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Inequalities and Asymptotics for Riccati Matrix Difference Operators

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In the first part inequalities for solutions of Riccati matrix difference equations are obtained which correspond to the linear Hamiltonian difference system

$$\Delta X_k = A_k X_{k+1} + B_k U_k, \qquad \Delta U_k = C_k X_{k+1} - A_k^T U_k,$$

where A_k, B_k, C_k, X_k, U_k are $n \times n$-matrices with symmetric B_k and C_k. If the matrices X_k are invertible, then the matrices $Q_k = U_k X_k^{-1}$ solve the Riccati matrix difference equation

$$Q_{k+1} = C_k + \left(I - A_k^T\right) Q_k (I + B_k Q_k)^{-1} (I - A_k).$$

In contrast to some recent papers dealing with these equations we do not assume that the matrices B_k are invertible. The second part of the paper deals with the asymptotic behaviour of solutions $Q_k(\lambda)$, as $|\lambda| \to \infty$, of the special Riccati matrix

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difference equation which corresponds to the Sturm–Liouville equation

$$\sum_{\mu=0}^{n} (-1)^{\mu} r_{\mu} \Delta^{2\mu} y_{k+1-\mu} = \lambda y_{k+1}$$

of even order $2n$ with constant coefficients $r_0, \dots, r_n$.

1. INTRODUCTION

In the first part of this paper we study inequalities for solutions of Riccati matrix difference equations which correspond to the linear Hamiltonian difference system

$$\Delta X_k = A_k X_{k+1} + B_k U_k, \qquad \Delta U_k = C_k X_{k+1} - A_k^T U_k$$

(abbreviated by LHDS) where $\Delta w_k = w_{k+1} - w_k$, and where A_k, B_k, C_k, X_k, U_k are real $n \times n$-matrices. We will assume throughout that B_k, C_k are symmetric and $I - A_k$ is invertible for all k.

Inequalities for solutions of Riccati equations play an important role in the oscillation theory of Hamiltonian systems. The history of these inequalities for continuous systems goes far back starting with the famous paper of Sturm [25], which gives inequalities for the zeros of solutions of linear second order differential equations. The results of Sturm were until now extended in various directions. A survey of these extensions and the basic facts of oscillation theory of linear Hamiltonian differential systems can be found in [23] (see also [20, 22]).

In the last years there has been made considerable effort to derive analogous results for discrete equations (as LHDS). Principal ideas of oscillation theory for the three term symmetric recurrence equation $-R_k x_{k+1} - R_{k-1} x_{k-1} + S_k x_k = 0$ for vectors $x_k \in \mathbb{R}^n$, for $n \times n$-matrices R_k, S_k with symmetric and invertible R_k were established in [4]. Extensions of these results to LHDS with positive definite matrices B_k are contained in [12–14]. However, the assumption of positive definiteness of B_k is rather strong, and such systems do not include Sturm–Liouville difference equations of higher order, which are treated in the second part (see also [1, 16]). In [3, 15] the authors conjecture that the results of the above mentioned papers remain valid for LHDS, where the matrices B_k may be singular. Their conjecture was answered affirmatively by M. Bohner in [7, 8]. In Bohner's work it is shown that, similarly to continuous Hamiltonian systems and to LHDS with positive definite B_k's, oscillation properties of general LHDS are intimately related to the corresponding

Riccati matrix difference equation

$$Q_{k+1} = C_k + \left(I - A_k^T\right) Q_k (I + B_k Q_k)^{-1} (I - A_k)$$

and to the non-negativity of the discrete quadratic functional

$$\mathscr{F}_0(x, u) = \sum_{k=0}^{N} \left\{ x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k \right\}$$

for admissible (x, u), i.e., with $\Delta x_k = A_k x_{k+1} + B_k u_k$.

Here we use these results in order to derive inequalities between matrices which are related to different LHDS. These inequalities generalize results given in [15]. The continuous version of our main result is given in [20, Chap. V].

In the second part we investigate the asymptotic behaviour of the solutions $Q_k(\lambda)$, as $|\lambda| \to \infty$, of the special Riccati matrix difference equation, which corresponds to the Sturm–Liouville difference equation

$$\sum_{\mu=0}^{n} (-1)^{\mu} r_{\mu} \Delta^{2\mu} y_{k+1-\mu} = \lambda y_{k+1}$$

of even order $2n$ with constant coefficients $r_0, \dots, r_n$. These asymptotics lead via results from the already mentioned work by M. Bohner to new inequalities for finite differences. These results for the special equation $(-1)^n \Delta^{2n} y_{k+1-n} = \lambda y_{k+1}$ are contained in [19], and the derivation of our results here is quite similar to [19]. For Sturm–Liouville *differential* equations and, more generally, for LHDS similar asymptotic results [20, Chap. 6] can be applied to derive bounds for a corresponding Rayleigh quotient. Such estimates lead to Rayleigh's principle [20, Sect. 7.7] (with applications to the optimal linear regulator [20, Theorem 8.1.1]), and they yield new variational inequalities [18, Corollary and Example]. In fact, our inequality here, Theorem 4 in Section 4, is a first result in this direction for the discrete case.

The paper is organized as follows. In the next section we recall basic properties of solutions of LHDS and their relations to the corresponding Riccati equations and the quadratic functionals. The main result of the first part—Sturm-type inequalities for different LHDS—is derived in Section 3, and it is used to prove an inequality between recessive solutions of associated Riccati equations. The section concludes with some comments concerning possible extensions and applications of the presented results. In Section 4 we state the main result on the asymptotics of the Sturm–Liouville difference equation with constant coefficients. This result is used to derive an inequality for finite differences. The concluding Section 5 is devoted to the proof of the asymptotics. The key for that proof is a certain matrix continued fraction, which is stated in Lemma 9. As indicated in Remark 3 this continued fraction may also be of separate interest in number theory.

2. AUXILIARY RESULTS

In this section we recall some basic concepts and properties of LHDS

$$\Delta X_k = A_k X_{k+1} + B_k U_k, \qquad \Delta U_k = C_k X_{k+1} - A_k^T U_k,\ k \in \mathbb{Z}, \quad \text{(H)}$$

where we assume throughout that the A_k, B_k, C_k are real $n \times n$-matrices satisfying

$$I - A_k \text{ are invertible and } B_k, C_k \text{ are symmetric for all } k, \tag{1}$$

and we denote $\tilde{A}_k := (I - A_k)^{-1}$. Moreover, we use the following notation. For a symmetric (and real) matrix D the inequality $D \geq 0$ ($D > 0$) means that D is non-negative (positive) definite. For any matrix V we denote by $V^\dagger$ its Moore–Penrose generalized inverse, i.e., the unique matrix $V^\dagger$ such that $VV^\dagger, V^\dagger V$ are symmetric, $V = VV^\dagger V$, and $V^\dagger = V^\dagger V V^\dagger$. Ker and Im denote the kernel and image of the matrix indicated, respectively. The symbol Δ denotes the forward difference operator, i.e., $\Delta w_k = w_{k+1} - w_k$.

Let $(X, U), (\tilde{X}, \tilde{U})$ (i.e., $X = \{X_k\}_{k \in \mathbb{Z}}$, etc.) be two solutions of (H). Then, under (1), $X_k^T \tilde{U}_k - U_k^T \tilde{X}_k$ is a constant matrix. A solution (X, U) of (H) is said to be *self-conjoined* if $X_k^T U_k - U_k^T X_k \equiv 0$. If, in addition, $\operatorname{rank}(X_k^T, U_k^T) \equiv n$, then (X, U) is called a *conjoined basis* of (H). For a conjoined basis (X, U) of (H) we say that an interval $(k, k+1]$ contains a *focal point* of X (or of (X, U)) whenever either $\operatorname{Ker} X_{k+1} \not\subset \operatorname{Ker} X_k$ or $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$ but $D_k := X_k X_{k+1}^\dagger \tilde{A}_k B_k \not\geq 0$. Note that if $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$, then the matrix D_k is symmetric as is shown in [7, Lemma 4]. The LHDS (H) is said to be *disconjugate* on an (integer) interval $[k, l] \cap \mathbb{Z}$ if the solution (X, U) of (H) with the initial conditions $X_k = 0, U_k = I$ (the so-called *principal solution* of (H) at k) has no focal point in $(k, l+1]$.

We say that (H) is *eventually disconjugate* if there exists $M \in \mathbb{N}$ such that (H) is disconjugate on $[M, l] \cap \mathbb{Z}$ for all $l > M$, and (H) is said to be *eventually controllable* if there exist $M, \kappa \in \mathbb{N}$ such that the only vector solution of (H) (i.e., $x_k, u_k \in \mathbb{R}^n$ solving (H)) with $x_k = 0$ for all $l \leq k \leq l + \kappa$, and for some $l \geq M$ is the trivial solution $(x, u) \equiv 0$. Finally, if (H) is eventually controllable, a conjoined basis (X, U) of (H) is called *recessive*, if there exists $l \in \mathbb{N}$ such that X_k is invertible, $D_k = X_k X_{k+1}^{-1} \tilde{A}_k B_k \geq 0$ for $k \geq l$ and if $\lim_{k\to\infty}(\sum_{j=l}^{k} X_{j+1}^{-1} \tilde{A}_j B_j X_j^{T-1})^{-1} = 0$. Note that, by [11], eventual controllability and eventual disconjugacy of (H) imply that the matrix $\sum_{j=l}^{k} X_{j+1}^{-1} \tilde{A}_j B_j X_j^{T-1}$ is invertible if k is sufficiently large. Eventual disconjugacy and controllability are also sufficient for the existence of the recessive solution, because under these assumptions both Reid's and Hartman's construction of this solution apply also to our LHDS (H) here (a discrete version of these originally continuous constructions for the special system $-R_k x_{k+1} - R_{k-1} x_{k-1} + S_k x_k = 0$ from the introduction may be found in [2, 4]).

We shall need the following lemmas, where we always assume (1). Moreover, we fix an interval $J = [0, N] \cap \mathbb{Z}$, $N \in \mathbb{N}$, and we put $J^* = J \cup \{N+1\}$. A pair (x, u) with $x, u : J^* \to \mathbb{R}^n$ is called (A, B)-*admissible* if it satisfies the *equation of motion*, i.e., $\Delta x_k = A_k x_{k+1} + B_k u_k$ for $k \in J$.

LEMMA 1 [5, Theorem 8.22]. *Let (X, U) be a conjoined basis of* (H) *such that* $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$ *for $k \in J$, and for $k \in J^*$ let Q_k be a symmetric $n \times n$-matrix satisfying $X_k^T Q_k X_k = U_k^T X_k$. Then, for admissible (x, u) with $x_0 \in \operatorname{Im} X_0$, we have that $x_k \in \operatorname{Im} X_k$ for all $k \in J^*$, $D_k z_k = X_k X_{k+1}^{\dagger} x_{k+1} - x_k$, and*

$$x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k = \Delta\left[x_k^T Q_k x_k\right] + z_k^T D_k z_k$$

for all $k \in J$, where

$$z_k = u_k - Q_k x_k \qquad \textit{and}$$

$$D_k = X_k X_{k+1}^{\dagger} \tilde{A}_k B_k = B_k - B_k \tilde{A}_k^T (Q_{k+1} - C_k) \tilde{A}_k B_k.$$

(Compare also [7, Remark 3(iii) and Theorem 1 (Picone's identity)].)

LEMMA 2. *Suppose that matrices A_k, B_k, C_k and $\underset{\sim}{A}_k, \underset{\sim}{B}_k, \underset{\sim}{C}_k$ satisfy* (1) *for $k \in J$, respectively, and put*

$$\mathscr{H}_k = \begin{pmatrix} -C_k & A_k^T \\ A_k & B_k \end{pmatrix}, \qquad \underset{\sim}{\mathscr{H}}_k = \begin{pmatrix} -\underset{\sim}{C}_k & \underset{\sim}{A}_k^T \\ \underset{\sim}{A}_k & \underset{\sim}{B}_k \end{pmatrix}, \qquad \tilde{D}_k = \underset{\sim}{B}_k\left(\underset{\sim}{B}_k^{\dagger} - B_k^{\dagger}\right)\underset{\sim}{B}_k.$$

Moreover, assume that $(x, \underset{\sim}{u})$ is $(\underset{\sim}{A}, \underset{\sim}{B})$-admissible and that

$$\operatorname{Ker} B_k \subset \operatorname{Ker} \underset{\sim}{B}_k, \qquad \operatorname{Im}(A_k - \underset{\sim}{A}_k) \subset \operatorname{Im}(B_k - \underset{\sim}{B}_k) \textit{ for } k \in J. \tag{2}$$

Then (x, u) with $u_k := B_k^{\dagger}(\Delta x_k - A_k x_{k+1})$ is (A, B)-admissible, and for $k \in J$,

$$x_{k+1}^T \underset{\sim}{C}_k x_{k+1} + \underset{\sim}{u}_k^T \underset{\sim}{B}_k \underset{\sim}{u}_k = x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k + \hat{z}_k^T (\mathscr{H}_k - \underset{\sim}{\mathscr{H}}_k) \hat{z}_k + \tilde{z}_k^T \tilde{D}_k \tilde{z}_k,$$

where $\hat{z}_k = \binom{I}{-P_k} x_{k+1}$, $\tilde{z}_k = P_k x_{k+1} + \underset{\sim}{u}_k$, *and* P_k *is any* $n \times n$*-matrix such that* $A_k - \underset{\sim}{A}_k = (B_k - \underset{\sim}{B}_k) P_k$ *holds*.

This lemma is an immediate consequence of [10, Lemma 7]. We need also the following relationship between the LHDS (H) and a corresponding discrete quadratic functional, which is the contents of [9, Theorem 3] (compare also [6, Theorem 7]).

LEMMA 3. *A conjoined basis* (X, U) *of* (H) *has no focal point in* $(0, N+1]$ *iff*

$$\mathscr{F}_0(x, u) + d^T X_0^T U_0 d > 0$$

with $\mathscr{F}_0(x, u) := \sum_{k=0}^{N} \{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\}$

for all (A, B)*-admissible* (x, u) *with* $x_0 = X_0 d \in \operatorname{Im} X_0$, $x_{N+1} = 0$, $x = \{x_k\}_{k=0}^{N+1} \not\equiv 0$.

Let us recall the relation between (H) and the Riccati matrix difference equation

$$Q_{k+1} = C_k + \left(I - A_k^T\right) Q_k (I + B_k Q_k)^{-1} (I - A_k). \tag{R}$$

Let (X, U) be a conjoined basis of (H). Then $Q_k = U_k X_k^{-1}$ is symmetric and solves (R) whenever X_k and X_{k+1} are invertible. In contrast to continuous Hamiltonian systems the mere existence of a symmetric solution of (R) on an interval $[N, \infty) \cap \mathbb{Z}$ does not imply the eventual disconjugacy of (H) because the matrices D_k may be indefinite. Observe $D_k = X_k X_{k+1}^{-1} \tilde{A}_k B_k = (I + B_k U_k X_k^{-1})^{-1} B_k = (I + B_k Q_k)^{-1} B_k$ if X_k and X_{k+1} are invertible. Therefore, we have by [11, Proposition 1(iv)] the following statement.

LEMMA 4. *Suppose that* (H) *is eventually controllable*. *Then* (H) *is eventually disconjugate if and only if there exists a symmetric solution* Q_k *of* (R) *for* $k \in [N, \infty) \cap \mathbb{Z}$ *with some* $N \in \mathbb{N}$ *such that* $(I + B_k Q_k)^{-1} B_k \geq 0$ *for all* $k \geq N$.

We need some auxiliary formulae on Sturm–Liouville difference equations of even order which will be treated in the last two sections. For $n \in \mathbb{N}$, we consider

$$L(y)_{k+1} := \sum_{\mu=0}^{n} (-1)^{\mu} r_{\mu} \Delta^{2\mu} y_{k+1-\mu} = \lambda y_{k+1}, \qquad 0 \leq k \leq N - n, \tag{SL}$$

with real parameter λ for some fixed $N \geq n$, where $r_0, \dots, r_n$ are given reals such that $r_n \neq 0$. This difference equation is intimately related to a corresponding LHDS, namely as follows (see [6, Proposition 5] or [11, Remark 2]). Vectors $x_k, u_k \in \mathbb{R}^n$ for $k \in J^* = [0, N+1] \cap \mathbb{Z}$ solve the special LHDS

$$\begin{cases} \Delta x_k = A x_{k+1} + B u_k, \\ \Delta u_k = C(\lambda) x_{k+1} - A^T u_k, \ k \in J = [0, N] \cap \mathbb{Z}, \\ \text{where } A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & & \ddots & \vdots \\ \vdots & & & 1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, \quad B = \frac{1}{r_n} \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{pmatrix}, \\ C(\lambda) = (-\lambda) \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} r_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_{n-1} \end{pmatrix}, \end{cases} \qquad (\mathrm{H}_{\mathrm{SL}})$$

if and only if there exists a solution $\{y_k\}_{k=1-n}^{N+1}$ of (SL) such that

$$(x_k)_\mu = \Delta^\mu y_{k-\mu} \qquad \text{for } \mu = 0, \dots, n-1, \ k \in J^*, \tag{3}$$

where $(\cdot)_\mu$ denotes the μth component of a vector, and then $y_{N+2}, \dots, y_{N+n+1}$ exist such that (SL) holds for $k \in J$ and such that

$$(u_k)_\mu = \sum_{\sigma=\mu+1}^{n} (-1)^{\sigma-\mu-1} r_\sigma \, \Delta^{2\sigma-\mu-1} y_{k+1-\sigma}$$
$$\text{for } \mu = 0, \dots, n-1, k \in J^*. \tag{4}$$

3. INEQUALITIES FOR DISCRETE RICCATI EQUATIONS

In this section we present a Sturm-type inequality for symmetric matrices, which are constructed from solutions of two LHDS of the form (H) and

$$\Delta X_k = \underset{\sim}{A}_k X_{k+1} + \underset{\sim}{B}_k U_k, \qquad \Delta U_k = \underset{\sim}{C}_k X_{k+1} - \underset{\sim}{A}_k^T U_k, \ k \in \mathbb{Z}, \quad (\underset{\sim}{\mathrm{H}})$$

where we assume that assumption (1) holds for both systems. We start with the following auxiliary identity.

LEMMA 5. *Assume* (1) *for* (H) *and* ($\underset{\sim}{\mathrm{H}}$), *and let* $(X, U), (\underset{\sim}{X}, \underset{\sim}{U})$ *be conjoined bases of* (H) *and* ($\underset{\sim}{\mathrm{H}}$), *respectively. Suppose* (2) *of Lemma* 2,

$\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$ *for* $k \in J$, *and let* $\operatorname{Im} \utilde{X}_0 \subset \operatorname{Im} X_0$. *Then, for* $k \in J$ *we have* $\operatorname{Im} \utilde{X}_{k+1} \subset \operatorname{Im} X_{k+1}$ *and*

$$\Delta\left[\utilde{X}_k^T \left(\utilde{Q}_k - Q_k \right) \utilde{X}_k \right] = Z_k^T D_k Z_k + \hat{Z}_k^T (\mathscr{H}_k - \utilde{\mathscr{H}}_k) \hat{Z}_k + \tilde{Z}_k^T \tilde{D}_k \tilde{Z}_k$$

with symmetric $Q_k, \utilde{Q}_k$ *such that* $X_k^T Q_k X_k = U_k^T X_k$, $\utilde{X}_k^T \utilde{Q}_k \utilde{X}_k = \utilde{U}_k^T \utilde{X}_k$, *with* $D_k, \tilde{D}_k, \mathscr{H}_k, \utilde{\mathscr{H}}_k$ *as in Lemmas* 1 *and* 2, *and with*

$$Z_k = B_k^{\dagger}(\Delta \utilde{X}_k - A_k \utilde{X}_{k+1}) - Q_k \utilde{X}_k,$$

$$\hat{Z}_k = \begin{pmatrix} I \\ -P_k \end{pmatrix} \utilde{X}_{k+1}, \quad \tilde{Z}_k = P_k \utilde{X}_{k+1} + \utilde{U}_k,$$

where P_k *satisfies* $A_k - \utilde{A}_k = (B_k - \utilde{B}_k) P_k$.

Proof. Let $c \in \mathbb{R}^n$ and put $x_k = \utilde{X}_k c$, $\utilde{u}_k = \utilde{U}_k c$, $u_k = B_k^{\dagger}(\Delta x_k - A_k x_{k+1})$. Then $(x, \utilde{u})$ solves $(\utilde{\mathrm{H}})$ so that

$$x_{k+1}^T \utilde{C}_k x_{k+1} + \utilde{u}_k^T \utilde{B}_k \utilde{u}_k = \Delta\left(x_k^T \utilde{u}_k \right) = c^T \left\{ \Delta\left(\utilde{X}_k^T \utilde{Q}_k \utilde{X}_k \right) \right\} c$$

holds for $k \in J$. By Lemma 2, (x, u) is (A, B)-admissible. Next, we prove $\operatorname{Im} \utilde{X}_{k+1} \subset \operatorname{Im} X_{k+1}$ by induction. First $\operatorname{Im} \utilde{X}_k \subset \operatorname{Im} X_k$ yields $X_k X_k^{\dagger} \utilde{X}_k = \utilde{X}_k$ (see [7, Remark 2(iii)] or [6, Lemma A5]), and $(\utilde{\mathrm{H}})$ combined with (2) implies in the same way that $\operatorname{Im}(\Delta \utilde{X}_k - A_k \utilde{X}_{k+1}) \subset \operatorname{Im} B_k$. Therefore, we get

$$\begin{aligned} \utilde{X}_{k+1} &= \utilde{X}_{k+1} - \tilde{A}_k \utilde{X}_k + \tilde{A}_k X_k X_k^{\dagger} \utilde{X}_k \\ &= X_{k+1} X_k^{\dagger} \utilde{X}_k + \tilde{A}_k (\Delta \utilde{X}_k - A_k \utilde{X}_{k+1}) - \left(X_{k+1} - \tilde{A}_k X_k \right) X_k^{\dagger} \utilde{X}_k \\ &= X_{k+1} X_k^{\dagger} \utilde{X}_k + \tilde{A}_k B_k \left\{ B_k^{\dagger} (\Delta \utilde{X}_k - A_k \utilde{X}_{k+1}) - U_k X_k^{\dagger} \utilde{X}_k \right\}. \end{aligned}$$

Since $\operatorname{Im}(\tilde{A}_k B_k) \subset \operatorname{Im} X_{k+1}$ by [7, Remark 2(ii)], it follows that $\operatorname{Im} \utilde{X}_{k+1} \subset \operatorname{Im} X_{k+1}$. Now, $x_k \in \operatorname{Im} \utilde{X}_k \subset \operatorname{Im} X_k$ yields by Lemma 1 that (use also $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$)

$$\begin{aligned} x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k &= \Delta\left[x_k^T Q_k x_k \right] + z_k^T D_k z_k \\ &= c^T \left\{ \Delta\left(\utilde{X}_k^T Q_k \utilde{X}_k \right) + Z_k^T D_k Z_k \right\} c, \end{aligned}$$

where $z_k = Z_k c = u_k - Q_k x_k$. By Lemma 2, the asserted identity follows. ∎

THEOREM 1 (Riccati Inequality). *Assume* (1) *for* (H) *and* $(\utilde{\mathrm{H}})$, *and let* (X, U) *and* $(\utilde{X}, \utilde{U})$ *be conjoined bases of* (H) *and* $(\utilde{\mathrm{H}})$, *respectively. Suppose* *that* $Q_k, \tilde{Q}_k$ *are symmetric with*

$$X_k^T Q_k X_k = U_k^T X_k, \qquad \tilde{X}_k^T \tilde{Q}_k \tilde{X}_k = \tilde{U}_k^T \tilde{X}_k$$

for $k \in J^* = [0, N+1] \cap \mathbb{Z}$, *and assume for* $k \in J^*$

$$\mathscr{H}_k \geq \tilde{\mathscr{H}}_k, \qquad \tilde{B}_k \geq \tilde{B}_k B_k^\dagger \tilde{B}_k, \qquad \operatorname{Ker} B_k \subset \operatorname{Ker} \tilde{B}_k. \tag{5}$$

If $\operatorname{Im} \tilde{X}_0 \subset \operatorname{Im} X_0$, $\tilde{X}_0^T(\tilde{Q}_0 - Q_0)\tilde{X}_0 \geq 0$, *and if* (X, U) *has no focal points in* $(0, N+1]$, *then* $(\tilde{X}, \tilde{U})$ *has no focal points in* $(0, N+1]$ *either, and*

$$\tilde{X}_k^T \big(\tilde{Q}_k - Q_k\big) \tilde{X}_k \geq 0 \qquad \textit{for all } k \in J^*.$$

Proof. By [20, Lemma 3.1.10], our assumption (5) implies (2) of Lemma 2. Now Lemma 5 yields (observe that $D_k \geq 0$, $\mathscr{H}_k - \tilde{\mathscr{H}}_k \geq 0$, $\tilde{D}_k \geq 0$ for $k \in J^*$ by the assumptions) $\Delta[\tilde{X}_k^T(\tilde{Q}_k - Q_k)\tilde{X}_k] \geq 0$ for $k \in J$ so that

$$\tilde{X}_k^T \big(\tilde{Q}_k - Q_k\big) \tilde{X}_k \geq \tilde{X}_0^T \big(\tilde{Q}_0 - Q_0\big) \tilde{X}_0 \geq 0 \qquad \text{for } k \in J^*.$$

It remains to show that $(\tilde{X}, \tilde{U})$ has no focal points in $(0, N+1]$. We prove this via Lemma 3. To begin with, let $(x, \tilde{u})$ be $(\tilde{A}, \tilde{B})$-admissible with $x_0 \in \operatorname{Im} \tilde{X}_0$, $x_{N+1} = 0$, and put $u_k = B_k^\dagger(\Delta x_k - A_k x_{k+1})$. Then, by Lemmas 1, 2, and the assumptions

$$\begin{aligned}
&\sum_{k=0}^{N} \left\{ x_{k+1}^T \tilde{C}_k x_{k+1} + \tilde{u}_k^T \tilde{B}_k \tilde{u}_k \right\} + x_0^T \tilde{Q}_0 x_0 \\
&\qquad \geq \mathscr{F}_0(x, u) + x_0^T \tilde{Q}_0 x_0 \\
&\qquad = x_{N+1}^T Q_{N+1} x_{N+1} + x_0^T \big(\tilde{Q}_0 - Q_0\big) x_0 + \sum_{k=0}^{N} z_k^T D_k z_k \geq 0,
\end{aligned}$$

and if we have always equality, then (use Lemma 1)

$$D_k z_k = X_k X_{k+1}^\dagger x_{k+1} - x_k = 0 \qquad \text{for } k \in J,$$

and $x_{N+1} = 0$ implies that $x_k = 0$ for all $k \in J^*$. Hence, Lemma 3 yields that $(\tilde{X}, \tilde{U})$ has no focal points in $(0, N+1]$. ∎

Now we apply Theorem 1 to obtain inequalities for so-called eventually minimal solutions of (R) and ($\tilde{\text{R}}$) corresponding to (H) and ($\tilde{\text{H}}$), respectively. The *eventually minimal* solution Q^- of (R) is $Q_k^- := U_k \tilde{X}_k^{-1}$, where (X, U) is the *recessive* solution as defined in Section 2. The terminology, eventually minimal solution, is justified by the fact that any other symmetric solution Q of (R) which exists on an interval $[l, \infty)$ satisfies $Q_k \geq Q_k^-$ for all $k \geq m$ with $m \geq l$ sufficiently large. This property of the solution Q^- for (H) with $B_k > 0$ for all k is proved in [15], but the same method applies also to (H) (or (R)) under the weaker assumptions here.

Moreover, recall that in the theory of continuous Hamiltonian systems

$$\dot{x} = A(t)x + B(t)u, \qquad \dot{u} = C(t)x - A^T(t)u \tag{$\mathrm{H_c}$}$$

and the corresponding Riccati equations

$$\dot{Q} + A^T(t)Q + QA(t) + Q^T B(t)Q - C(t) = 0 \tag{$\mathrm{R_c}$}$$

the terminology *principal* and *distinguished* solution is used instead of the "discrete" terminology recessive and eventually minimal solution. In contrast to the discrete case, any symmetric solution $Q(t)$ of $(\mathrm{R_c})$ on an interval (a, ∞) must satisfy $Q(t) \geq Q^-(t)$ for *all* t for which it exists. More generally, if (X, U) and $(\tilde{X}, \tilde{U})$ are two conjoined bases of $(\mathrm{H_c})$ and if

$$U(t_1)X^{-1}(t_1) \geq \tilde{U}(t_1)\tilde{X}^{-1}(t_1), \qquad U(t_2)X^{-1}(t_2) \ngeq \tilde{U}(t_2)\tilde{X}^{-1}(t_2)$$

for some $t_1 < t_2$, then $\tilde{X}(t)$ must be singular for some $t \in (t_1, t_2)$ (where it is assumed that $B(t) \geq 0$, see [20]). For discrete Riccati equations we derive a similar result.

LEMMA 6. *Suppose that $Q, \tilde{Q}$ and B are (real) symmetric $n \times n$-matrices such that $I + BQ$ and $I + B\tilde{Q}$ are invertible, and such that*

$$(I + BQ)^{-1}B \geq 0 \qquad \text{and} \qquad \Delta := Q(I + BQ)^{-1} - \tilde{Q}\big(I + B\tilde{Q}\big)^{-1} \geq 0.$$

Then $Q \geq \tilde{Q}$.

Proof. First, by our assumptions, $(I + BQ)^{-1}B$, $(I + B\tilde{Q})^{-1}B$, $Q(I + BQ)^{-1}$, and $\tilde{Q}(I + B\tilde{Q})^{-1}$ are symmetric. Thus the identities

$$\begin{aligned} Q - \tilde{Q} &= (I + BQ)^T Q(I + BQ)^{-1}\big(I + B\tilde{Q}\big) - \tilde{Q} - QB\tilde{Q} \\ &= \big(I + B\tilde{Q}\big)^T \Delta\big(I + B\tilde{Q}\big) + \big(Q - \tilde{Q}\big)B\Delta\big(I + B\tilde{Q}\big) \\ &\geq \big(Q - \tilde{Q}\big)B\Delta\big(I + B\tilde{Q}\big); \end{aligned}$$

$$\begin{aligned} \big(Q - \tilde{Q}\big)B\Delta\big(I + B\tilde{Q}\big) &= \big(Q - \tilde{Q}\big)B\big\{Q(I + BQ)^{-1}\big(I + B\tilde{Q}\big) - \tilde{Q}\big\} \\ &= \big(Q - \tilde{Q}\big)B\big\{I - Q(I + BQ)^{-1}B\big\}\big(Q - \tilde{Q}\big) \\ &= \big(Q - \tilde{Q}\big)B(I + QB)^{-1}\big(Q - \tilde{Q}\big) \geq 0 \end{aligned}$$

hold by our assumptions. Hence $Q \geq \tilde{Q}$. ∎

Next, observe that any two solutions $Q, \tilde{Q}$ of (R) satisfy

$$Q_{k+1} - \tilde{Q}_{k+1} = (I - A_k)^T \times \left\{ Q_k(I + B_k Q_k)^{-1} - \tilde{Q}_k\left(I + B_k \tilde{Q}_k\right)^{-1} \right\}(I - A_k).$$

Hence, we obtain the following statement.

LEMMA 7. *Assume* (1), *and suppose that* $Q_k, \tilde{Q}_k, Q_{k+1}, \tilde{Q}_{k+1}$ *are symmetric and satisfy* (R) *for some fixed* $k \in \mathbb{Z}$ *with* $(I + B_k Q_k)^{-1} B_k \geq 0$. *Then* $Q_{k+1} \geq \tilde{Q}_{k+1}$ *implies that* $Q_k \geq \tilde{Q}_k$.

THEOREM 2. *Assume* (1) *for* (H) *and* ($\underset{\sim}{\mathrm{H}}$), *suppose that* (H) *and* ($\underset{\sim}{\mathrm{H}}$) *are eventually controllable, that* (H) *is eventually disconjugate, and suppose that* (5) *of Theorem* 1 *holds for all sufficiently large* k. *Then the eventually minimal solutions* Q^- *and* $\underset{\sim}{Q}^-$ *of* (R) *and of*

$$Q_{k+1} = \underset{\sim}{C}_k + \left(I - \underset{\sim}{A}_k^T\right) Q_k (I + \underset{\sim}{B}_k Q_k)^{-1} (I - \underset{\sim}{A}_k), \tag{$\underset{\sim}{\mathrm{R}}$}$$

respectively, satisfy $\underset{\sim}{Q}_k^- \leq Q_k^-$ *for all* $k \geq N$ *with* N *sufficiently large*.

Proof. First, as before in Theorem 1, (5) implies (2). Next, the eventual disconjugacy of (H) yields the same for ($\underset{\sim}{\mathrm{H}}$) via Theorem 1. Moreover, by the discussion in Section 2 (preceding Lemma 1), our assumptions guarantee the existence of the recessive solutions of (H) and ($\underset{\sim}{\mathrm{H}}$), and therefore the existence of the recessive solutions Q^- and $\underset{\sim}{Q}^-$. Now, let $N \in \mathbb{N}$ such that the recessive solutions (X, U) and $(\underset{\sim}{X}, \underset{\sim}{U})$ of (H) and ($\underset{\sim}{\mathrm{H}}$), respectively, have no focal points in $[N, \infty) \cap \mathbb{Z}$. For fixed $m \geq N$ we consider the conjoined basis $(\underset{\sim}{X}^*, \underset{\sim}{U}^*)$ of (H) with $\underset{\sim}{X}_m^* = I$, $\underset{\sim}{U}_m^* = Q_m^- = U_m X_m^{-1}$. Then, by Theorem 1, $(\underset{\sim}{X}^*, \underset{\sim}{U}^*)$ has no focal points in (m, ∞), $\underset{\sim}{Q}_k = \underset{\sim}{U}_k^* \underset{\sim}{X}_k^{*-1}$ exists, and $\underset{\sim}{Q}_k \geq Q_k^-$ for $k \geq m$. Moreover, there exists $l > m$ with

$$\underset{\sim}{Q}_k \geq \underset{\sim}{Q}_k^- \qquad \text{for all } k \geq l$$

by the discussion after Theorem 1. Hence, by Lemma 7, $\underset{\sim}{Q}_k \geq \underset{\sim}{Q}_k^-$ for all $k \geq m$, in particular, $Q_m^- = \underset{\sim}{Q}_m \geq \underset{\sim}{Q}_m^-$. This yields our assertion. ∎

Let us conclude this section with some comments.

Remark 1. (i) Our Riccati inequality, Theorem 1, is the discrete version of [20, Theorem 5.1.2]. This theorem plays an important role in the investigation of the asymptotic behavior of the continuous Riccati matrix equation (R_c) and in the investigation of eigenvalue problems associated with (H_c).

(ii) For solutions of the continuous Riccati equation we have the following statement: For any two symmetric solutions $Q, \tilde{Q}$ of $(\mathrm{R}_{\mathrm{c}})$, which exist on an interval $[t_1, t_2]$, we have that $\operatorname{ind}(Q - \tilde{Q})(t)$ is constant on $[t_1, t_2]$, where ind denotes the index, i.e., the number of negative eigenvalues of a matrix. Observe that the statement of Lemma 7 in this terminology reads as $\operatorname{ind}(Q_{k+1} - \tilde{Q}_{k+1}) = 0$ implies that $\operatorname{ind}(Q_k - \tilde{Q}_k) = 0$. We conjecture that the more general result in the continuous case holds also for the discrete equation (R).

(iii) Theorem 2 concerns inequalities for minimal solutions of (R) and $(\underset{\sim}{\mathrm{R}})$ near infinity. Note that Theorem 1 states essentially that $Q_k \geq \underset{\sim}{Q}_k$ for solutions Q and $\underset{\sim}{Q}$ of (R) and $(\underset{\sim}{\mathrm{R}})$, while the assertion of Theorem 2 is the *reversed* inequality $\underset{\sim}{Q}_k^- \leq Q_k^-$ for the eventually minimal solutions of (R) and $(\underset{\sim}{\mathrm{R}})$. In [2], C. Ahlbrandt introduced the concept of maximal solutions near minus infinity (also called *primordially maximal* solution) of the special Riccati equation (R) which corresponds to the recurrence equation $-R_k x_{k+1} - R_{k-1} x_{k-1} + S_k x_k = 0$ from the introduction. He proved that any solution which exists on $(-\infty, N] \cap \mathbb{Z}$ is less than or equal to the primordially maximal solution Q^+ near $-\infty$. He gave an explicit formula for this solution in terms of matrix continued fractions. His construction of the primordially maximal solution applies likely to the more general equations (R) which are considered in this paper. We conjecture that, similarly to the case considered in [2, 4], any solution Q of (R) which exists on $\mathbb{Z}$ satisfies $Q_k^- \leq Q_k \leq Q_k^+$ except for at most finitely many $k \in \mathbb{Z}$. Moreover, Lemma 7 suggests that this number of "exceptions" is closely related to the number of focal points of the conjoined basis (X, U) of (H) which determines the Q_k.

(iv) In [15], L. Erbe and P. Yan prove the following statement for the Riccati equation (R) with positive definite B_k's: Let W_k be symmetric matrices such that

$$\begin{aligned} R[W_k] := {} & \Delta W_k + C_k + A_k^T W_k + W_k A_k - A_k^T W_k A_k \\ & + (I - A_k)^T W_k \left(B_k^{-1} + W_k\right)^{-1} (I - A_k) = 0, \end{aligned}$$

and let V_k be symmetric matrices with $R[V_k] \leq 0$. Then $V_k + B_k^{-1} > 0$ for $k \in J$ and $V_0 \leq W_0$ imply that $V_k \geq W_k$ for $k \in J$. This result follows from Theorem 1 by putting $\underset{\sim}{A}_k = A_k$, $\underset{\sim}{B}_k = B_k$, $\underset{\sim}{C}_k = -R[V_k]$. One may verify directly that all assumptions of Theorem 1 hold. In [15], the inequality is used to study algebraic Riccati equations. Our Theorems 1 and 2 combined with results in [21] may be used to study algebraic Riccati equations without assuming that B is positive definite.

4. ASYMPTOTICS: RESULTS

We now turn our attention to Sturm–Liouville difference equations of the form

$$L(y)_{k+1} := \sum_{\mu=0}^{n} (-1)^{\mu} r_{\mu} \Delta^{2\mu} y_{k+1-\mu} = \lambda y_{k+1}, \qquad 0 \le k \le N-n \quad \text{(SL)}$$

with $\lambda \in \mathbb{R}$ for some fixed $N \ge n$ and given reals $r_0, \dots, r_n$ with $r_n \neq 0$.

As is well known and stated more precisely at the end of Section 2, (SL) is equivalent to the LHDS of the form $(\mathrm{H}_{\mathrm{SL}})$. For fixed $n \times n$-matrices X_0 and U_0 that are independent of $\lambda \in \mathbb{R}$ and that satisfy $X_0^T U_0 = U_0^T X_0$ and $\operatorname{rank}(X_0^T, U_0^T) = n$ we consider for each $\lambda \in \mathbb{R}$ the solution $(X(\lambda), U(\lambda))$ of $(\mathrm{H}_{\mathrm{SL}})$ with $X_0(\lambda) = X_0$ and $U_0(\lambda) = U_0$. Of course the quotient $Q_k(\lambda) = U_k(\lambda) X_k^{-1}(\lambda)$ is a solution of the corresponding Riccati matrix difference equation provided $X_k(\lambda)$ is invertible. The main result of this section gives the asymptotic expansion of $Q_k(\lambda)$ as $|\lambda|$ tends to infinity in terms of negative powers of λ. In order to state this result we introduce the following $n \times n$-matrices via their (μ, ν)th entries $0 \le \mu, \nu \le n-1$. These entries are certain binomial coefficients $\binom{\alpha}{\beta}$, and for convenience we assume throughout $\binom{\alpha}{\beta} = 0$ for integers α, β with $\beta < 0$ or $\alpha < \beta$. We need some notation

$$\begin{cases} (T)_{\mu\nu} = (-1)^{\nu} \binom{n-1-\mu}{\nu}, \\ \left(\tilde{T}_{\sigma}\right)_{\mu\nu} = (-1)^{n-1-\mu-\nu} \binom{2\sigma}{\sigma+n+\nu-\mu}, \\ (H_{\sigma})_{\mu\nu} = (-1)^{\nu} \binom{\sigma}{\sigma+\nu-\mu}, \\ \left(\tilde{H}_{\sigma}\right)_{\mu\nu} = \begin{cases} (-1)^{\nu} \binom{\sigma}{\sigma+\mu-\nu} & \text{if } \mu \ge n-\sigma \\ 0 & \text{otherwise.} \end{cases} \end{cases} \tag{6}$$

With the above notation (6) our main result now reads as follows.

THEOREM 3. *Suppose $(X(\lambda), U(\lambda))$ are conjoined bases of $(\mathrm{H}_{\mathrm{SL}})$ for $\lambda \in \mathbb{R}$ such that the initial values $X_0 = X_0(\lambda)$ and $U_0 = U_0(\lambda)$ are fixed matrices independent of λ. Then, for sufficiently large $|\lambda|$, $X_k(\lambda)$ is invertible when $k \ge 3n$, and the quotient $Q_k(\lambda) = U_k(\lambda) X_k^{-1}(\lambda)$ satisfies uniformly for $k \ge 3n$,*

$$Q_k(\lambda) = T^T\left\{-\lambda I + G_0 + \lambda^{-1} G_1 + \lambda^{-2} G_2 + O(\lambda^{-3})\right\} T \qquad \text{as } |\lambda| \to \infty,$$

where we used notation (6) *and*

$$\begin{cases} \tilde{T} = \sum_{\sigma=0}^{n} r_\sigma \tilde{T}_\sigma, \qquad G_0 = \sum_{\sigma=0}^{n} r_\sigma H_\sigma^T H_\sigma, \qquad G_1 = \tilde{T}^T \tilde{T}, \\ G_2 = \sum_{\sigma=0}^{n} r_\sigma \tilde{T}^T \left\{ H_\sigma^T H_\sigma + \tilde{H}_\sigma^T \tilde{H}_\sigma \right\} \tilde{T}. \end{cases} \tag{7}$$

Moreover, T, $\tilde{T}$, *and* H_σ, $0 \le \sigma \le n$, *are invertible*.

Remark 2. We shall give a proof of the above result in the next section. However, here we wish to remark that $(H_\sigma)_{\mu\nu} = 0$ for $\nu > \mu$ and $(H_\sigma)_{\nu\nu} = (-1)^\nu$ so that H_σ is invertible for $0 \le \sigma \le n$. Also, $\mu < \nu$ implies $2\sigma \le \sigma + n < \sigma + n + \nu - \mu$ so that $(\tilde{T}_\sigma)_{\mu\nu} = 0$ for $\mu < \nu$. Since $(\tilde{T}_\sigma)_{\nu\nu} = (-1)^{n-1}\binom{2\sigma}{\sigma+n} \ne 0$ only for $\sigma = n$ and since $r_n \ne 0$, $\tilde{T}$ is invertible too. Finally, an easy calculation shows that T is invertible and that $(T^{-1})_{\mu\nu} = (-1)^{n-1-\nu}\binom{\mu}{n-1-\nu}$ holds. These considerations of course imply that G_1 and $H_\sigma^T H_\sigma$, $0 \le \sigma \le n$, are positive definite so that G_0 and G_2 are positive definite as well provided that, e.g., $r_\sigma \ge 0$ holds for all $0 \le \sigma \le n$.

Let us briefly give a hint how the matrix T shows up when dealing with Eq. (SL). The transformation formulae (3) and (4) at the end of Section 2 may be equivalently rewritten as

$$\begin{aligned} (x_k)_\mu = \Delta^\mu y_{k-\mu} &= \sum_{s=0}^{\mu} \binom{\mu}{s} (-1)^{s-\mu} y_{k-\mu+s} \\ &= \sum_{s=0}^{n-1} \binom{\mu}{n-1-s} (-1)^{n-1-s} y_{k+1-n+s} = \sum_{s=0}^{n-1} (T^{-1})_{\mu s} (\tilde{x}_k)_s \end{aligned}$$

so that

$$\begin{aligned} x_k = T^{-1}\tilde{x}_k \qquad &\text{with } (\tilde{x}_k)_\mu = y_{k+1-n+\mu}, \\ (T^{-1})_{\mu\nu} &= (-1)^{n-1-\nu} \binom{\mu}{n-1-\nu} \end{aligned} \tag{8}$$

and

$$\begin{aligned} (u_k)_\mu = &\sum_{\sigma=\mu+1}^{n} (-\Delta)^{\sigma-\mu-1} \{ r_\sigma \Delta^\sigma y_{k+1-\sigma} \} \\ &= \sum_{\nu=0}^{n-1} \sum_{\sigma=\mu+1}^{n} r_\sigma \binom{2\sigma-\mu-1}{\nu+\sigma-n} (-1)^{\nu+n} y_{k+1-n+\nu} \\ &\quad + \sum_{\nu=0}^{n-\mu-1} \sum_{\sigma=\mu+1}^{n} r_\sigma \binom{2\sigma-\mu-1}{\nu+\sigma} (-1)^{\nu} y_{k+1+\nu} \end{aligned}$$

so that

$$\begin{cases} u_k = T_1 \tilde{x}_k + T_2 \tilde{\tilde{x}}_k \qquad \text{with } \left(\tilde{\tilde{x}}_k\right)_\mu = y_{k+1+\mu} \text{ and} \\ T_1 = \sum_{\sigma=0}^{n} r_\sigma T_{1\sigma}, \qquad T_2 = \sum_{\sigma=0}^{n} r_\sigma T_{2\sigma}, \text{ where} \\ (T_{1\sigma})_{\mu\nu} = \begin{cases} (-1)^{\nu+n} \binom{2\sigma - \mu - 1}{\sigma + \nu - n} & \text{if } \mu \le \sigma - 1 \\ 0 & \text{otherwise,} \end{cases} \\ (T_{2\sigma})_{\mu\nu} = \begin{cases} (-1)^{\nu} \binom{2\sigma - \mu - 1}{\sigma + \nu} & \text{if } \mu \le \sigma - 1 \\ 0 & \text{otherwise.} \end{cases} \end{cases} \tag{9}$$

As already mentioned before, the proof of Theorem 3 will be delayed to Section 5. For the remainder of this section we state and prove an application of Theorem 3, namely an inequality for certain finite differences. Its proof requires a combination of the above result on asymptotics, of the Reid Roundabout Theorem, and of the discrete Picone formula from Section 2.

THEOREM 4. *Let $n \in \mathbb{N}$, $r_\sigma \ge 0$ for $0 \le \sigma \le n-1$, $r_n > 0$, and let X_0 and U_0 be $n \times n$-matrices with $\operatorname{rank}(X_0^T, U_0^T) = n$, $X_0^T U_0 = U_0^T X_0$, and $X_0^T U_0 \ge 0$. Then there exists $c_0 = c_0(n) > 0$ such that for all $N \ge 3n-1$, $c \ge c_0$, and for all $y_{1-n}, \dots, y_{N+1} \in \mathbb{R}$ with*

$$x_0 = \left(\Delta^\mu y_{-\mu}\right)_{\mu=0}^{n-1} = X_0 d \in \operatorname{Im} X_0 \tag{10}$$

the inequality

$$\begin{aligned} \sum_{k=0}^{N} \sum_{\sigma=0}^{n} r_\sigma \{\Delta^\sigma y_{k+1-\sigma}\}^2 + c \sum_{k=1}^{N-n+1} y_k^2 + d^T X_0^T U_0 d \\ \ge \tilde{x}_{N+1}^T \left\{ G_0 - \frac{1}{c} G_1 \right\} \tilde{x}_{N+1} \end{aligned} \tag{11}$$

holds, where (see (8)) $\tilde{x}_{N+1} = (y_{N+2-n}, \dots, y_{N+1})^T$, and where $G_0, G_1 > 0$ are defined by (7). The inequality is sharp in the following sense, by using the same notation, and with the same c_0.

SUPPLEMENT. *There exists* $c_1 = c_1(n) > 0$ *such that for all* $N \geq 3n - 1$, $c \geq c_0$ *there exist* $y_{1-n}, \ldots, y_{N+1} \in \mathbb{R}$ *with* (10) *such that*

$$\begin{cases} \sum_{k=0}^{N} \sum_{\sigma=0}^{n} r_\sigma \{\Delta^\sigma y_{k+1-\sigma}\}^2 + c \sum_{k=1}^{N+1-n} y_k^2 + d^T X_0^T U_0 d \\ < \tilde{x}_{N+1}^T G_0 \tilde{x}_{N+1} - \left\{1 - \frac{c_1}{c}\right\} \frac{1}{c} \tilde{x}_{N+1}^T G_1 \tilde{x}_{N+1}. \end{cases} \tag{12}$$

Proof. Let $(X(\lambda), U(\lambda))$ be the conjoined basis of $(\mathrm{H}_{\mathrm{SL}})$ with initial values X_0 and U_0 as in our theorem. If $\lambda < 0$, then

$$\mathscr{F}(y, \lambda) := \sum_{k=0}^{N} \sum_{\sigma=0}^{n} r_\sigma \{\Delta^\sigma y_{k+1-\sigma}\}^2 + (-\lambda) \sum_{k=0}^{N} y_{k+1}^2 + d^T X_0^T U_0 d$$

is positive for all $\{y_k\}_{k=1-n}^{N+1} \neq 0$ with (10) and with $y_{N+2-n} = \cdots = y_{N+1} = 0$. Hence

$$\tilde{\mathscr{F}}(x, u) = \sum_{k=0}^{N} \left\{x_{k+1}^T C(\lambda) x_{k+1} + u_k^T B u_k\right\} + x_0^T X_0 X_0^\dagger U_0 X_0^\dagger x_0$$

is positive for all with respect to $(\mathrm{H}_{\mathrm{SL}})$ admissible pairs (x, u) with $x_0 \in \operatorname{Im} X_0$, $x_{N+1} = 0$, and $x \neq 0$. The Reid Roundabout Theorem, Lemma 3 from Section 2, now implies that $(X(\lambda), U(\lambda))$ has no focal points in $(0, N+1]$, i.e., that

$$\operatorname{Ker} X_{k+1}(\lambda) \subset \operatorname{Ker} X_k(\lambda), \quad D_k(\lambda) = X_k(\lambda) X_{k+1}^\dagger(\lambda) \tilde{A} B \geq 0 \qquad \text{for all } 0 \leq k \leq N$$

holds. The discrete Picone identity, Lemma 1 from Section 2, now in turn implies

$$\begin{aligned} \tilde{\mathscr{F}}(x, u) &= \sum_{k=0}^{N} \left\{\Delta\left(x_k^T Q_k(\lambda) x_k\right) + z_k^T(\lambda) D_k(\lambda) z_k(\lambda)\right\} + x_0^T Q_0 x_0 \\ &= x_{N+1}^T Q_{N+1}(\lambda) x_{N+1} + \sum_{k=0}^{N} z_k^T(\lambda) D_k(\lambda) z_k(\lambda) \\ &\geq x_{N+1}^T Q_{N+1}(\lambda) x_{N+1} \end{aligned}$$

for all with respect to $(\mathrm{H}_{\mathrm{SL}})$ admissible pairs (x, u) with $x_0 \in \operatorname{Im} X_0$, where we put $Q_k(\lambda) = X_k(\lambda) X_k^\dagger(\lambda) U_k(\lambda) X_k^\dagger(\lambda)$ and $z_k(\lambda) = u_k - Q_k(\lambda) x_k$. By Theorem 3 there exist $c_0 = c_0(n) > 0$ and $c_1 = c_1(n) > 0$ such that

$X_k(\lambda)$ is invertible and

$$0 < Q_k(\lambda) - T^T\left\{cI + G_0 - \frac{1}{c}G_1\right\}T < \frac{c_1}{c^2}T^TG_1T \tag{13}$$

hold for all $k \geq 3n$ and for all $c = -\lambda \geq c_0$ (note $G_1 > 0$). Now, if $N + 1 \geq 3n$ and $c \geq c_0$, the above considerations together with the left part of inequality (13) imply

$$\begin{aligned}
&\sum_{k=0}^{N}\sum_{\sigma=0}^{n} r_\sigma\{\Delta^\sigma y_{k+1-\sigma}\}^2 + (-\lambda)\sum_{k=0}^{N-n} y_{k+1}^2 + d^TX_0^T U_0 d \\
&\quad = \mathscr{F}(y, \lambda) - c\sum_{k=N-n+1}^{N} y_{k+1}^2 \\
&\quad = \mathscr{F}(y, \lambda) - c\tilde{x}_{N+1}^T\tilde{x}_{N+1} \\
&\quad \geq \tilde{x}_{N+1}^T(T^{-1})^T Q_{N+1}(\lambda)T^{-1}\tilde{x}_{N+1} - c\tilde{x}_{N+1}^T\tilde{x}_{N+1} \\
&\quad = \tilde{x}_{N+1}^T(T^{-1})^T\{Q_{N+1}(\lambda) - cT^TT\}T^{-1}\tilde{x}_{N+1} \\
&\quad \geq \tilde{x}_{N+1}^T\left\{G_0 - \frac{1}{c}G_1\right\}\tilde{x}_{N+1}
\end{aligned}$$

for all $N \geq 3n - 1$, for all $c \geq c_0$, and for all $y_{1-n}, \dots, y_{N+1} \in \mathbb{R}$ with (10), and this is the assertion (11).

Now, to prove the Supplement, we consider the solution (x, u) of $(\mathrm{H}_{\mathrm{SL}})$ with $x_0 = X_0 d$, $u_0 = U_0 d$ for some $d \in \mathbb{R}^n$, $d \neq 0$ so that $x_k = X_k d$, $u_k = U_k d$, and $z_k = u_k - X_k X_k^\dagger U_k X_k^\dagger x_k = 0$ hold for all $0 \leq k \leq N + 1$. But then we have

$$\begin{aligned}
\mathscr{F}(y, \lambda) - c\sum_{k=N-n+1}^{N} y_{k+1}^2 &= \tilde{x}_{N+1}^T(T^{-1})^T\{Q_{N+1}(\lambda) - cT^TT\}T^{-1}\tilde{x}_{N+1} \\
&< \tilde{x}_{N+1}^T\left\{G_0 - \frac{1}{c}G_1 + \frac{c_1}{c^2}G_1\right\}\tilde{x}_{N+1} \\
&= \tilde{x}_{N+1}^TG_0\tilde{x}_{N+1} - \left\{1 - \frac{c_1}{c}\right\}\frac{1}{c}\tilde{x}_{N+1}^TG_1\tilde{x}_{N+1}
\end{aligned}$$

for all $N \geq 3n - 1$ and for all $c \geq c_0$ by the right part of inequality (13), and this is assertion (12) so that our proof is complete. ∎

5. ASYMPTOTICS: PROOFS

Let us in this last section give the proof of our main result on asymptotics of solutions of Riccati matrix difference equations, Theorem 3. We will proceed somewhat similar to the considerations in [19, Sect. 5] where Eq. (SL) already has been examined for the case of $r_0 = \cdots = r_{n-1} = 0$ and $r_n = 1$. We therefore recommend to have this work ready for reference.

First of all, let us construct a fundamental matrix of (SL). To do so, we compute the characteristic polynomial of (SL). It is given by

$$P(x) = (-1)^{n+1}\lambda x^n + \sum_{\mu=0}^{n} (-1)^{n-\mu} r_\mu \sum_{\nu=0}^{2\mu} \binom{2\mu}{\nu} (-1)^\nu x^{n+\nu-\mu}$$

$$= x^n \left\{ (-1)^{n+1}\lambda + \sum_{\mu=0}^{n} (-1)^{n-\mu} \frac{r_\mu}{x^\mu} (x-1)^{2\mu} \right\}$$

so that

$$P(x) = x^n \left\{ (-1)^{n+1}\lambda + \sum_{\mu=0}^{n} (-1)^{n-\mu} r_\mu \left(x + \frac{1}{x} - 2 \right)^\mu \right\} = x^{2n} P\left(\frac{1}{x}\right). \tag{14}$$

We let $-\lambda r_n = |\lambda r_n| e^{i\pi l}$ with $l \in \{0, 1\}$ and put for $\lambda \neq 0$

$$\rho = \rho(\lambda) = \sqrt[n]{|\lambda / r_n|}, \qquad \varepsilon_\nu = \exp\left\{ \frac{2\pi i \nu}{n} + \pi i \frac{n-1-l}{n} \right\} \quad \text{for } 0 \le \nu \le n. \tag{15}$$

If ρ is sufficiently large, $\tilde{P}(y) := (-1)^{n+1}\lambda + \sum_{\mu=0}^{n} (-1)^{n-\mu} r_\mu y^\mu$ possesses n distinct zeros, denoted by $\rho\tilde{\varepsilon}_\nu(\lambda)$, $0 \le \nu \le n$. Now, $\varepsilon_\nu^n = (-1)^n \lambda r_n / |\lambda r_n|$ and (note (15))

$$0 = \tilde{P}(\rho\tilde{\varepsilon}_\nu(\lambda)) = \rho^n \left\{ (-1)^{n+1}\lambda\rho^{-n} + r_n \tilde{\varepsilon}_\nu^n(\lambda) + \frac{1}{\rho} \sum_{\mu=0}^{n-1} (-1)^{n-\mu} r_\mu \rho^{\mu+1-n} \tilde{\varepsilon}_\nu^\mu(\lambda) \right\}$$

$$\sim \rho^n \left\{ (-1)^{n+1} |\lambda r_n| / \lambda + r_n \tilde{\varepsilon}_\nu^n(\lambda) \right\} = (-1)^{n+1}\lambda \left\{ 1 - \varepsilon_\nu^n \tilde{\varepsilon}_\nu^n(\lambda) \right\} \quad \text{as } \rho \to \infty$$

imply that $\tilde{\varepsilon}_\nu(\lambda) \to \varepsilon_\nu$ for all $0 \le \nu < n$ as $|\lambda| \to \infty$. From now on we assume that ρ is sufficiently large and put

$$\delta_\nu = \delta_\nu(\lambda) = \frac{1}{\rho} + \frac{\tilde{\varepsilon}_\nu}{2}\left\{1 + \sqrt{1 + \frac{4}{\rho\tilde{\varepsilon}_\nu}}\right\}, \qquad x_\nu = \rho\delta_\nu \text{ for } 0 \le \nu < n, \tag{16}$$

where we use the principal value of the square root. Hence

$$\delta_\nu(\lambda) \to \varepsilon_\nu \qquad \text{as } |\lambda| \to \infty \text{ for all } 0 \le \nu < n, \tag{17}$$

and $x_\nu, 1/x_\nu$, $0 \le \nu < n$ are the $2n$ distinct zeros of P because of

$$\begin{aligned} x_\nu + \frac{1}{x_\nu} - 2 = \frac{1}{x_\nu}(x_\nu - 1)^2 &= \frac{1}{x_\nu}\left[\frac{\rho\tilde{\varepsilon}_\nu}{2}\left\{1 + \sqrt{1 + \frac{4}{\rho\tilde{\varepsilon}_\nu}}\right\}\right]^2 \\ &= \frac{\rho\tilde{\varepsilon}_\nu}{x_\nu}\left[\frac{\rho\tilde{\varepsilon}_\nu}{4}\left\{1 + 2\sqrt{1 + \frac{4}{\rho\tilde{\varepsilon}_\nu}} + 1 + \frac{4}{\rho\tilde{\varepsilon}_\nu}\right\}\right] = \rho\tilde{\varepsilon}_\nu \end{aligned}$$

and because of (14). Note that (17) is exactly the crucial relationship (21) from [19]. As in [19, formula (22)] we now proceed by introducing some notation:

$$\begin{cases} V = (v_{\mu\nu}), \quad \tilde{V} = \left(\dfrac{1}{v_{\mu\nu}}\right), \quad v_{\mu\nu} = v_{\mu\nu}(\lambda) = \delta_\nu^{\mu}(\lambda) \\ \qquad \text{for } 0 \le \mu, \nu < n, \\ \Delta = \operatorname{diag}(\delta_0, \dots, \delta_{n-1}), \quad D(\alpha) = \operatorname{diag}(1, \alpha, \dots, \alpha^{n-1}) \\ \qquad \text{for } \alpha \in \mathbb{C}, \\ \Phi = D(\rho)V\Delta^{-1}V^{-1}D(1/\rho), \quad \tilde{\Phi} = D(1/\rho)\tilde{V}\Delta\tilde{V}^{-1}D(\rho), \\ \Psi = \Phi^{-n}, \tilde{\Psi} = \tilde{\Phi}^{-n}, J_1 = I - \rho^{-2n}\Psi^{-1}\tilde{\Psi}, J_2 = I - \rho^{-2n}\tilde{\Psi}\Psi^{-1}. \end{cases} \tag{18}$$

This notation implies the following statements.

LEMMA 8. *Assume notation* (6), (9), *and* (18). *Then we have*

(i) $\Phi^k = D(\rho)V\Delta^{-k}V^{-1}D(1/\rho)$, $\tilde{\Phi}^k = D(1/\rho)\tilde{V}\Delta^k\tilde{V}^{-1}D(\rho)$, $k \in \mathbb{Z}$; $\Psi = D(\rho)V\Delta^n V^{-1}D(1/\rho)$, $\tilde{\Psi} = D(1/\rho)\tilde{V}\Delta^{-n}\tilde{V}^{-1}D(\rho)$;

(ii) $\rho^{-k}\Phi^k = (A^T)^k + \rho^{-n}A^{n-k}\Psi^{-1}$, $\rho^{k-n}\Phi^{-k} = (A^T)^{n-k}\Psi + \rho^{-n}A^k$, $\rho^{-k}\tilde{\Phi}^{-k} = A^k + \rho^{-n}(A^T)^{n-k}\tilde{\Psi}$, $\rho^{k-n}\tilde{\Phi}^k = A^{n-k}\tilde{\Psi}^{-1} + \rho^{-n}(A^T)^k$, $0 \le k \le n$;

(iii) *If* J_1 *and* J_2 *are invertible, then the fundamental matrix* $W(k)$ *of* (SL) *with the initial condition* $W(0) = I$ *is given by*

$$W(k) = \begin{pmatrix} W_1(k) & W_2(k) \\ W_1(k+n) & W_2(k+n) \end{pmatrix},$$

where $W_1(k) = \{\rho^{-k}\tilde{\Phi}^{-k} - \rho^{k-2n}\Phi^{-k}\Psi^{-1}\tilde{\Psi}\}J_1^{-1}$, $W_2(k) = \{\rho^{k-n}\Phi^{-k}\Psi^{-1} - \rho^{-k-n}\tilde{\Phi}^{-k}\Psi^{-1}\}J_2^{-1}$;

(iv) *If* $(X = X(\lambda), U = U(\lambda))$ *is a conjoined basis of* $(\mathrm{H}_{\mathrm{SL}})$ *for some* $\lambda \in \mathbb{R}$, *then for* $0 \le k \le N+1$,

$$X_k = \rho^{k-n}T^{-1}\left\{I + \rho^{n-2k}\tilde{\Phi}^{-k}\tilde{J}J^{-1}\Phi^k\Psi\right\}\Psi^{-1}\Phi^{-k}J,$$

$$U_k = T_1TX_k + \rho^kT_2\left\{I + \rho^{-n-2k}\tilde{\Phi}^{-k}\tilde{\Psi}\tilde{J}J^{-1}\Phi^k\right\}\Phi^{-k}J$$

so that $U_kX_k^{-1}$ *is given by*

$$T_1T + \rho^nT_2\left\{I + \rho^{-n-2k}\tilde{\Phi}^{-k}\tilde{\Psi}\tilde{J}J^{-1}\Phi^k\right\}^{-1}\Psi\left\{I + \rho^{n-2k}\tilde{\Phi}^{-k}\tilde{J}J^{-1}\Phi^k\Psi\right\}^{-1}T,$$

where

$$\begin{cases} J = J_2^{-1}T_2^{-1}\{U_0 - T_1TX_0\} - \rho^{-n}\tilde{\Psi}J_1^{-1}TX_0, \\ \tilde{J} = J_1^{-1}TX_0 - \rho^{-n}\Psi^{-1}J_2^{-1}T_2^{-1}\{U_0 - T_1TX_0\} \end{cases} \tag{19}$$

provided the occurring inverse matrices exist.

Proof. Part (i), i.e., [19, formula (23)], follows directly from (18), while (ii) and (iii), i.e., [19, Lemmas 3 and 2], follow exactly as in [19]. Finally, (iv) follows as in the proof of [19, Lemma 5] with (iii) and

$$\begin{pmatrix} X_k \\ U_k \end{pmatrix} = \begin{pmatrix} T^{-1} & 0 \\ T_1 & T_2 \end{pmatrix} W(k) \begin{pmatrix} T & 0 \\ -T_2^{-1}T_1T & T_2^{-1} \end{pmatrix} \begin{pmatrix} X_0 \\ U_0 \end{pmatrix}.$$

(See (8) and (9) from Section 4 and note that T and T_2 are invertible.) ∎

LEMMA 9. *The formulae*

$$\Gamma = \rho^{-n}G - \lambda\rho^{-n}I - \rho^{-2n}\tilde{T}^T\Gamma^{-1}\tilde{T},$$

$$\tilde{\Gamma} = \rho^{-n}G - \lambda\rho^{-n}I - \rho^{-2n}\tilde{T}\tilde{\Gamma}^{-1}\tilde{T}^T$$

hold, where

$$\begin{cases} \tilde{T} \text{ is defined by } (7), \qquad \Gamma = \tilde{T}\Psi, \tilde{\Gamma} = \tilde{T}^T \tilde{\Psi}^{-1}, G = \sum_{\sigma=0}^{n} r_\sigma G_{(\sigma)}; \\ \left(G_{(\sigma)}\right)_{\mu\nu} = (-1)^{\mu-\nu} \binom{2\sigma}{\sigma+\nu-\mu} \qquad \text{for } 0 \le \mu, \nu < n, 0 \le \sigma \le n. \end{cases} \tag{20}$$

Proof. Let $z_0, \dots, z_{n-1}$ denote any zeros of P from (14) and put $Z = (z_\nu^\mu)$ and $\tilde{\Delta} = \operatorname{diag}(z_0, \dots, z_{n-1})$. For $0 \le \mu, \nu < n$ we then have

$$\begin{aligned} (-1)^n \lambda z_\nu^\mu &= \sum_{\sigma=0}^{n} (-1)^{n-\sigma} r_\sigma \left\{ z_\nu + \frac{1}{z_\nu} - 2 \right\}^\sigma z_\nu^\mu \\ &= \sum_{\sigma=0}^{n} (-1)^{n-\sigma} r_\sigma z_\nu^{\mu-\sigma} \sum_{s=0}^{2\sigma} \binom{2\sigma}{s} z_\nu^s (-1)^{2\sigma-s} \\ &= \sum_{\sigma=0}^{n} (-1)^{n-1} r_\sigma \{\Sigma_1 + \Sigma_2 + \Sigma_3\}, \end{aligned}$$

where for $0 \le \sigma \le n$,

$$\begin{aligned} \Sigma_1 &= z_\nu^{\mu-\sigma} \sum_{s=0}^{\sigma-\mu-1} (-1)^{s+\sigma-1} \binom{2\sigma}{s} z_\nu^s \\ &= \sum_{s=0}^{n-1} (-1)^{n-1-s-\mu} \binom{2\sigma}{\sigma+n+\mu-s} z_\nu^{s-n} = \left(\tilde{T}_\sigma^T Z \tilde{\Delta}^{-n}\right)_{\mu\nu}, \\ \Sigma_2 &= z_\nu^{\mu-\sigma} \sum_{s=\sigma-\mu}^{n+\sigma-\mu-1} (-1)^{s+\sigma-1} \binom{2\sigma}{s} z_\nu^s \\ &= \sum_{s=0}^{n-1} (-1)^{s-\mu-1} \binom{2\sigma}{s+\sigma-\mu} z_\nu^s = -\left(G_{(\sigma)} Z\right)_{\mu\nu}, \\ \Sigma_3 &= z_\nu^{\mu-\sigma} \sum_{s=n+\sigma-\mu}^{2\sigma} (-1)^{s+\sigma-1} \binom{2\sigma}{s} z_\nu^s \\ &= \sum_{s=0}^{n-1} (-1)^{n-1-\mu-s} \binom{2\sigma}{\sigma+n+s-\mu} z_\nu^{s+n} = \left(\tilde{T}_\sigma Z \tilde{\Delta}^n\right)_{\mu\nu}. \end{aligned}$$

Therefore we have

$$\{(-\lambda) I + G\} Z = \tilde{T}^T Z \tilde{\Delta}^{-n} + \tilde{T} Z \tilde{\Delta}^n. \tag{21}$$

Now, first, if $z_\nu = \rho\delta_\nu$ (see (16)), then $Z = D(\rho)V$, $\tilde{\Delta} = \rho\Delta$, and $\Psi = \rho^{-n} Z \tilde{\Delta}^n Z^{-1}$ by (18) and Lemma 8(i) so that (21) yields by denoting $\Gamma = \tilde{T}\Psi$

$$(-\lambda)I + G = \tilde{T}^T \rho^{-n} \Psi^{-1} + \tilde{T}\rho^n \Psi = \rho^{-n} \tilde{T}^T \Gamma^{-1} \tilde{T} + \rho^n \Gamma$$

and hence the first formula of our assertion. Next, if $z_\nu = 1/(\rho\delta_\nu)$, we have $Z = D(1/\rho)\tilde{V}$, $\tilde{\Delta} = \rho^{-1}\Delta^{-1}$, and $\tilde{\Psi} = \rho^n Z \tilde{\Delta}^n Z^{-1}$ again by (18) and Lemma
8(i) so that again (21) yields by denoting $\tilde{\Gamma} = \tilde{T}^T \tilde{\Psi}^{-1}$

$$(-\lambda)I + G = \tilde{T}^T \rho^n \tilde{\Psi}^{-1} + \tilde{T}\rho^{-n} \tilde{\Psi} = \rho^n \tilde{\Gamma} + \rho^{-n} \tilde{T} \tilde{\Gamma}^{-1} \tilde{T}^T$$

and hence the second formula of our assertion. ∎

Remark 3. By (17), (18), and Lemma 8(i) we have $\Psi, \tilde{\Psi}, \Psi^{-1}, \tilde{\Psi}^{-1} = O(\rho^{n-1})$ as $\rho \to \infty$ trivially. The above Lemma 9 yields (let $a \in \{-1, 1\}$ denote the sign of $-\lambda r_n$ and note that $\rho^n = |\lambda / r_n|$ by (15))

$$\begin{cases} \Psi = a r_n \tilde{T}^{-1} + \rho^{-n} \tilde{T}^{-1} G - \rho^{-2n} \tilde{T}^{-1} \tilde{T}^T \Psi^{-1}, \\ \tilde{T}^T \tilde{\Psi}^{-1} = a r_n I + \rho^{-n} G - \rho^{-2n} \tilde{T} \tilde{\Psi}. \end{cases}$$

Hence $\Psi(\rho) \to a r_n \tilde{T}^{-1}$ and $\tilde{T}^T \tilde{\Psi}^{-1}(\rho) \to a r_n I$ as $\rho \to \infty$ and therefore $\Psi, \tilde{\Psi}, \Psi^{-1}, \tilde{\Psi}^{-1} = O(1)$ as $\rho \to \infty$. Thus we have

$$\begin{cases} \Psi = a r_n \tilde{T}^{-1} + \rho^{-n} \tilde{T}^{-1} G + O(\rho^{-2n}), \\ \tilde{\Psi} = (a r_n)^{-1} \tilde{T}^T - (a r_n)^{-2} \rho^{-n} G \tilde{T}^T + O(\rho^{-2n}) \qquad \text{as } \rho \to \infty. \end{cases} \tag{22}$$

Note also that, if we define modified matrices Γ_*, G_*, and T_* by

$$\Gamma_* = \rho^n D(-1) \Gamma D(-1), \qquad G_* = D(-1) G D(-1),$$
$$T_* = (-1)^{n-1} D(-1) \tilde{T} D(-1),$$

then Lemma 9 implies $\Gamma_* = (-\lambda)I + G_* - a r_n T_*^T \Gamma_*^{-1} T_*$. Thus we have obtained a *matrix continued fraction* for Γ_*, where all matrix elements of G_*, T_*, $a r_n T_*$, and $(-\lambda)I$ are integers provided that $r_0, \dots, r_n$, and λ are integers. Moreover, they are all non-negative provided $r_0, \dots, r_n$, and $-\lambda$ are non-negative. G_* is symmetric and, as we shall see later, positive definite if $r_0, \dots, r_{n-1} \geq 0$ and $r_n > 0$, $\lambda < 0$.

We may now finish the proof of Theorem 3 as follows.

Proof of Theorem 3. First of all, (18) and (22) imply $J_1 = I + O(\rho^{-2n})$ and $J_2 = I + O(\rho^{-2n})$ as $\rho \to \infty$ so that J_1 and J_2 are invertible for large ρ (which we assume throughout anyway). Next (19) and (22) yield

$$J = O(1) \qquad \text{and} \qquad \tilde{J} = O(1) \text{ as } \rho \to \infty \tag{23}$$

and, moreover, again by using (19) and (22) (with $a = \operatorname{sgn}(-\lambda r_n)$ as above),

$$J = T_2^{-1}(U_0 - T_1 T X_0) - a\frac{\rho^{-n}}{r_n}\tilde{T}^T T X_0 + O(\rho^{-2n}) \qquad \text{as } \rho \to \infty. \tag{24}$$

Let us define $\mathscr{A} := T_2^{-1}(U_0 - T_1 T X_0)$ and $\mathscr{B} := \tilde{T}^T T X_0 = T_2^T X_0$. Note that $T_2 = T^T \tilde{T}$ holds by Lemma 10 (iv) below. Hence

$$\begin{cases} \operatorname{rank}(\mathscr{A}^T, \mathscr{B}^T) = \operatorname{rank}\left(X_0^T, U_0^T\right) = n, \text{ and} \\ \mathscr{A}^T \mathscr{B} = U_0^T X_0 - X_0^T (T_1 T)^T X_0 \text{ is symmetric.} \end{cases}$$

Note that $T_1 T$ is symmetric by Lemma 10 (v) below. Therefore, by [17, Proposition 2], $\mathscr{A} - \varepsilon\mathscr{B}$ is invertible for small $|\varepsilon| > 0$ and $(\mathscr{A} - \varepsilon\mathscr{B})^{-1} = O(1/|\varepsilon|)$ as $\varepsilon \to 0$. Thus

$$J \text{ is invertible and } J^{-1} = O(\rho^n) \text{ as } \rho \to \infty \tag{25}$$

by (24). Now suppose $k = 3n + l \geq 3n$. Then all inverses from Lemma 8(iv) exist, and we obtain from Lemma 8(i), (ii), (iv), Lemma 9, and from (18), (22), (23), (25) that the following asymptotics for $Q_k(\lambda) - T_1 T$ hold uniformly as $\rho \to \infty$:

$$\begin{aligned}
&\rho^n T_2 \left\{ I + \rho^{-7n} \rho^{-l} \tilde{\Phi}^{-l} \tilde{\Psi}^4 \tilde{J} J^{-1} \rho^{-l} \Phi^l \Psi^{-3} \right\} \\
&\qquad \times \Psi \left\{ I + \rho^{-5n} \rho^{-l} \tilde{\Phi}^{-l} \tilde{\Psi}^3 \tilde{J} J^{-1} \rho^{-l} \Phi^l \Psi^{-2} \right\} T \\
&\quad = \rho^n T_2 \Psi T + O(\rho^{-3n}) \\
&\quad = \rho^n T_2 \left\{ a r_n \tilde{T}^{-1} + \rho^{-n} \tilde{T}^{-1} G - \rho^{-2n} \tilde{T}^{-1} \tilde{T}^T \Psi^{-1} \right\} T + O(\rho^{-3n}) \\
&\quad = \rho^n T_2 \left\{ a r_n \tilde{T}^{-1} + \rho^{-n} \tilde{T}^{-1} G - \rho^{-2n} \tilde{T}^{-1} \tilde{T}^T \left[\frac{a}{r_n} \tilde{T} - \frac{1}{r_n^2} \rho^{-n} G \tilde{T} \right] \right\} T \\
&\qquad + O(\rho^{-3n}) \\
&\quad = \rho^n T_2 \tilde{T}^{-1} \left\{ a r_n I + \rho^{-n} G - a\frac{\rho^{-2n}}{r_n} \tilde{T}^T \tilde{T} + \frac{\rho^{-3n}}{r_n^2} \tilde{T}^T G \tilde{T} \right\} T \\
&\qquad + O(\rho^{-3n}) \\
&\quad = T_2 \tilde{T}^{-1} \left\{ -\lambda I + G + \frac{1}{\lambda} \tilde{T}^T \tilde{T} + \frac{1}{\lambda^2} \tilde{T}^T G \tilde{T} \right\} T + O(\rho^{-3n}).
\end{aligned}$$

This yields the statement of our Theorem 3 because of the binomial identities of the concluding Lemma 10. ∎

LEMMA 10. *By elementary calculations* (*see* [24]), *we have for* $0 \le \sigma \le n$ *with the notation* (6), (7), (9), *and* (20):

(i) $T_{2\sigma} = T^T \tilde{T}_\sigma$ (ii) $G_{(\sigma)} = H_\sigma^T H_\sigma + \tilde{H}_\sigma^T \tilde{H}_\sigma$

(iii) $T^T \tilde{H}_\sigma^T \tilde{H}_\sigma = -T_{1\sigma}$ (iv) $T_2 \tilde{T}^{-1} T = T^T T$

(v) $T_1 T + T_2 \tilde{T}^{-1} G T = T^T G_0 T$ (vi) $T_2 \tilde{T}^{-1} \tilde{T}^T \tilde{T} T = T^T G_1 T$

(vii) $T_2 \tilde{T}^{-1} \tilde{T}^T G \tilde{T} T = T^T G_2 T.$

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