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Higher Order Dynamic Equations on Measure Chains: Wronskians, Disconjugacy, and Interpolating Families of Functions

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This paper introduces generalized zeros and hence disconjugacy of nth order linear dynamic equations, which cover simultaneously as special cases (among others) both differential equations and difference equations. We also define Markov, Fekete, and Descartes interpolating systems of functions. The main result of this paper states that disconjugacy is equivalent to the existence of any of the above interpolating systems of solutions and that it is also equivalent to a certain factorization representation of the operator. The results in this paper unify the corresponding theories of disconjugacy for nth order linear ordinary differential equations and for nth order linear difference equations.

Key Words: time scales; measure chains; disconjugacy; Markov system; Frobenius factorization.

1. INTRODUCTION

The theory of dynamic systems on measure chains is undergoing rapid development as it provides a unifying structure for the study of differential equations in the continuous case and the study of finite difference equa-

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tions in the discrete case. In this paper we continue this development by defining disconjugacy of nth order linear dynamic equations on measure chains and developing the introductory theory. Disconjugacy of nth order linear differential equations is well developed and we refer the reader to the often cited monograph by Coppel [6] for both the development of the theory and a summary of its rich history. For nth order linear difference equations, we refer the reader to Hartman's paper [9] which has generated so much activity in the study of difference equations.

In this paper, we shall define disconjugacy of nth order linear dynamic equations on measure chains and we shall define Markov, Fekete, and Descartes interpolating systems of functions on measure chains (see also [6, 7, 9]). The primary purpose of the paper is to obtain the fundamental result that disconjugacy, the existence of any one of the above interpolating systems of solutions, and a factorization representation of the nth order linear dynamic operator are equivalent on compact measure chains.

The paper is organized in the following way. In Section 2, we will provide preliminary material with respect to the calculus on measure chains. The development is not exhaustive; we refer the reader to [1−4, 8, 13, 14] for more extensive developments. We present sufficient material so that the paper is self-contained. In Section 3, we shall define a generalized zero (GZ) of higher order and develop an analogue of Rolle's theorem associated with GZs of higher order. In Section 4, we present a brief discussion about the theory of initial value problems and prove lemmas related to unique solvability of initial value problems (with so-called well-posed equations) and continuation of solutions to all of the measure chain. In Section 5 we introduce and examine Wronskian determinants, while in Section 6 we define interpolating families of functions (Markov, Fekete, and Descartes) and obtain some fundamental results. Finally, in Section 7, we define disconjugacy and state and prove the main result of the paper.

The development in Section 7 follows the analogous development in [6, 9]. We shall focus on those arguments that are specific to the development on measure chains and refer the reader to [6, 9] if the argument is completely analogous.

There is already some development in the study of higher order dynamic equations on measure chains. Anderson [5] has initiated the study of right disfocality and Henderson and Prasad [12] and Henderson [11] have initiated the study of Lidstone type boundary value problems. In each of these studies, the authors develop their methods from the methods for first or second order problems. Here, in contrast, we study the general nth order equation and develop the corresponding methods.

2. CALCULUS ON MEASURE CHAINS

A measure chain $\mathbb{T}$ is any closed subset of $\mathbb{R}$. For our purposes, we shall also assume that $\mathbb{T}$ is bounded, and thus put

$$a = \min\{t : t \in \mathbb{T}\} \quad \text{and} \quad b = \max\{t : t \in \mathbb{T}\}.$$

Define $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

where $\inf \emptyset := b$ and $\sup \emptyset := a$. For an "interval" $[t_1, t_2] \cap \mathbb{T}$ with $t_1, t_2 \in \mathbb{T}$ and $t_1 < t_2$ we shall simply write $[t_1, t_2]$ so that $\mathbb{T} = [a, b]$ using this notation. $t \in \mathbb{T}$ is called left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. Define $\mu : \mathbb{T} \to \mathbb{R}_0^+$ by $\mu(t) = \sigma(t) - t$. Next, put $\mathbb{T}^\kappa = \mathbb{T}$ if b is left-dense and $\mathbb{T}^\kappa = \mathbb{T} \setminus \{b\}$ if b is left-scattered. We will write $\mathbb{T}^\kappa = \mathbb{T} \setminus (\rho(b), b]$.

We say that a function f defined on $\mathbb{T}$ is *differentiable* at $t \in \mathbb{T}$ if for all $\varepsilon > 0$ there is a neighborhood U of t such that for some α the inequality

$$|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| < \varepsilon|\sigma(t) - s|$$

is true for all $s \in U$, and in this case we write $f^\Delta(t) = \alpha$. Note that in right-dense points $f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$ provided this limit exists and in right-scattered points $f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$ provided f is continuous at t.

Remark 2.1. Note that if $\mathbb{T} = \mathbb{R}$, we have for $t \in \mathbb{R}$

$$\sigma(t) = \rho(t) = t, \qquad \mu(t) = 0, \qquad \text{and} \qquad f^\Delta(t) = f'(t)$$

if $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function; hence dynamic equations on this time scale are ordinary differential equations. If, on the other hand, $\mathbb{T} = \mathbb{Z}$, then for $t \in \mathbb{Z}$

$$\sigma(t) = t + 1, \qquad \rho(t) = t - 1, \qquad \mu(t) = 1, \qquad \text{and} \qquad f^\Delta(t) = \Delta f(t)$$

if $f : \mathbb{Z} \to \mathbb{R}$ is a sequence where $\Delta f(t) = f(t+1) - f(t)$ is the usual forward difference operator; hence dynamic equations on this time scale are ordinary difference equations.

LEMMA 2.1. *Let f and g be functions on $\mathbb{T}$ and let $t \in \mathbb{T}^\kappa$. Then*:

(i) *if f is differentiable at t, then f is continuous at t*;

(ii) *if t is right-scattered and f is continuous at t, then*

$$f^\Delta(t) = (f(\sigma(t)) - f(t))/\mu(t);$$

(iii) *if* $f^{\Delta}(t)$ *exists, then* $f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t)$;

(iv) *if* $f^{\Delta}(\rho(t))$ *exists and if* t *is left-scattered, then*

$$f^{\Delta}(\rho(t)) = (f(t) - f(\rho(t)))/\mu(\rho(t));$$

(v) *if* $f^{\Delta}(t)$ *exists on* $\mathbb{T}^{\kappa}$ *and* f *is invertible on* $\mathbb{T}$, *then*

$$(f^{-1})^{\Delta}(t) = -(f(\sigma(t))^{-1}f^{\Delta}(t)f^{-1}(t) \qquad \text{on } \mathbb{T}^{\kappa}.$$

LEMMA 2.2. *If* f *and* g *are differentiable at* $t \in \mathbb{T}^{\kappa}$, *then*

$$(fg)^{\Delta}(t) = g(\sigma(t))f^{\Delta}(t) + g^{\Delta}(t)f(t); \tag{1}$$

$$(f/g)^{\Delta}(t) = (g(t)f^{\Delta}(t) - g^{\Delta}(t)f(t))/[g(\sigma(t))g(t)], \tag{2}$$

where (2) *is valid provided* $g(t)g(\sigma(t)) \neq 0$.

Remark 2.2. We lose a standard tool in the study of higher order equations on measure chains because of Lemma 2.2. The right hand side in (1) or (2) is a function of g^{σ}, and so, in general, one cannot take higher order derivatives of products or ratios. This provides an interesting case where the space of two times differentiable functions on $\mathbb{T}$ is not closed under multiplication. Let

$$\mathbb{T} = \{t_n : n \in \mathbb{N}_0\} \cup \{0, -1\}, \qquad t_n = (1/2)^{2^n}. \tag{3}$$

One can show directly that if g^{σ} is differentiable at $t = 0$, then $g^{\Delta}(0) = 0$ and $|(g(t_n) - g(0))/t_n^2|$ is bounded in a neighborhood of $t = 0$. As might be implied by (1), one can also show directly that t^2 is not twice differentiable at $t = 0$. As t^2 is the product of t and t, this shows that the space of two times differentiable functions is not closed under multiplication on this measure chain.

If f has an antiderivative F, i.e., $F^{\Delta} = f$, then we define an integral by $\int_r^s f(t)\Delta t = F(s) - F(r)$. It is known that any rd-continuous function possesses an antiderivative. Here, a function is called rd-continuous if it is continuous in right-dense points and if its left-sided limit exists in left-dense points. We refer the reader to [2, 4, 8, 13] for further discussion.

In order to study higher order dynamic equations, define inductively $f^{\Delta^k} = (f^{\Delta^{k-1}})^{\Delta}$, $\sigma^k(t) = \sigma(\sigma^{k-1})(t)$, and $\rho^k(t) = \rho(\rho^{k-1})(t)$. For the sake of notation, let $f^{\Delta^0} = f$ and $\sigma^0(t) = \rho^0(t) = t$. For a nonnegative integer i, let $\mathbb{T}^{\kappa^i} = \mathbb{T} \setminus (\rho^i(b), b]$. Note that $\mathbb{T}^{\kappa^0} = \mathbb{T}$ and that $\mathbb{T}^{\kappa^1} = \mathbb{T}^{\kappa}$. Finally, we also put $f^{\sigma} = f \circ \sigma$.

3. GENERALIZED ZEROS OF HIGHER ORDER

We now define a generalized zero (GZ) of order greater than or equal to k.

DEFINITION 3.1. $t = a$ is a GZ of order greater than or equal to k of y if

$$y^{\Delta^j}(a) = 0, \qquad j = 0, \dots, k-1.$$

$a < t \in \mathbb{T}^{\kappa^{k-1}}$ is a GZ of order greater than or equal to k of y if

$$y^{\Delta^j}(t) = 0, \qquad j = 0, \dots, k-1$$

or

$$y^{\Delta^j}(t) = 0, \qquad j = 0, \dots, k-2, \qquad y^{\Delta^{k-1}}(\rho(t))\, y^{\Delta^{k-1}}(t) < 0. \tag{4}$$

Remark 3.1. Suppose (4) holds. Then t is left-scattered and $\sigma(\rho(t)) = t$. In particular, employing Lemma 2.1(iv),

$$\begin{aligned} y^{\Delta^{k-1}}(\rho(t)) &= \left(y^{\Delta^{k-2}}(t) - y^{\Delta^{k-2}}(\rho(t))\right)/\mu(\rho(t)) \\ &= -y^{\Delta^{k-2}}(\rho(t))/\mu(\rho(t)). \end{aligned}$$

It follows inductively that

$$y^{\Delta^{k-1}}(\rho(t)) = (-1)^{k-1} \mu(\rho(t))^{-(k-1)} y(\rho(t)). \tag{5}$$

Hence (4) is equivalent to

$$y^{\Delta^j}(t) = 0, \qquad j = 0, \dots, k-2, (-1)^{k-1} y(\rho(t))\, y^{\Delta^{k-1}}(t) < 0. \tag{6}$$

We state the following result without its technical but easy proof.

LEMMA 3.1. *Let $j \geq 0$ be an integer and let $t \in \mathbb{T}^{\kappa^j}$. Then*

$$y^{\Delta^i}(t) = 0, \qquad i = 0, \dots, j,$$

if, and only if,

$$y^{\Delta^i}\left(\sigma^l(t)\right) = 0, \qquad i = 0, \dots, j-l, l = 0, \dots, j.$$

Moreover,

$$y^{\Delta^{j+1-l}}\left(\sigma^l(t)\right) = \prod_{s=0}^{l-1} \mu(\sigma^s(t))\, y^{\Delta^{j+1}}(t).$$

Remark 3.2. If y has a GZ of order greater than or equal to k at t we shall say that y has at least k GZs, counting multiplicities. Note that if y has a GZ of order greater than or equal to k at t then, as a corollary to Lemma 3.1, y has a GZ of order greater than or equal to $k-1$ at $\sigma(t)$. In order to avoid redundancies as we count GZs, if y has a GZ of order greater than or equal to k_1 at t_1 and y has a GZ of order greater than or equal to k_2 at t_2 and $\sigma^{k_1-1}(t_1) < t_2$, we shall say that y has at least $k_1 + k_2$ GZs, counting multiplicities.

We shall also have need of the following version of Rolle's theorem.

LEMMA 3.2. *If y has at least $k \in \mathbb{N}$ GZs on $\mathbb{T}$, counting multiplicities, then y^Δ has at least $k-1$ GZs on $\mathbb{T}$, counting multiplicities.*

Proof. We shall show the following two statements:

(i) If y has a GZ of order greater than or equal to $m \in \mathbb{N}$ at t, then y^Δ has a GZ of order greater than or equal to $m-1$ at t.

(ii) If y has a GZ of order greater than or equal to $m \in \mathbb{N}$ at t and if y has a GZ of order greater than or equal to 1 at s with $\sigma^{m-1}(t) < s$, then y^Δ has at least m GZs in $[t, s)$, counting multiplicities.

Once statements (i) and (ii) are established, the claim of the theorem follows by partitioning $\mathbb{T} = [a, b]$ appropriately.

Statement (i) clearly holds and hence it remains to prove (ii). In fact, taking into account (i), to prove (ii) it suffices to show the following two statements:

(iii) If $y(r) = 0$ and y^Δ has no GZ in $[r, s)$, where $r < \rho(s)$, then y has no GZ at s.

(iv) If $y(\rho(r))y(r) < 0$ and y^Δ has no GZ in $[r, s)$, where $r < \rho(s)$, then y has no GZ at s.

First, if the assumptions of (iii) hold, then $y^\Delta(\tau) > 0$ for all $\tau \in [r, s)$ or $y^\Delta(\tau) < 0$ for all $\tau \in [r, s)$ so that

$$y(\rho(s))y(s) = \left\{\int_r^{\rho(s)} y^\Delta(\tau)\Delta\tau\right\}\left\{\int_r^s y^\Delta(\tau)\Delta\tau\right\} > 0.$$

Second, if the assumptions of (iv) hold, then $\rho(r) < r$ and

$$y(\rho(r))y^\Delta(\rho(r)) = y(\rho(r))(y(r) - y(\rho(r)))/\mu(\rho(r)) < 0$$

and hence, since $y^\Delta(\tau)$ is of constant sign on $[\rho(r), s)$, we have

$$y(\rho(r))y^\Delta(\tau) < 0 \qquad \text{for all} \qquad \tau \in [\rho(r), s).$$

Then

$$y(\rho(r))y(t) = y(\rho(r))\left\{y(r) + \int_r^t y^{\Delta}(\tau)\Delta\tau\right\} < 0 \quad \text{for} \quad t \in \{\rho(s), s\}$$

so that $y^2(\rho(r))y(\rho(s))y(s) > 0$ and hence $y(\rho(s))y(s) > 0$. ∎

4. INITIAL VALUE PROBLEMS

We shall study the linear nth order dynamic equation on a measure chain

$$Ly = 0 \qquad \text{with} \qquad Ly := y^{\Delta^n} + \sum_{i=1}^{n} q_i\left(y^{\Delta^{n-i}}\right)^{\sigma}, \tag{7}$$

where q_i is rd-continuous, $i = 1, \dots, n$. Moreover, we shall assume that $1 + \mu q_1 \neq 0$ on $\mathbb{T}$. y is said to be a solution of (7) on $\mathbb{T}$ if y satisfies $Ly(t) = 0$, $t \in \mathbb{T}^{\kappa^n}$, and $y^{\Delta^j} \in C(\mathbb{T}^{\kappa^j})$, $j = 0, \dots, n-1$.

In order to guarantee that the solution space of (7) is n-dimensional, we must guarantee that solutions of (7) are uniquely determined in backward time. See [4, 8] for discussions in the setting of systems of first order equations. One transforms (7) to a system of first order equations in a standard way,

$$z_i = y^{\Delta^{i-1}}, \qquad i = 1, \dots, n.$$

To calculate z_n^{Δ} employ Lemma 2.1(iii). Let A denote the coefficient matrix of the corresponding system of first order dynamic equations and let I denote the $n \times n$ identity. Aulbach and Hilger [4] call A regressive if for each $t \in \mathbb{T}^{\kappa^n}$, $I + \mu(t)A(t)$ is invertible. If A is regressive and rd-continuous, then one obtains the existence and uniqueness of solutions of initial value problems; moreover, these solutions extend to $\mathbb{T}$. If A is regressive and rd-continuous, we shall say that (7) is *well posed*. Note that if (7) is well posed, then (7) is uniquely determined (up to leading coefficient equal to 1) by n linearly independent solutions.

Equation (7.1) is equivalent to a first order system $z^{\Delta} = A(t)z$, where

$$I + \mu(t)A(t)$$

$$= \begin{pmatrix} 1 & \mu(t) & & & \\ & 1 & \mu(t) & & \\ & & \ddots & \ddots & \\ & & & 1 & \mu(t) \\ -\mu(t)p_n(t) & \cdots & \cdots & -\mu(t)p_2(t) & 1 - \mu(t)p_1(t) \end{pmatrix}$$

and $p_n = q_n/(1 + \mu q_1)$, $p_i = (q_i + \mu q_{i+1})/(1 + \mu q_1)$, $i = 1, \ldots, n-1$. Thus, A is regressive if and only if

$$1 + \sum_{i=1}^{n} (-\mu(t))^i p_i(t) \neq 0, \qquad t \in \mathbb{T}^{\kappa^n}$$

holds. Hence we have the following result.

THEOREM 4.1. *Equation* (7) *is well posed if the q_i are rd-continuous for all $1 \leq i \leq n$ and*

$$1 + \mu(t) q_1(t) \neq 0, \qquad t \in \mathbb{T}^{\kappa^n}. \tag{8}$$

In the case of differential equations, $\mu = 0$ and (8) is trivially satisfied. In the case of difference equations, Hartman [9] applies the binomial expansion to express higher order differences as linear combinations of y^{σ^i}. In Hartman's setting the test for regression reduces to requiring that the product of the first and last coefficients of the difference equation does not vanish. In the abstract setting of measure chains, we shall employ (8).

LEMMA 4.1. *Assume that for $i = 1, \ldots, k+1$, u_i is $k + 1 - i$ times differentiable and does not vanish on $\mathbb{T}^{\kappa^{i-1}}$. Then a dynamic equation of the factored form*

$$u_{k+1}\left(u_k\left(\cdots\left(u_2(u_1 y)^{\Delta}\right)^{\Delta} \cdots\right)^{\Delta}\right)^{\Delta}(t) = 0, \qquad t \in \mathbb{T}^{\kappa^k} \tag{9}$$

is well posed.

Proof. By introducing "quasi-derivatives"

$$z_1 = u_1 y; \qquad z_i = u_i z_{i-1}^{\Delta}, \qquad i = 2, \ldots, k+1,$$

Eq. (9) can be equivalently written as $z^{\Delta} = A(t) z$, where

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_k \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} 0 & \frac{1}{u_2} & & \\ & \ddots & \ddots & \\ & & 0 & \frac{1}{u_k} \\ & & & 0 \end{pmatrix}.$$

Hence A is rd-continuous and $\det(I - \mu(t) A(t)) \equiv 1$. ∎

5. WRONSKIAN DETERMINANTS

Let $y_1, \ldots, y_n$ denote n functions that are n times continuously differentiable on $\mathbb{T}^{\kappa^n}$. Define

$$D(i_1, \ldots, i_k)(t) = \det \begin{pmatrix} y_{i_1}(t) & \cdots & y_{i_k}(t) \\ y_{i_1}^{\Delta}(t) & \cdots & y_{i_k}^{\Delta}(t) \\ \vdots & & \vdots \\ y_{i_1}^{\Delta^{k-1}}(t) & \cdots & y_{i_k}^{\Delta^{k-1}}(t) \end{pmatrix}.$$

We will write $W_k = D(1, \ldots, k)$ to denote a usual Wronskian determinant. Hence

$$W_k = \det \begin{pmatrix} y_1 & \cdots & y_k \\ \vdots & & \vdots \\ y_1^{\Delta^{k-2}} & \cdots & y_k^{\Delta^{k-2}} \\ y_1^{\Delta^{k-1}} & \cdots & y_k^{\Delta^{k-1}} \end{pmatrix} = \det \begin{pmatrix} y_1^{\sigma} & \cdots & y_k^{\sigma} \\ \vdots & & \vdots \\ \left(y_1^{\Delta^{k-2}}\right)^{\sigma} & \cdots & \left(y_k^{\Delta^{k-2}}\right)^{\sigma} \\ y_1^{\Delta^{k-1}} & \cdots & y_k^{\Delta^{k-1}} \end{pmatrix}, \tag{10}$$

where the last equality follows from Lemma 2.1(iii).

We will have need in Section 7 to differentiate determinant functions. The following result can be verified easily.

LEMMA 5.1. *We have*

$$\left[\det(a_{ij})_{1 \le i, j \le n}\right]^{\Delta} = \sum_{k=1}^{n} \det\left(b_{ij}^{(k)}\right)_{1 \le i, j \le n},$$

where

$$b_{ij}^{(k)} = \begin{cases} a_{ij}^{\sigma} & \text{if } i < k \\ a_{ij}^{\Delta} & \text{if } i = k \\ a_{ij} & \text{if } i > k. \end{cases}$$

An immediate consequence of the above Lemma 5.1 is the formula

$$W_k^{\Delta} = \det \begin{pmatrix} y_1^{\sigma} & \cdots & y_k^{\sigma} \\ \vdots & & \vdots \\ \left(y_1^{\Delta^{k-2}}\right)^{\sigma} & \cdots & \left(y_k^{\Delta^{k-2}}\right)^{\sigma} \\ y_1^{\Delta^{k}} & \cdots & y_k^{\Delta^{k}} \end{pmatrix}, \tag{11}$$

and from there we easily derive

$$W_n^{\Delta} = \det\begin{pmatrix} y_1^{\sigma} & \cdots & y_n^{\sigma} \\ \vdots & & \vdots \\ \left(y_1^{\Delta^{n-2}}\right)^{\sigma} & \cdots & \left(y_n^{\Delta^{n-2}}\right)^{\sigma} \\ Ly_1 & \cdots & Ly_n \end{pmatrix} - q_1(t)W_n^{\sigma}.$$

Hence the following Abel's type result holds.

THEOREM 5.1. *If $y_1, \ldots, y_n$ are n solutions of* (7), *then W satisfies the first order linear dynamic equation*

$$W_n^{\Delta} = -q_1(t)W_n^{\sigma}.$$

Also note that Theorem 5.1 and Lemma 2.1(iii) imply $(1 + \mu(t)q_1(t))W_n^{\sigma} = W_n$ or, if (7) is well posed, $W_n^{\sigma} = (1/(1 + \mu(t)q_1(t)))W_n$.

6. INTERPOLATING FAMILIES OF FUNCTIONS

We shall employ some common terminology for interpolating families of functions.

DEFINITION 6.1. We shall say that $\{y_1, \ldots, y_n\}$ is a

(i) *Markov system* on $\mathbb{T}$ if

$$W_k(t) > 0, \qquad t \in \mathbb{T}^{\kappa^{k-1}}, k = 1, \ldots, n;$$

(ii) *Fekete system* on $\mathbb{T}$ if

$$D(i, \ldots, i + k - 1)(t) > 0,$$
$$t \in \mathbb{T}^{\kappa^{k-1}}, i = 1, \ldots, n - k + 1, k = 1, \ldots, n;$$

(iii) *Descartes system* on $\mathbb{T}$ if

$$D(i_1, \ldots, i_k)(t) > 0, \qquad t \in \mathbb{T}^{\kappa^{k-1}}, 1 \le i_1 < \cdots < i_k \le n, k = 1, \ldots, n.$$

LEMMA 6.1. *Assume that $\{y_1, \ldots, y_n\}$ is a Markov system on $\mathbb{T}$. Then for each $k = 1, \ldots, n$, the kth order linear dynamic equation on a time scale,*

$$L_k y(t) = \left(W(y_1, \ldots, y_k, y)/W_k^{\sigma}\right)(t) = 0, \qquad t \in \mathbb{T}^{\kappa^k}$$

is well posed, and $\{y_1, \ldots, y_k\}$ is a basis of the solution space of

$$L_k y(t) = 0, \qquad t \in \mathbb{T}^{\kappa^k}.$$

Proof. Each y_i is assumed to be n times continuously differentiable, so the rd-continuity assumption in L_k is satisfied. First, as in (10), composite each entry in $W(y_1, \ldots, y_k, y)$ except in the last row with σ. Then expand $W(y_1, \ldots, y_k, y)$ along the last column and write

$$L_k y(t) = y^{\Delta^k}(t) + \sum_{i=1}^{k} q_i(t)\left(y^{\Delta^{k-i}}\right)^{\sigma}(t) = 0, \qquad t \in \mathbb{T}^{\kappa^k},$$

where $q_1 = -W_k^{\Delta}/W_k^{\sigma}$, and this follows from (11). Therefore

$$1 + \mu q_1 = 1 - \mu W_k^{\Delta}/W_k^{\sigma} = \left(W_k^{\sigma} - \mu W_k^{\Delta}\right)/W_k^{\sigma} = W_k/W_k^{\sigma} \neq 0 \quad (12)$$

(where we have used Lemma 2.1(iii), i.e., (8) holds. ∎

It is interesting to compare (12) to Hartman's Proposition 2.7 in [9, p. 8].

The following lemma follows directly from determinant identities and is due to Fekete (see [6]). We omit the proof.

LEMMA 6.2. *$\{y_1, \ldots, y_n\}$ is a Fekete system on $\mathbb{T}$ if and only if $\{y_1, \ldots, y_n\}$ is a Descartes system on $\mathbb{T}$.*

Remark 6.1. Analogous details are derived in [6, 9]. The proof relies only on an application of Sylvester's identity,

$$\begin{aligned} &D\left(i_2, \ldots, i_{j-1}, i_1 + j - 1, i_j, \ldots, i_{k-1}\right) D(i_1, \ldots, i_k) \\ &\quad = D(i_1, \ldots, i_{k-1}) D\left(i_2, \ldots, i_{j-1}, i_1 + j - 1, i_j, \ldots, i_k\right) \\ &\qquad D(i_2, \ldots, i_k) D\left(i_1, \ldots, i_{j-1}, i_1 + j - 1, i_j, \ldots, i_{k-1}\right), \end{aligned}$$

where $j \in \{2, \ldots, k\}$ denotes the least integer such that $i_1 + (j-1) < i_j$. Fekete implies Descartes follows by a double induction on k and $i_k - i_1$.

7. DISCONJUGACY

We say that (7) is *disconjugate* on $\mathbb{T}$ if (7) is well posed, and if y is a solution of (7) on $\mathbb{T}$ and y has greater than or equal to n GZs, counting multiplicities, then $y \equiv 0$. We also say that L given in (7) has a *Frobenius factorization* on $\mathbb{T}$ if $L = M$ for some operator M defined by

$$My(t) = v_{n+1}\left((1/v_n)\left(\cdots\left((1/v_2)(y/v_1)^{\Delta}\right)^{\Delta}\cdots\right)^{\Delta}\right)^{\Delta}(t), \qquad t \in \mathbb{T}^{\kappa^n},$$

where each v_i is positive and rd-continuous on $\mathbb{T}^{\kappa^{i-1}}$, $i = 1, \ldots, n+1$.

The purpose of this paper is to state and prove the following theorem.

THEOREM 7.1. *The following are equivalent*:

(i) *Equation* (7) *is disconjugate on* $\mathbb{T}$;

(ii) *there exists a Markov system of solutions of* (7) *on* $\mathbb{T}$;

(iii) *there exists a Fekete system of solutions of* (7) *on* $\mathbb{T}$;

(iv) *there exists a Descartes system of solutions of* (7) *on* $\mathbb{T}$;

(v) *L defined by* (7) *has a Frobenius factorization on* $\mathbb{T}$.

The equivalence of (iii) and (iv) is the contents of Lemma 6.2. We shall prove below that (ii) implies (iii) (see Lemma 7.2 and Theorem 7.5) and is equivalent to (v) (see Theorem 7.2), and that (i) implies (ii) (see Lemma 7.1 and Theorem 7.3). To complete the process, we shall prove that (v) implies (i) (see Theorem 7.4). Since (iii) implies (ii) trivially, the equivalence of items (i) through (v) will be established.

We begin with the argument that (ii) and (v) are equivalent. Details are analogous to those found in [6] for the continuous case and [9] for the discrete case. We supply details specific to the measure chains case.

THEOREM 7.2. *Parts* (ii) *and* (v) *are equivalent*.

Proof. First, assume (ii) and let $\{y_1, \dots, y_n\}$ denote a Markov system of solutions of (7) on $\mathbb{T}$. Note that by Lemma 6.1,

$$L_k y = W(y_1, \dots, y_k, y)/W_k^\sigma = 0$$

denotes the uniquely determined well-defined kth order linear dynamic equation with leading coefficient 1 that has $y_1, \dots, y_k$ as k linearly independent solutions. Also note that for $k = 0, \dots, n-1$ (with $W_0 = 1$),

$$L_{k+1} y = (W_{k+1}/W_k^\sigma)\big((W_k^\sigma/W_{k+1}) L_k y\big)^\Delta. \tag{13}$$

To see this, set the right hand side of (13) equal to 0 and note that $y_1, \dots, y_{k+1}$ satisfy this $(k+1)$st order linear dynamic equation. Moreover, to see that the leading coefficient is 1, note that by (2) of Lemma 2.2

$$L_{k+1} y = \big(W^\Delta(y_1, \dots, y_k, y) W_{k+1} - W(y_1, \dots, y_k, y) W_{k+1}^\Delta\big)/(W_k^\sigma W_{k+1}^\sigma),$$

and hence (apply Lemma 5.1 twice) the leading coefficient of $y^{\Delta^{k+1}}$ is

$$\big(W_k^\sigma W_{k+1} + \mu W_k^\sigma W_{k+1}^\Delta\big)/(W_k^\sigma W_{k+1}^\sigma) = 1.$$

This implies (v) if we set $v_1 = y_1$, $v_{n+1} = W_n/W_{n-1}^\sigma$, and

$$v_k = (W_k W_{k-2}^\sigma)/(W_{k-1}^\sigma W_{k-1}) \qquad \text{for } k = 2, \dots, n.$$

Assume (v). Set $y_1 = v_1$. Inductively, let y_k denote a solution of

$$W(y_1, \dots, y_{k-1}, y) = W_{k-1}^\sigma \prod_{i=1}^{k} v_i.$$

Then $\{y_1, \ldots, y_n\}$ is a Markov system. To see that it is in fact a Markov system of solutions of (7) on $\mathbb{T}$ solve for v_k at each step to obtain

$$v_k = (W_k W_{k-2}^{\sigma})/(W_{k-1}^{\sigma} W_{k-1}).$$

So at each step, y_k satisfies

$$\left((1/v_k)\Big(\cdots\big((1/v_2)(y/v_1)^{\Delta}\big)^{\Delta}\cdots\Big)^{\Delta}\right)^{\Delta}(t) = 0$$

and the proof of the equivalence of (ii) and (v) is complete. ∎

Remark 7.1. Note that our statement of (v) is not an immediate analogue of the factorization statement found in [6]. In particular, we do not characterize the differentiability properties of the v_i coefficients (which can be done and leads to long formulae involving products and sums of the various v_i's). The proof of (v) implies (i), given above, is valid if $\ker L = \ker M$, and this is guaranteed by our condition $L = M$. But care must be taken in applying this equivalence. The fact that an equation is given in factored form does not imply automatically that it is disconjugate, even if it has a system of solutions with positive Wronskians. This can be concluded only if the M of such an equation is equal to an L of the form given in (7). One example to illustrate this is the equation $(y/(t+2))^{\Delta^2} = 0$ on the time scale (3) from Remark 2.2. It has two solutions $y_1(t) = t + 2$ and $y_2(t) = t(t+2)$, so W_1 and W_2 are positive. However, it cannot be written in the form (7), and hence the condition $L = M$ is violated.

We now argue that (i) implies (ii).

LEMMA 7.1. *Assume* (i). *Then there exists a set of solutions,* $y_1, \ldots, y_n$ *of* (7) *on* $\mathbb{T}$ *such that* $W_n(t) > 0$, $a \le t \in \mathbb{T}^{\kappa^{n-1}}$, *and* $W_k(t) > 0$, $\sigma^{n-k-1}(a) < t \in \mathbb{T}^{\kappa^{k-1}}$, $k = 1, \ldots, n-1$.

Proof. Let y_k denote a solution of (7), satisfying the partial set of initial conditions

$$y_k^{\Delta^i}(a) = 0, \qquad i = 0, \ldots, n-k-1, (-1)^{k-1} y_k^{\Delta^{n-k}}(a) > 0.$$

By construction, a is a GZ of order greater than or equal to $n - k$ for any linear combination of $y_1, \ldots, y_k$.

If W_k vanishes for some $\sigma^{n-k-1}(a) < t \in \mathbb{T}^{\kappa^{k-1}}$, then there exists a nontrivial linear combination of $y_1, \ldots, y_k$ with a GZ of order greater than or equal to k at t. Thus, disconjugacy is violated.

If $W_k(\rho(c))W_k(c) < 0$ for some $1 \le k \le n$ and some $a < c \in \mathbb{T}^{\kappa^{k-1}}$, then clearly $k \ne 1$, because otherwise y_1 would have a GZ at c, a contradiction to (i). Note that in this case c is left-scattered so that

$\sigma(\rho(c)) = c$. Since $W_{k-1}(c) \neq 0$, the system

$$\left(y_i^{\Delta^{j-1}}(c)\right)_{1 \le i, j \le k-1} \alpha = -\left(y_k^{\Delta^{j-1}}(c)\right)_{1 \le j \le k-1} \tag{14}$$

has a unique solution $\alpha = (\alpha_1, \ldots, \alpha_{k-1})^T \neq 0$. Put $y = \alpha_1 y_1 + \cdots + \alpha_{k-1} y_{k-1} + y_k$. Then

$$y(c) = \cdots = y^{\Delta^{k-2}}(c) = 0. \tag{15}$$

Employ (10), replace y_k in the kth column of $W_k(\rho(c))$ and of $W_k(c)$ with y, expand along the kth column, and use (15) to obtain

$$0 > W_k(\rho(c))W_k(c) = \left(y^{\Delta^{k-1}}(\rho(c))W_{k-1}(c)\right)\left(y^{\Delta^{k-1}}(c)W_{k-1}(c)\right). \tag{16}$$

Now (16), coupled with (15), implies that y satisfies (4) at $t = c$. That is, y has a GZ of order greater than or equal to k at c which again contradicts disconjugacy.

In particular, each W_k is of strict sign for $\sigma^{n-k-1}(a) < t \in \mathbb{T}^{\kappa^{k-1}}$. To see that $W_k(t) > 0$ for $\sigma^{n-k-1}(a) < t \in \mathbb{T}^{\kappa^{k-1}}$, we note that $W_1(t) = y_1(t) > 0$ for $t > \sigma^{n-2}(a)$ (use, e.g., Taylor's formula from [1, Theorem 2]) and proceed by induction. Let $k > 1$ and suppose $c > \sigma^{n-k}(a)$ is left-scattered (if there are no left-scattered points, then apply again [1] to obtain the desired conclusion). Now define α as in (14) and put $y = \sum_{i=1}^{k-1} \alpha_i y_i + y_k$ so that (15) holds. Hence y has a GZ of order greater than or equal to $k-1$ at c and (by construction) a GZ of order greater than or equal to $n-k$ at a. Therefore y has already at least $n-1$ GZs in $\mathbb{T}$, but because of disconjugacy, it cannot have another one. Hence, since

$$(-1)^{k-1} y^{\Delta^{n-k}}(a) > 0,$$

it follows that $(-1)^{k-1} y(\rho(c)) > 0$, and therefore by (5) that $y^{\Delta^{k-1}}(\rho(c)) > 0$. So, as in (16), $W_k(\rho(c)) = y^{\Delta^{k-1}}(\rho(c))W_{k-1}(c) > 0$, and hence $W_k(t) > 0$ for all $t > \sigma^{n-k-1}(a)$. ∎

THEOREM 7.3. *Part* (i) *implies* (ii).

Proof. Let $j = 0$ if $a = \sigma(a)$, let $j = n-1$ if $\sigma^{n-2}(a) < \sigma^{n-1}(a)$, or let $j \in \{1, \ldots, n-2\}$ be such that $\sigma^{j-1}(a) < \sigma^j(a) = \sigma^{j+1}(a)$. For each $k = 1, \ldots, n$, let $y_k(x; t)$ denote the solution of (7) satisfying the following set of initial conditions. If $n - k < j$, assume

$$y_k(x; \sigma^i(a)) = (-1)^{k-1} x^{n-k-i}/(n-k-i)!, \qquad i = 0, \ldots, n-k,$$

$$y_k(x; \sigma^i(a)) = 0, \qquad i = n-k+1, \ldots, j,$$

$$y_k^{\Delta^i}(x; \sigma^j(a)) = 0, \qquad i = 0, \ldots, n-j-1.$$

If $n - k \geq j$, assume

$$y_k(x; \sigma^i(a)) = (-1)^{k-1} x^{n-k-i}/(n-k-i)!, \qquad i = 0, \dots, j,$$

$$y_k^{\Delta^i}(x; \sigma^j(a)) = (-1)^{k-1} x^{n-k-(i+j)}/(n-k-(i+j))!, \qquad i = 0, \dots, n-k-j,$$

$$y_k^{\Delta^i}(x; \sigma^j(a)) = 0, \qquad i = n-k-j+1, \dots, n-j-1.$$

Note that

$$W_l(x; \sigma^j(a)) = \det\{a_{ik}\},$$

where

$$a_{ik} = (-1)^{k-1}(d^{i-1}/dx^{i-1}) x^{n-k-j}/(n-k-j)!, \qquad l = 1, \dots, n-j.$$

Apply Lemma 2.1(iii) and note that

$$W_l(x; \sigma^{j-i}(a)) = \prod_{h=1}^{i} \left(\mu(\sigma^{j-h}(a)) \right)^{l-h} \det\{a_{ik}\},$$

where

$$a_{ik} = (-1)^{k-1}(d^{i-1}/dx^{i-1}) x^{n-k-j+i}/(n-k-j+i)!, \qquad l = 1, \dots, n-j+i,$$

$i, k = 1, \dots, l$.

For $x > 0$, each $W_k(x; \sigma^i(a)) > 0$, $i = 0, \dots, j$, since each determinant is obtained from a well-known Markov family of polynomials [9]. For $x = 0$, Lemma 7.1 applies and $W_k(0; t) > 0$, $\sigma^{n-k-1}(a) < t \in \mathbb{T}^{\kappa^{k-1}}$, $k = 1, \dots, n$. By continuous dependence of solutions of initial value problems (see [14, Theorem 2.6.2]), for $x > 0$ and sufficiently small, $W_k(x; t) > 0$, $t \in \mathbb{T}^{\kappa^{k-1}}$, $k = 1, \dots, n$, since $\mathbb{T}$ is compact. ▮

We now argue that (v) implies (i). This argument is modeled after Hartman's argument of [9, p. 15, (a) implies (c)].

THEOREM 7.4. *Part* (v) *implies* (i).

Proof. Assume (v) and assume for the sake of contradiction that (7) is not disconjugate on $\mathbb{T}$. Let $\{y_1, \dots, y_n\}$ be as in the proof of the equivalence of (ii) and (v) and let y denote a nontrivial solution of (7) having at least n GZs in $\mathbb{T}$. By repeated applications of Rolle's theorem Lemma 3.2, it follows from (13) that each $L_j y$ has at least $n - j$ GZs on $\mathbb{T}$. To make this observation, replace $L_k y$ in (13) with the ratio of two Wronskian determinants in order that the derivative can be calculated. Thus, there exists $k \geq 2$ such that $y = \sum_{i=1}^{k} c_i y_i$ and $c_k \neq 0$. But $L_{k-1} y = c_k W_k / W_{k-1}^{\sigma}$ which is of constant sign and contradicts that $L_{k-1} y$ has $n-k+1$ GZs on $\mathbb{T}$. ∎

We now address that (ii) implies (iii) and model our development after that of Coppel [6, Lemma 8, p. 95].

LEMMA 7.2. *Let $\{y_1, \dots, y_n\}$ be a Markov system on $\mathbb{T}$ such that for some $c \in \mathbb{T}^{\kappa^{n-1}}$,*

$$y_k^{\Delta^j}(c) = 0, \qquad j = 0, \dots, k-2, k = 2, \dots, n.$$

Then

$$D(j, \dots, j+k-1)(t) > 0, \qquad \sigma^{j-2}(c) < t \in \mathbb{T}^{\kappa^{k-1}},$$

$j = 2, \dots, n-k+1$, $k = 1, \dots, n$.

Proof. First, note the identity

$$\left(\frac{D(k, \dots, n)}{D(k-1, \dots, n-1)} \right)^{\Delta} = \frac{D(k, \dots, n-1)^{\sigma} D(k-1, \dots, n)}{D(k-1, \dots, n-1)^{\sigma} D(k-1, \dots, n-1)}, \tag{17}$$

if $D(k-1, \dots, n-1)^{\sigma} D(k-1, \dots, n-1) \neq 0$. The verification of (17) is analogous to the verification of (13) and we refer the reader to [6, Lemma 4 and Lemma 8]. Putting $f = D(k, \dots, n)/W_{n-k+1}$, we conclude

$$\begin{aligned} f^{\Delta} &= \left(\frac{D(k, \dots, n)}{W_{n-k+1}} \right)^{\Delta} = \left(\frac{D(k-1, \dots, n-1) D(k, \dots, n)}{W_{n-k+1} D(k-1, \dots, n-1)} \right)^{\Delta} \\ &= \left(\frac{D(k-1, \dots, n-1)}{W_{n-k+1}} \right)^{\sigma} \left(\frac{D(k, \dots, n)}{D(k-1, \dots, n-1)} \right)^{\Delta} \\ &\quad + \left(\frac{D(k-1, \dots, n-1)}{W_{n-k+1}} \right)^{\Delta} \frac{W_{n-k+1}}{D(k-1, \dots, n-1)} f. \end{aligned} \tag{18}$$

Hence f satisfies an initial value problem of the form

$$f^{\Delta} = c_1 f + c_2, \qquad f(\sigma^{k-1}(a)) = 0.$$

The proof of the lemma follows by a double induction on n and k. Also, by the equivalence of (iii) and (iv) it is sufficient at each step to prove that

$D(k,\ldots,n) > 0, t > \sigma^{k-1}(a)$. For $n = 1$ there is nothing to prove and for $n > 1$, $k = 1$ there is nothing to prove. For $n > 1$ and $k > 1$ apply (17) and the induction hypotheses to see that

$$(D(k,\ldots,n)/D(k-1,\ldots,n-1))^{\Delta} > 0$$

for $t > \sigma^{k-1}(a)$. Apply (18) and the induction hypotheses to see that in the initial value problem given above, $c_1, c_2 > 0$, for $t > \sigma^{k-1}(a)$. It follows by a variation of parameters representation of f (see [4]) that $f > 0$ for $t > \sigma^{k-1}(a)$ and the proof of Lemma 7.2 is complete. ∎

THEOREM 7.5. *Part* (ii) *implies* (iii).

Proof. The construction for the proof here is analogous to the construction of the proof of Theorem 7.3 and we omit the details here. ∎

We close the paper with two immediate corollaries of Theorem 7.1. The first is a generalization of Rolle's theorem and the second states that the product of disconjugate operators is disconjugate. Each are corollaries of the factorization criterion. See [6].

COROLLARY 7.1. *Assume* (7) *is disconjugate on* $\mathbb{T}$ *and assume that* y *is* n *times continuously differentiable and has at least* $n+1$ *GZs on* $\mathbb{T}$. *Then* Ly *has at least one GZ on* $\mathbb{T}^{\kappa^n}$.

COROLLARY 7.2. *Assume* $L_i y = 0$ *is disconjugate on* $\mathbb{T}$, $i = 1, 2$. *Then the product* $L_1(L_2 y) = 0$ *is disconjugate on* $\mathbb{T}$.

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