mathcal{I}^\kappa$. Now we prove $v(t) \ge w(t)$ on $\mathcal{I}$. Let z be a solution of $\mathcal{L}_{R,P}[z] = 0$ that generates v. By using Picone's identity, we have

$$\mathcal{F}_{R,P}(\xi;a,t) = [v(s)|\xi(s)|^\alpha]_a^t + \int_a^t \left\{R|\xi|^\alpha - v(|\xi|^\alpha)^\Delta + \mathcal{S}[v,R]|\xi^\sigma|^\alpha\right\}(s)\,\Delta s$$

for $t \in \mathcal{I}$. Therefore $\mathcal{F}_{R,P}(z;a,t) = [v(s)|z(s)|^\alpha]_a^t$ for $t \in \mathcal{I}$. On the other hand, again by Picone's identity,

$$\mathcal{F}(z;a,t) = [w(s)|z(s)|^\alpha]_a^t + \int_a^t \tilde{G}(z,w)(s)\,\Delta s \ge [w(s)|z(s)|^\alpha]_a^t,$$

$t \in \mathcal{I}$, where $\tilde{G}$ is as in the proof of Theorem 5.1. Clearly $\mathcal{F}_{R,P}(z;a,t) \geq \mathcal{F}(z;a,t)$, and hence

$$v(t)|z(t)|^\alpha - v(a)|z(a)|^\alpha \geq w(t)|z(t)|^\alpha - w(a)|z(a)|^\alpha.$$

This in turn yields

$$(v(t) - w(t))|z(t)|^\alpha \geq (v(a) - w(a))|z(a)|^\alpha \geq 0.$$

Since $z(t) \neq 0$ on $\mathcal{I}$, we obtain from the above inequality that $v(t) \geq w(t)$ for all $t \in \mathcal{I}$. □

11 Examples

We start this section by an application of our oscillation and nonoscillation criteria and of the Sturm type comparison theorem. The discussion on the definition on generalized zeros is also given there.

Example 11.1. Consider the equation

$$(\Phi(y^\Delta))^\Delta - C\Phi(y^\sigma) = 0, \tag{11.1}$$

where C is any real positive constant. Equation (11.1) is nonoscillatory (i.e., there exists $a \in \mathbb{T}$ such that $y(t)y^\sigma(t) > 0$ for $t \in \mathcal{I}_a$, where y is any solution of (11.1)) by Theorem 6.1, since equation $(\Phi(y^\Delta))^\Delta = 0$ is nonoscillatory. Indeed, the space of all solutions of this one-term equation has a linear structure and the functions $y_1(t) = 1,\ y_2(t) = t$ form its basis. Note that the nonoscillation of this equation can be detected also by Theorem 9.1. Now, multiplying (11.1) by -1 we obtain

$$(-\Phi(y^\Delta))^\Delta + C\Phi(y^\sigma) = 0, \tag{11.2}$$

which is equivalent to (11.1). Equation $(\Phi(y^\Delta))^\Delta + C\Phi(y^\sigma) = 0$ is oscillatory by Theorem 8.1 and hence (11.2) is also oscillatory by Theorem 6.1. Another argument is that $(-\Phi(y^\Delta))^\Delta = 0$ is oscillatory and thus (11.2) is oscillatory by Theorem 6.1. So we arrive at the strange thing: (11.1) is at the same time oscillatory and nonoscillatory. Is it true? How can it be possible? It is because we apply two different definitions of a generalized zero (which depend on the sign of $r(t)$ and hence on the relevant equation). The solution spaces of (11.1) and (11.2) are the same, but in the latter case we have $r(t) \equiv -1$ and thus $r(t)y(t)y^\sigma(t) < 0$ at each $t \in \mathcal{I}_a$. As it can be easily seen, nonproportional solutions of (11.2) have common generalized zeros (not just zeros) at each $t \in \mathbb{T}$. In particular, this is unusual in the continuous case (i.e., when $\mathbb{T} = \mathbb{R}$). Thus to obtain a reasonable oscillation theory, it seems to be convenient to assume $r(t) > 0$ at all right-dense points $t \in \mathbb{T}$. Also observe that in the continuous case, once $r(t)$ is negative at a point t, then it is negative at all points because of its continuity. Multiplying the equation involving such $r(t)$ by -1 we get the "natural" form with $r(t) > 0$.

Next we give an application of Theorem 10.2 and of Theorem 9.1.

Example 11.2. Let $\mathbb{T} = \mathbb{Z}$. Then $\mu(t) \equiv 1$, $f^\Delta(t) = \Delta f(t)$ and $\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t)$. Further, let $r(t) = \left[(t+1)^{\beta-1} - t^{\beta-1}\right]^{1-\alpha}$ and

$$p(t) = \frac{\gamma}{t(t+1)} + \frac{\lambda(-1)^t}{t},$$

where γ and λ are real constants. It is easy to see that $p(t)$ changes sign for $\lambda \neq 0$. Moreover,

$$\gamma - \lambda < t \sum_{s=t}^{\infty} p(s) < \gamma + \lambda \tag{11.3}$$

and

$$\sum_{s=0}^{t-1} r^{1-\beta}(s) = t^{\beta-1} \to \infty$$

as $t \to \infty$. By Theorem 9.1, if $\gamma \geq \lambda > 0$ and

$$\gamma + \lambda < \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1},$$

then (7.2) holds and (3.1) is nonoscillatory because of (11.3) and

$$\mathcal{A}(t) = t \sum_{s=t}^{\infty} p(s).$$

Consequently, the equation

$$(r(t)\Phi(y^\Delta))^\Delta + \left(\frac{\gamma q(t)}{t(t+1)} + \frac{\lambda(-1)^t q(t)}{t}\right)\Phi(y^\sigma) = 0,$$

where $q(t)$ is any nonincreasing sequence between 0 and 1, is also nonoscillatory by Theorem 10.2.

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