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Impulsive differential equations: Periodic solutions and applications☆

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ABSTRACT

This paper deals with the periodic solutions problem for impulsive differential equations. By using Lyapunov's second method and the contraction mapping principle, some conditions ensuring the existence and global attractiveness of unique periodic solutions are derived, which are given from impulsive control and impulsive perturbation points of view. As an application, the existence and global attractiveness of unique periodic solutions for Hopfield neural networks are discussed. Finally, two numerical examples are provided to demonstrate the effectiveness of the proposed results.

1. Introduction

As is well known, impulsive differential equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. They have been extensively studied in the past several years, see Baĭnov and Simeonov (1993), Haddad, Chellaboina, and Nersesov (2006), Ignatyev (2008), Lakshmikantham, Baĭnov, and Simeonov (1989), Li (2012), Nieto and O'Regan (2009), Samoĭlenko and Perestyuk (1995), Stamova and Stamov (2001) and the references cited therein. A very basic and important qualitative problem in the study of impulsive differential equations concerns the existence and attractiveness of periodic solutions. Many important and interesting results on this topic have been reported, see Cooke and Kroll (2002), Huseynov (2010), Nieto (2002), Shen, Li, and Wang (2006), Stamov (2009) and Wang, Yu, and Niu (2012) for recent works.

On the other hand, impulsive control theory has become a very important direction in the theory of impulsive differential equations, stimulated by their numerous applications to problems arising in orbital transfer of satellite (Prussing, Wellnitz, & Heckathorn, 1989), ecosystems management (Liu & Rohlf, 1998), electrical engineering (Yang & Chua, 1997), and so on. More related to this matter, we mention (Li, 2010; Li & Rakkiyappan, 2013; Yang, 1999) and the references in these works.

In this paper, we shall investigate the periodic solutions problem for impulsive differential equations via Lyapunov's second method and contraction mapping principle. Some sufficient conditions ensuring the existence and global attractiveness of periodic solutions are derived from impulsive control and impulsive perturbation points of view, respectively. Especially, our results show that impulsive control may contribute to the existence and attractiveness of periodic solutions. In addition, we develop our theoretical results to study the existence and attractiveness of periodic solutions for Hopfield neural networks. The rest of this paper is organized as follows. In Section 2, we introduce some notations and definitions. In Section 3, we present the main results on periodic solutions problem of the addressed equations. In Section 4, two numerical examples and their computer simulations are given in order to show the effectiveness of our methods. Finally, we shall make some concluding remarks in Section 5.

2. Preliminaries

Notation 2.1. Let $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{N}$ denote the set of the real numbers, the set of the positive numbers, and the set of the positive integers,

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respectively. Moreover, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the n-dimensional and $n \times m$-dimensional real spaces, respectively, equipped with the Euclidean norm $\|\cdot\|$, and $[\cdot]$ denotes the integer function. For any interval $J \subseteq \mathbb{R}$ and any set $S \subseteq \mathbb{R}^k$, $1 \le k \le n$, we put $\mathrm{C}(J, S) = \{\phi : J \to S$ is continuous $\}$, $\mathrm{PC}(J, S) = \{\varphi : J \to S$ is continuous everywhere except at a finite number of points t, at which $\varphi(t^+), \varphi(t^-)$ exist and $\varphi(t^+) = \varphi(t)\}$, and $\mathbb{K} = \{a \in \mathrm{C}(\mathbb{R}_+, \mathbb{R}_+) : a(0) = 0$ and $a(s) > 0$ for $s > 0$ and a is strictly increasing in $s\}$.

Consider the impulsive problem

$$\begin{cases} x'(t) = f(t, x(t)), \quad t \in [t_{k-1}, t_k), \\ \Delta x|_{t=t_k} = x(t_k) - x(t_k^-) = I_k(t_k, x(t_k^-)), \quad k \in \mathbb{N}, \\ x(t_0) = x_0, \end{cases} \tag{2.1}$$

where $x_0 \in \mathbb{R}^n$, x' denotes the right-hand derivative of x, the impulse times $\{t_k\}_{k\in\mathbb{N}}$ satisfy $0 \le t_0 < t_1 < \cdots < t_k < \cdots$ and $\lim_{k\to\infty} t_k = \infty$, $f \in \mathrm{C}([t_{k-1}, t_k) \times \mathbb{R}^n, \mathbb{R}^n)$, and $I_k \in \mathrm{C}([t_0, \infty) \times \mathbb{R}^n, \mathbb{R}^n)$.

Definition 2.2. The function $V : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ belongs to the class ϑ_0 provided

(i) V is continuous on each of the sets $[t_{k-1}, t_k) \times \mathbb{R}^n \times \mathbb{R}^n$ and all limits
$$\lim_{(t,x,y)\to(t_k^-,\hat{x},\hat{y})} V(t, x, y) = V(t_k^-, \hat{x}, \hat{y}) \quad \text{exist;}$$
(ii) $V(t, x, y)$ is locally Lipschitz in x and y, i.e., for given (t, x, y), there exists a neighborhood $U = U(t, x, y)$ and constants $L_1 = L_1(t, x, y)$, $L_2 = L_2(t, x, y)$ such that
$$|V(\tau, u, v) - V(\tau, \tilde{u}, \tilde{v})| \le L_1\|u - \tilde{u}\| + L_2\|v - \tilde{v}\| \quad \text{for } (\tau, u, v), (\tau, \tilde{u}, \tilde{v}) \in U;$$
(iii) $V(t, x, y) \equiv 0$ for any $t \ge t_0$ as $x = y \in \mathbb{R}^n$.

Definition 2.3. Let $V \in \vartheta_0$. For any $(t, x, y) \in [t_{k-1}, t_k) \times \mathbb{R}^n \times \mathbb{R}^n$, the upper right-hand Dini derivative of V along the solution of (2.1) is defined by

$$D^+V(t, x, y) = \limsup_{\delta\to 0^+} \frac{1}{\delta} \{V(t + \delta, x + \delta f(t, x), y + \delta f(t, y)) - V(t, x, y)\}.$$

Definition 2.4. A map $x : \mathbb{R}_+ \to \mathbb{R}^n$ is said to be an ω-periodic solution of (2.1) provided

(i) x satisfies (2.1) and is a piecewise continuous map with first-class discontinuity points;
(ii) x satisfies $x(t + \omega) = x(t)$ for $t \ne t_k$ and $x(t_k + \omega^+) = x(t_k^+)$ for $k \in \mathbb{N}$.

Definition 2.5. Let $x^* = x^*(t, t_0, x_0^*)$ be an ω-periodic solution of (2.1) with initial value (t_0, x_0^*). Then x^* is said to be globally attractive if for any solution $x = x(\cdot, t_0, x_0)$ of (2.1) through (t_0, x_0), $|x - x^*| \to 0$ as $t \to \infty$.

3. Main results

Theorem 3.1. *Assume there exist $w_1, w_2 \in \mathbb{K}$, $\lambda \in \mathrm{PC}(\mathbb{R}_+, \mathbb{R})$, $V \in \vartheta_0$, $\omega > 0$, $q \in \mathbb{N}$, $\{\eta_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+$ with*

(i) $w_1(\|x - y\|) \le V(t, x, y) \le w_2(\|x - y\|)$ *for* $(t, x, y) \in [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n$;
(ii) $D^+V(t, x, y) \le \lambda(t)V(t, x, y)$ *for* $(t, x, y) \in [t_{k-1}, t_k) \times \mathbb{R}^n \times \mathbb{R}^n$;
(iii) $V(t_k, x + I_k(t_k, x), y + I_k(t_k, y)) \le \eta_k V(t_k^-, x, y)$ *for* $k \in \mathbb{N}$;
(iv) $f(t + \omega, \cdot) = f(t, \cdot), I_k(t + \omega, \cdot) = I_k(t, \cdot), I_{k+q}(t, \cdot) = I_k(t, \cdot), t_{k+q} = t_k + \omega, k \in \mathbb{N}$

and

$$\left(\prod_{t_0<t_k\le t} \eta_k\right) \exp\left(\int_{t_0}^t \lambda(s)\mathrm{d}s\right) \to 0 \quad \text{as } t \to \infty. \tag{3.1}$$

Then (2.1) has a unique ω-periodic solution which is globally attractive.

Proof. Let $x = x(\cdot, t_0, x_0)$ and $y = y(\cdot, t_0, y_0)$ be the solutions of (2.1) through (t_0, x_0) and (t_0, y_0), respectively, where $x_0 \ne y_0$. Set $V(t) = V(t, x(t), y(t))$. It follows from (ii) and (iii) (see Samoĭlenko & Perestyuk, 1995 for detailed information) that

$$V(t) \le V(t_0) \left(\prod_{k=1}^m \eta_k\right) \exp\left(\int_{t_0}^t \lambda(s)\mathrm{d}s\right) \quad \text{for } t \in [t_m, t_{m+1}) \text{ and } m \in \mathbb{N},$$

i.e.,

$$V(t) \le V(t_0) \left(\prod_{t_0<t_k\le t} \eta_k\right) \exp\left(\int_{t_0}^t \lambda(s)\mathrm{d}s\right) \quad \text{for } t > t_0.$$

By (i), we get

$$\|x(t) - y(t)\| \le w_1^{-1}\left(w_2(\|x_0 - y_0\|)\left(\prod_{t_0<t_k\le t} \eta_k\right) \times \exp\left(\int_{t_0}^t \lambda(s)\mathrm{d}s\right)\right) \quad \text{for } t > t_0,$$

which, together with (3.1), yields that there exists $T \ge t_0$ such that

$$\|x(t) - y(t)\| \le \frac{1}{2}\|x_0 - y_0\| \quad \text{for } t > T. \tag{3.2}$$

Consider a simple operator from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by

$$\mathcal{F} : u_0 \to u(t_0 + \omega, t_0, u_0),$$

where $u(\cdot, t_0, u_0)$ is a solution of (2.1) through (t_0, u_0). It then can be deduced that

$$\mathcal{F}^k u_0 = u(t_0 + k\omega, t_0, u_0), \quad k \in \mathbb{N}. \tag{3.3}$$

Choosing $k = [T + 1]/\omega + 1$ and considering (3.2) and (3.3), we have

$$\|\mathcal{F}^k x_0 - \mathcal{F}^k y_0\| = \|x(t_0 + k\omega, t_0, x_0) - y(t_0 + k\omega, t_0, y_0)\| \le \frac{1}{2}\|x_0 - y_0\|.$$

Hence, $\mathcal{F}$ is a contraction mapping in the Banach space $\mathbb{R}^n$. Using Banach's fixed point theorem, there exists a unique $u^* \in \mathbb{R}^n$ such that $\mathcal{F}u^* = u^*$, which implies that there exists a unique $u^* \in \mathbb{R}^n$ such that $x(t_0 + \omega, t_0, u^*) = u^*$, where $x = x(\cdot, t_0, u^*)$ is a solution of (2.1) through (t_0, u^*). Define an ω-periodic extension of x by

$$\hat{x}(t) = \begin{cases} x(t) & \text{if } t \in [t_0, t_0 + \omega), \\ x(t - n\omega) & \text{if } t \in [t_0 + n\omega, t_0 + (n + 1)\omega), \ n \in \mathbb{N}. \end{cases}$$

It is clear that $\hat{x}$ is ω-periodic. Moreover, for any $t \in [t_0 + n\omega, t_0 + (n + 1)\omega)$, when $t \ne t_k$, $k \in \mathbb{N}$, we have

$$\hat{x}'(t) = x'(t - n\omega) = f(t - n\omega, x(t - n\omega)) = f(t, x(t - n\omega)) = f(t, \hat{x}(t)),$$

and when $t = t_k$, $k \in \mathbb{N}$, we have

$$\begin{aligned}\hat{x}(t_k) &= x(t_k - n\omega) = x(t_{k-nq}) \\ &= x(t_{k-nq}^-) + I_{k-nq}(t_{k-nq}, x(t_{k-nq}^-)) \\ &= x((t_k - n\omega)^-) + I_k(t_k - n\omega, x(t_k - n\omega)^-) \\ &= \hat{x}(t_k^-) + I_k(t_k, \hat{x}(t_k^-)).\end{aligned}$$

Thus $\hat{x}$ is an ω-periodic solution of (2.1) through (t_0, u^*). By the existence–uniqueness of solutions of (2.1), $\hat{x} = x = x(\cdot, t_0, u^*)$. Hence, we have proven that (2.1) has an ω-periodic solution. Furthermore, we claim that the ω-periodic solution $x = x(\cdot, t_0, u^*)$ is globally attractive. In fact, for any another solution $y = y(\cdot, t_0, y_0)$ of (2.1), by the above discussion, we find

$$\begin{aligned}\|x(t) - y(t)\| \le\ & w_1^{-1}\left(w_2(\|u^* - y_0\|)\left(\prod_{t_0<t_k\le t}\eta_k\right)\right. \\ &\left.\times \exp\left(\int_{t_0}^{t}\lambda(s)ds\right)\right) \to 0 \quad \text{as } t \to \infty.\end{aligned}$$

Thus the proof is complete. □

Remark 3.2. In Chellaboina, Bhat, and Haddad (2003), an invariance principle was presented for dynamical systems possessing left-continuous flows, especially for impulsive systems. Some invariant set stability theorems for nonlinear impulsive dynamical systems were derived, which can be used to investigate stability of limit cycles and periodic orbits, especially for state-dependent impulsive systems. However, due to the assumptions that

$$V(s(t, x_0)) \le V(s(\tau, x_0)), \quad \tau \le t \quad \text{or} \quad V(x + f_d(x)) - V(x) \le 0,$$

the results on dynamics are only valid for the impulsive perturbation point of view. In Liang, Liu, and Xiao (2011), the existence of periodic solution for impulsive delay differential equations was studied via some inequality techniques. One of the necessary but undesirable bases is that the solutions of the system are ultimately bounded. In Zhang, Yan, and Zhao (2008), some results for existence of positive periodic solutions of impulsive differential equations with or without delays were studied via some fixed point theorems. But all those results (Liang et al., 2011; Zhang et al., 2008) are not suitable for the attractiveness of the periodic solution. In this paper, Theorem 3.1 provides some Lyapunov conditions for the existence and global attractiveness of the unique ω-periodic solution of (2.1). In particular, when $\lambda \in \mathrm{PC}(\mathbb{R}_+, \mathbb{R}_+)$, Theorem 3.1 is given from the impulsive control point of view. That is, (2.1) may originally have no ω-periodic solution or the periodicity may be unknown, but it admits a unique ω-periodic solution which is globally attractive under proper impulsive control. When $\lambda \in \mathrm{PC}(\mathbb{R}_+, \mathbb{R}_-)$, it is given from the impulsive perturbation point of view. That is, if the corresponding continuous system of (2.1) (i.e., without impulsive effects) originally admits a periodic solution which is globally attractive, then (2.1) can keep the existence and global attractiveness of the ω-periodic solution under certain impulsive perturbations, i.e., robustness. Therefore, the development results in this paper make up for some deficiencies of the existing results (Chellaboina et al., 2003; Liang et al., 2011; Zhang et al., 2008).

Remark 3.3. If $\lambda(t)$ in (3.1) satisfies some special conditions, then one may find that it is possible that the impulsive constants η_k in Theorem 3.1 are large enough. That is, Theorem 3.1 is valid for impulsive differential equations with large impulse effects. In particular, if $\lambda(t) \equiv \lambda \in \mathbb{R}$, then we can state the following corollaries of Theorem 3.1.

Corollary 3.4. *Assume (i)–(iv) of Theorem 3.1. If there exists $\mu \in (0, 1)$ such that $\eta_k \exp(\lambda(t_k - t_{k-1})) \le \mu$, then (2.1) has a unique ω-periodic solution which is globally attractive.*

Corollary 3.5. *Assume (i)–(iv) in Theorem 3.1. If $\lambda > 0$ and there exists $\mu \in (0, 1)$ such that $\eta_k \le \mu \exp(-\lambda\omega)$, then (2.1) has a unique ω-periodic solution which is globally attractive.*

Next, we shall apply the previous theoretical results to the Hopfield neural networks

$$\begin{cases} x_i'(t) = -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) + J_i(t) \\ \quad \text{if } t \in [t_{k-1}, t_k), \\ x_i(t_k) = G_{ik}(t_k, x_i(t_k^-)) \\ \quad \text{if } i \in \Lambda,\ k \in \mathbb{N}, \end{cases} \tag{3.4}$$

where $\Lambda = \{1, 2, \ldots, n\}$, $n \ge 2$, is the number of neurons in the network, x_i is the state of the ith unit at time t, a_{ij} is the connection weight of the unit j on the unit i at time t, f_j is the activation function of the neurons, J_i is the input of the unit i at time t, $c_i > 0$ is the rate with which the ith unit resets its potential to the resting state in isolation when disconnected from the network and external inputs at time t, and G_{ik} is the impulsive function. Assume

(H_1) c_i, a_{ij} and J_i are all continuously periodic functions defined on $[t_0, \infty)$ with common period $\omega > 0$, $i, j \in \Lambda$;
(H_2) $G_{ik}(t + \omega, \cdot) = G_{ik}(t, \cdot), G_{i(k+q)}(t, \cdot) = G_{ik}(t, \cdot), t_{k+q} = t_k + \omega$, $k \in \mathbb{N}$, where $q \in \mathbb{N}$;
(H_3) There exist constants $l_i > 0$, $\Gamma_{ik} > 0$ such that

$$\begin{aligned}&|f_i(u) - f_i(v)| \le l_i|u - v|, \\ &|G_{ik}(t, u) - G_{ik}(t, v)| \le \Gamma_{ik}|u - v|, \\ &\quad (u, v) \in \mathbb{R}^2,\ i \in \Lambda,\ k \in \mathbb{N};\end{aligned}$$

(H_4) $c_i^* = \min_{t\in[t_0,t_0+\omega]} c_i(t)$, $a_{ij}^* = \max_{t\in[t_0,t_0+\omega]} |a_{ij}(t)|$.

Theorem 3.6. *Assume that (H_1)–(H_4) hold. Then there exists a unique ω-periodic solution of (3.4) which is globally attractive provided there exist constants $\varepsilon_i > 0$, $i \in \Lambda$, and $\mu \in (0, 1)$ such that*

$$\begin{aligned}&\left(\max_{i\in\Lambda}\Gamma_{ik}\right)\exp\left(\left(-\min_{i\in\Lambda}c_i^* + \sum_{i=1}^{n}\varepsilon_i\max_{j\in\Lambda}\frac{a_{ij}^* l_j}{\varepsilon_j}\right)(t_k - t_{k-1})\right) \\ &\quad \le \mu, \quad k \in \mathbb{N}.\end{aligned} \tag{3.5}$$

Proof. Let $x = (x_1, x_2, \ldots, x_n)^T$ and $y = (y_1, y_2, \ldots, y_n)^T$ be two solutions of (2.1) with different initial values. Consider the Lyapunov function $V(t, x, y) = \sum_{i=1}^{n}\varepsilon_i|x_i - y_i|$. By direct computations, we can deduce that

$$\begin{aligned}D^+V(t, x, y) =\ & \sum_{i=1}^{n}\varepsilon_i\mathrm{sgn}(x_i - y_i)\Bigg\{-c_i(t)(x_i - y_i) \\ &+ \sum_{j=1}^{n}a_{ij}(t)[f_j(x_j) - f_j(y_j)]\Bigg\} \\ \le\ & -\sum_{i=1}^{n}\varepsilon_i c_i(t)|x_i - y_i| + \sum_{i=1}^{n}\sum_{j=1}^{n}\varepsilon_i|a_{ij}(t)|l_j|x_j - y_j| \\ \le\ & -\left(\min_{i\in\Lambda}\right)c_i^*\sum_{i=1}^{n}\varepsilon_i|x_i - y_i| \\ &+ \sum_{i=1}^{n}\varepsilon_i\left(\max_{j\in\Lambda}\frac{a_{ij}^* l_j}{\varepsilon_j}\right)\sum_{j=1}^{n}\varepsilon_j|x_j - y_j| \\ =\ & \left\{-\min_{i\in\Lambda}c_i^* + \sum_{i=1}^{n}\varepsilon_i\max_{j\in\Lambda}\left|\frac{a_{ij}^* l_j}{\varepsilon_j}\right|\right\}V(t, x, y).\end{aligned} \tag{3.6}$$

In addition, we have

$$\begin{aligned} V(t_k, x(t_k), y(t_k)) &= \sum_{i=1}^{n} \varepsilon_i |x_i(t_k) - y_i(t_k)| \\ &= \sum_{i=1}^{n} \varepsilon_i |G_{ik}(t_k, x_i(t_k^-)) - G_{ik}(t_k, y_i(t_k^-))| \\ &\le \sum_{i=1}^{n} \varepsilon_i \Gamma_{ik} |x_i(t_k^-) - y_i(t_k^-)| \\ &\le \left(\max_{i\in\Lambda} \Gamma_{ik}\right) V(t_k^-, x(t_k^-), y(t_k^-)). \end{aligned} \tag{3.7}$$

Considering (3.5), (3.6), and (3.7), it is easy to check that all conditions in Corollary 3.4 are satisfied. Thus (3.4) has a unique ω-periodic solution which is globally attractive. □

Corollary 3.7. *Assume that* (H$_1$)–(H$_4$) *hold. Then there exists a unique ω-periodic solution of* (3.4) *which is globally attractive provided there exist constants $\varepsilon_i > 0$, $i \in \Lambda$, and $\mu \in (0, 1)$ such that*

$$\min_{i\in\Lambda} c_i^* < \sum_{i=1}^{n} \varepsilon_i \max_{j\in\Lambda} \frac{a_{ij}^* l_j}{\varepsilon_j}$$

and

$$\max_{i\in\Lambda} \Gamma_{ik} \le \mu \exp\left(\left(\min_{i\in\Lambda} c_i^* - \sum_{i=1}^{n} \varepsilon_i \max_{j\in\Lambda} \frac{a_{ij}^* l_j}{\varepsilon_j}\right)\omega\right), \quad k \in \mathbb{N}.$$

Remark 3.8. So far, many interesting results on stability of the periodic solution of impulsive neural networks have been reported, see Allegretto, Papaini, and Forti (2010) and Wang, Liao, and Li (2007). One may find that most of the existing results focus on the stability problem of the periodic solution from the impulsive perturbations point of view. It is known that, in theory and practice, impulsive control has been widely used to stabilize some unstable systems and synchronize some chaotic systems (see Antunes, Hespanha, & Silvestre, 2013 and Stamova, 2009). The main idea of impulsive control of RNNs is to introduce impulsive effects into the topological structure of the networks and then change the states of the systems. However, there is little work on the impulsive control problem of periodic solution of neural networks (Li & Song, 2013) due to the deficiency of theoretical work on impulsive control systems. Based on Theorem 3.1, Corollary 3.7 in this paper provides a sufficient condition which can guarantee the existence, uniqueness, and global attractiveness of the periodic solution of impulsive neural networks even if the network models may originally have no periodic solution or the periodicity may be unknown, even if the system is originally unstable or divergent.

4. Examples

Example 4.1. Consider the 2D Hopfield neural networks

$$\begin{cases} \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} -0.3 + 0.01\sin t & 0 \\ 0 & -0.4 + 0.02\cos t \end{pmatrix} \\ \quad \times \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 1 + 0.2\cos t & -1 + 0.5\sin t \\ 1 - 0.2\sin t & 1 + 0.1\cos t \end{pmatrix} \\ \quad \times \begin{pmatrix} \tanh(x_1(t)) \\ \tanh(x_2(t)) \end{pmatrix} + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, \quad t \ne t_k, \\ \begin{pmatrix} x_1(t_k) \\ x_2(t_k) \end{pmatrix} = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.2 \end{pmatrix} \begin{pmatrix} x_1(t_k^-) \\ x_2(t_k^-) \end{pmatrix}, \quad k \in \mathbb{N}, \end{cases} \tag{4.1}$$

where $t_k = (k\pi)/8$ for $k \in \mathbb{N}$. Obviously, the right-hand side of the first equation of (4.1) is 2π-periodic (i.e., $\omega = 2\pi$). Choose $q = 16$, $\Gamma_{1k} = 0.3$, $\Gamma_{2k} = 0.2$, and $l_1 = l_2 = 1$. Then it can be deduced that conditions (H$_1$)–(H$_4$) hold. In addition, let $\mu = 0.8$ and $\varepsilon_1 = \varepsilon_2 = 1$. Then

$$\begin{aligned} &\left(\max_{i\in\Lambda} \Gamma_{ik}\right) \exp\left(\left(-\min_{i\in\Lambda} c_i^* + \sum_{i=1}^{n} \varepsilon_i \max_{j\in\Lambda} \frac{a_{ij}^* l_j}{\varepsilon_j}\right)(t_k - t_{k-1})\right) \\ &= 0.3\exp\left(\frac{2.41\pi}{8}\right) \simeq 0.7729 < \mu, \quad k \in \mathbb{N}. \end{aligned}$$

Hence, by Theorem 3.6, (4.1) has a unique 2π-periodic solution which is globally attractive. This is shown in Fig. 4.1(a) and (b). Moreover, if there is no impulsive effect, i.e., $x(t_k) = x(t_k^-)$, then Theorem 3.6 is invalid. In this case, it is interesting to see that (4.1) has no 2π-periodic solution which is globally attractive. This is shown in Fig. 4.1(c) and (d). This means that impulse contributes to the existence and attractiveness of the unique periodic solution.

Example 4.2. Consider the 2D neural networks

$$\begin{cases} \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \\ \quad + \begin{pmatrix} -0.1 + 0.2\sin(\pi t) & 0.15 \\ -0.6 & 0.15 + 0.05\cos(\pi t) \end{pmatrix} \\ \quad \times \begin{pmatrix} \sin(x_1(t)) \\ \cos(x_2(t)) \end{pmatrix} + \begin{pmatrix} -\frac{5}{2}\cos(\pi t) \\ \frac{4}{3}\sin(\pi t) \end{pmatrix}, \quad t \ne t_k, \\ \begin{pmatrix} x_1(t_k) \\ x_2(t_k) \end{pmatrix} = \begin{pmatrix} e^{0.5} & 0 \\ 0 & e^{0.6} \end{pmatrix} \begin{pmatrix} x_1(t_k^-) \\ x_2(t_k^-) \end{pmatrix}, \quad k \in \mathbb{N}, \end{cases} \tag{4.2}$$

where $t_k = 0.5k$ for $k \in \mathbb{N}$. In this case, we know $\omega = 2$. Choose $q = 4$, $\Gamma_{1k} = e^{0.5}$, $\Gamma_{2k} = e^{0.6}$, and $l_1 = l_2 = 1$. Then it can be deduced that conditions (H$_1$)–(H$_4$) hold. In addition, let $\mu = 0.95$, $\varepsilon_1 = 1.2$, and $\varepsilon_2 = 0.6$. Then

$$\begin{aligned} &\left(\max_{i\in\Lambda} \Gamma_{ik}\right) \exp\left(\left(-\min_{i\in\Lambda} c_i^* + \sum_{i=1}^{n} \varepsilon_i \max_{j\in\Lambda} \frac{a_{ij}^* l_j}{\varepsilon_j}\right)(t_k - t_{k-1})\right) \\ &\simeq 0.9048 < \mu, \quad k \in \mathbb{N}. \end{aligned}$$

Hence, by Theorem 3.6, (4.2) has a unique 2-periodic solution which is globally attractive. This is shown in Fig. 4.2(a) and (b). Moreover, if we let $\varepsilon_1 = \varepsilon_2$, then it can be derived that

$$\begin{aligned} &\left(\max_{i\in\Lambda} \Gamma_{ik}\right) \exp\left(\left(-\min_{i\in\Lambda} c_i^* + \sum_{i=1}^{n} \varepsilon_i \max_{j\in\Lambda} \frac{a_{ij}^* l_j}{\varepsilon_j}\right)(t_k - t_{k-1})\right) \\ &\simeq 1.0513 > 1, \quad k \in \mathbb{N}, \end{aligned}$$

which implies that Theorem 3.6 is invalid. Thus the choice of ε_i is very important in applications.

5. Conclusion

In this paper, we have proposed several sufficient conditions for the existence and global attractiveness of unique periodic solutions of impulsive differential equations. The results were given from impulsive control and impulsive perturbation points of view and established by Lyapunov's second method and the Banach contraction mapping principle. They complement and improve some existing results. Numerical examples have been given in order to demonstrate the effectiveness of the presented theoretical results. Here we also point out that it is possible to develop the ideas in this paper for the impulsive control problem of periodic solutions for impulsive functional differential equations. Some further research in this direction will be done in the future.

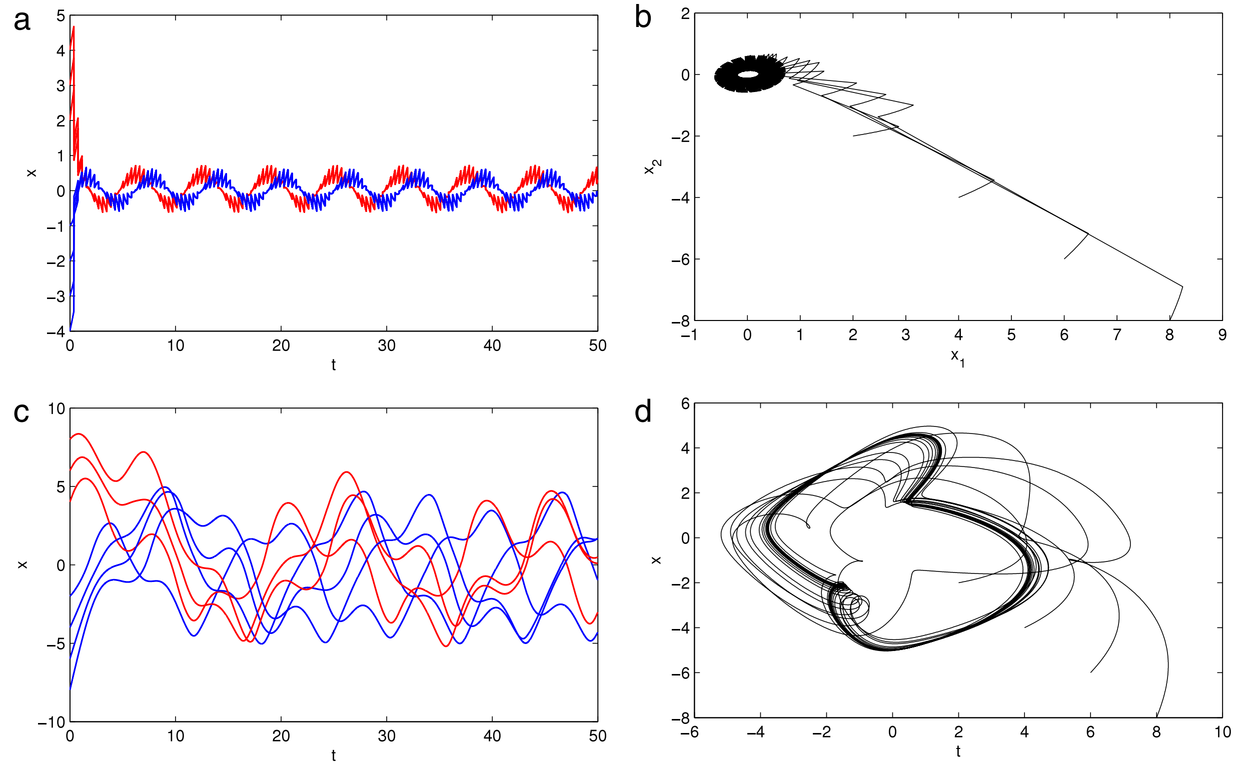

Fig. 4.1. (a) State trajectory of (4.1) on [0, 50]. (b) Phase portrait of 2π-periodic solutions of (4.1) on [0, 100]. (c) State trajectory of (4.1) without impulses on [0, 50]. (d) Phase portrait of (4.1) without impulses on [0, 100].

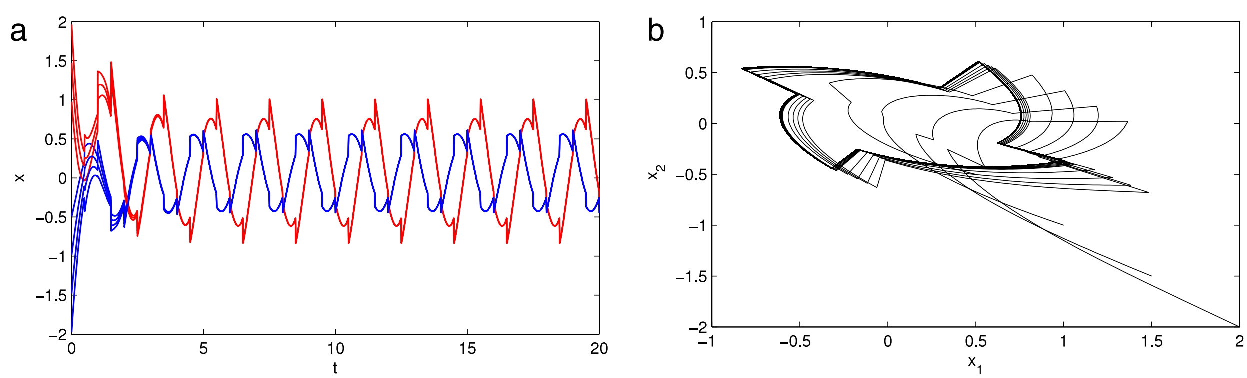

Fig. 4.2. (a) State trajectory of (4.2) on [0, 20]. (b) Phase portrait of 2π-periodic solutions of (4.2) on [0, 100].

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