INHOMOGENEOUS DISCRETE VARIATIONAL PROBLEMS.

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Abstract. We prove a relationship between a general inhomogeneous discrete variational problem, positive semidefiniteness of its second variation, and a certain inhomogeneous boundary value problem. This result gives both necessary and sufficient conditions for the solvability of our discrete variational problem.

1. INTRODUCTION.

The purpose of this paper is to emphasize the importance of positive definiteness of homogeneous discrete quadratic functionals

$$\mathcal{F}_2(\eta,\xi)=\sum_{k=0}^{N}\left\{\eta_{k+1}^T C_k\eta_{k+1}+\xi_k^T B_k\xi_k\right\}+\begin{pmatrix}-\eta_0\\ \eta_{N+1}\end{pmatrix}^T S\begin{pmatrix}-\eta_0\\ \eta_{N+1}\end{pmatrix},$$

i.e., of the question when

$$\left\{\begin{array}{c}\mathcal{F}_2(\eta,\xi)>0 \quad \text{for all} \quad (\eta,\xi) \quad \text{with}\\ \Delta\eta_k=A_k\eta_{k+1}+B_k\xi_k\ (0\le k\le N),\ \binom{-\eta_0}{\eta_{N+1}}\in \operatorname{Im}R^T,\ \text{and}\ \eta\neq 0\end{array}\right.$$

holds. Here, $\eta_k\in\mathbb{R}^n$ for all $0\le k\le N+1$, $\Delta\eta_k=\eta_{k+1}-\eta_k$, $\xi_k\in\mathbb{R}^n$, A_k,B_k,C_k are $n\times n$-matrices with $I-A_k$ invertible and B_k,C_k symmetric for all $0\le k\le N$, and S,R are $2n\times 2n$-matrices such that S is symmetric.

Recently, characterizations of positive definiteness in this sense have been given by the author in [2, 3] involving disconjugacy of linear Hamiltonian difference systems

$$\Delta\eta_k=A_k\eta_{k+1}+B_k\xi_k,\quad \Delta\xi_k=C_k\eta_{k+1}-A_k^T\xi_k \qquad (0\le k\le N).$$

In this paper we will not discuss those characterizations but show that it indeed makes sense to give them by presenting how the question of positive definiteness of homogeneous discrete quadratic functionals arises naturally while trying to solve certain general inhomogeneous discrete variational problems. For more special versions of this task we refer to the paper [1] by C. Ahlbrandt and to Chapter 8 of the book [4] by W. Kelley and A. Peterson. The continuous analogue of it can be found in Section 2.3 of the book [5] by W. Kratz, and we will in fact refer to this work again later on.

We shortly summarize the setup of this paper. The next section contains notation and terminology as well as the introduction of both the inhomogeneous discrete variational problem and the inhomogeneous discrete boundary value problem. We give the main results on the relationship of these two problems to positive semidefiniteness of homogeneous discrete quadratic functionals in Section 3. In Section 4 we prove two auxiliary lemmas while the last section is devoted to the proofs of our main results, Theorem 1 and Theorem 2 of Section 3 below.

2. NOTATION AND TERMINOLOGY.

Let $J := [0, N] \cap \mathbb{Z}$, $J^* := J \cup \{N+1\}$, and let be given real matrix- and vector-valued functions on J

$$Q,\ \tilde{Q},\ P,\ \mathcal{A},\ \mathcal{B},\ q,\ p,\ c$$

of type $n \times n,\ n \times m,\ m \times m,\ n \times n,\ n \times m,\ n \times 1,\ m \times 1,\ n \times 1$ as well as real (constant) matrices and vectors

$$S,\ S^*,\ s,\ s^*$$

of type $2n \times 2n,\ 2n \times 2n,\ 2n \times 1,\ 2n \times 1$ such that the assumptions

$$Q, P \text{ symmetric, } P, \tilde{\mathcal{B}} := \mathcal{B}^T \mathcal{B} \text{ invertible on } J; S \text{ symmetric; } s^* \in \operatorname{Im} S^*$$

hold. With this setting and for $x = (x_k)_{k \in J^*}$ and $v = (v_k)_{k \in J}$ we define an inhomogeneous discrete quadratic functional by

$$\mathcal{F}(x, v) := \sum_{k=0}^{N} \left\{ x_{k+1}^T Q_k x_{k+1} + 2 x_{k+1}^T \tilde{Q}_k v_k + v_k^T P_k v_k + 2 x_{k+1}^T q_k + 2 p_k^T v_k \right\} + \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T \left\{ S \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} + 2s \right\}$$

and consider a general inhomogeneous discrete variational problem

$$\text{(VP)} \qquad \mathcal{F}(x,v) \to \min, \quad (x,v) \text{ admissible}, \quad x \in \tilde{\mathcal{R}}.$$

Here, (x,v) is called admissible if it satisfies $\Delta x_k = \mathcal{A}_k x_{k+1} + \mathcal{B}_k v_k + c_k$ for all $k \in J$, and we write $x \in \tilde{\mathcal{R}}$ in case of $S^* \binom{-x_0}{x_{N+1}} = s^*$. We put

$$\begin{cases} A = \mathcal{A} - \mathcal{B}P^{-1}\tilde{Q}^T, \quad B = \mathcal{B}P^{-1}\mathcal{B}^T, \quad C = Q - \tilde{Q}P^{-1}\tilde{Q}^T, \\ \qquad a = c - \mathcal{B}P^{-1}p, \quad b = q - \tilde{Q}P^{-1}p \qquad \text{(on } J) \end{cases}$$

and consider an inhomogeneous boundary value problem of the form

$$\text{(BP)} \quad \begin{cases} \Delta x_k = A_k x_{k+1} + B_k u_k + a_k, \ \Delta u_k = C_k x_{k+1} - A_k^T u_k + b_k \\ \qquad (x,u) \in \mathcal{R}, \end{cases}$$

where $(x,u) \in \mathcal{R}$ indicates that

$$(RS + S^*)\begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} + R\begin{pmatrix} u_0 \\ u_{N+1} \end{pmatrix} = s^* - Rs$$

holds with a matrix R satisfying $\operatorname{rank}(R \ \ S^*) = 2n$ and $\operatorname{Im} R^T = \operatorname{Ker} S^*$ (such an R exists as is shown in [5, Corollary 3.1.3]).

3. MAIN RESULTS.

With the notation from Section 2 we can now give both a sufficient and a necessary condition for a pair $(\hat{x}, \hat{v})$ to be a solution of the variational problem (VP). These main results of this paper read as follows.

Theorem 1 *(Sufficiency).*
Let $(\hat{x}, \hat{u})$ solve (BP) and let $\mathcal{F}_2$ be positive semidefinite. Then $(\hat{x}, \hat{v})$ solves (VP), where $\hat{v}$ is defined by

$$\hat{v}_k := P_k^{-1}(\mathcal{B}_k^T \hat{u}_k - \tilde{Q}_k^T \hat{x}_{k+1} - p_k), \quad k \in J.$$

Theorem 2 *(Necessity).*
Let $(\hat{x}, \hat{v})$ solve (VP). Then $\mathcal{F}_2$ is positive semidefinite and $\mathcal{F}_1(\hat{x}, \hat{u}) = 0$, where $\hat{u}$ is defined by

$$\hat{u}_k := \mathcal{B}_k \tilde{\mathcal{B}}_k^{-1}(P_k \hat{v}_k + \tilde{Q}_k^T \hat{x} + p_k), \quad k \in J.$$

In the following definition we summarize the description of the terms used in the above statements.

Definition (First and Second Variation).
The first variation of the discrete variational problem (VP) is given by

$$\begin{aligned}\mathcal{F}_1(x,u,\eta,\xi) \ := \ & \sum_{k=0}^{N} \eta_{k+1}^T \left\{ C_k x_{k+1} - A_k^T u_k + b_k - \Delta u_k \right\} \\ & + \begin{pmatrix} -\eta_0 \\ \eta_{N+1} \end{pmatrix}^T \left\{ \begin{pmatrix} u_0 \\ u_{N+1} \end{pmatrix} + S \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} + s \right\},\end{aligned}$$

and we write $\mathcal{F}_1(x,u)=0$ whenever

$$\begin{cases} \mathcal{F}_1(x,u,\eta,\xi) = 0 \quad \text{holds for all} \quad (\eta,\xi) \quad \text{with} \\ \Delta\eta_k = A_k \eta_{k+1} + B_k \xi_k \ (k \in J) \text{ and } \binom{-\eta_0}{\eta_{N+1}} \in \operatorname{Im} R^T. \end{cases}$$

The second variation of (VP) is given by

$$\mathcal{F}_2(\eta,\xi) := \sum_{k=0}^{N} \left\{ \eta_{k+1}^T C_k \eta_{k+1} + \xi_k^T B_k \xi_k \right\} + \begin{pmatrix} -\eta_0 \\ \eta_{N+1} \end{pmatrix}^T S \begin{pmatrix} -\eta_0 \\ \eta_{N+1} \end{pmatrix},$$

and we write $\mathcal{F}_2 \geq 0$ and call $\mathcal{F}_2$ positive semidefinite whenever

$$\begin{cases} \mathcal{F}_2(\eta,\xi) \geq 0 \quad \text{holds for all} \quad (\eta,\xi) \quad \text{with} \\ \Delta\eta_k = A_k \eta_{k+1} + B_k \xi_k \ (k \in J) \text{ and } \binom{-\eta_0}{\eta_{N+1}} \in \operatorname{Im} R^T. \end{cases}$$

4. SOME AUXILIARY RESULTS.

We approach our ultimate goal of proving Theorems 1 and 2 by giving two auxiliary results. We use the notation introduced before.

Lemma 1. If $v_k = P_k^{-1}(\mathcal{B}_k^T u_k - \tilde{Q}_k^T x_{k+1} - p_k)$ holds for all $k \in J$, then we have for all $k \in J$

$$\begin{aligned} & x_{k+1}^T Q_k x_{k+1} + 2x_{k+1}^T \tilde{Q}_k v_k + v_k^T P_k v_k + 2x_{k+1}^T q_k + 2p_k^T v_k \\ & = \ x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k + 2x_{k+1}^T b_k - p_k^T P_k^{-1} p_k \end{aligned}$$

and

$$\mathcal{A}_k x_{k+1} + \mathcal{B}_k v_k + c_k = A_k x_{k+1} + B_k u_k + a_k.$$

Proof. The above assumptions yield

$$\begin{aligned}
& x_{k+1}^T Q_k x_{k+1} + 2x_{k+1}^T \tilde{Q}_k v_k + v_k^T P_k v_k + 2x_{k+1}^T q_k + 2p_k^T v_k \\
&= x_{k+1}^T Q_k x_{k+1} + 2x_{k+1}^T q_k + \left\{ v_k^T P_k + 2x_{k+1}^T \tilde{Q}_k + 2p_k^T \right\} v_k \\
&= x_{k+1}^T Q_k x_{k+1} + 2x_{k+1}^T q_k \\
&\quad + \left\{ u_k^T \mathcal{B}_k + x_{k+1}^T \tilde{Q}_k + p_k^T \right\} P_k^{-1} \left\{ \mathcal{B}_k^T u_k - \tilde{Q}_k^T x_{k+1} - p_k \right\} \\
&= x_{k+1}^T \left\{ Q_k - \tilde{Q}_k P_k^{-1} \tilde{Q}_k^T \right\} x_{k+1} + u_k^T \mathcal{B}_k P_k^{-1} \mathcal{B}_k^T u_k \\
&\quad + 2x_{k+1}^T \left\{ q_k - \tilde{Q}_k P_k^{-1} p_k \right\} - p_k^T P_k^{-1} p_k \\
&= x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k + 2x_{k+1}^T b_k - p_k^T P_k^{-1} p_k
\end{aligned}$$

and

$$\begin{aligned}
& \mathcal{A}_k x_{k+1} + \mathcal{B}_k v_k + c_k = \mathcal{A}_k x_{k+1} + \mathcal{B}_k P_k^{-1} (\mathcal{B}_k^T u_k - \tilde{Q}_k^T x_{k+1} - p_k) + c_k \\
&= (\mathcal{A}_k - \mathcal{B}_k P_k^{-1} \tilde{Q}_k^T) x_{k+1} + \mathcal{B}_k P_k^{-1} \mathcal{B}_k^T u_k + c_k - \mathcal{B}_k P_k^{-1} p_k \\
&= A_k x_{k+1} + B_k u_k + a_k
\end{aligned}$$

for all $k \in J$ so that the assertion of the lemma follows. ■

Lemma 2. Let $\Delta \eta_k = A_k \eta_{k+1} + B_k \xi_k$ be satisfied for all $k \in J$. If $v_k = P_k^{-1}(\mathcal{B}_k^T u_k - \tilde{Q}_k^T x_{k+1} - p_k)$ and $\zeta_k = P_k^{-1}(\mathcal{B}_k^T \xi_k - \tilde{Q}_k^T \eta_{k+1})$ hold on J, then we have

$$\mathcal{F}(x + \eta, v + \zeta) - \mathcal{F}(x, v) = 2\mathcal{F}_1(x, u, \eta, \xi) + \mathcal{F}_2(\eta, \xi).$$

Proof. The above assumptions yield

$$v_k + \zeta_k = P_k^{-1} \left\{ \mathcal{B}_k^T (u_k + \xi_k) - \tilde{Q}_k^T (x_{k+1} + \eta_{k+1}) - p_k \right\}$$

for all $k \in J$, and then it follows by applying Lemma 1 that

$$\begin{aligned}
& \mathcal{F}(x + \eta, v + \zeta) - \mathcal{F}(x, v) \\
&= \sum_{k=0}^{N} \{ (x_{k+1}^T + \eta_{k+1}^T) C_k (x_{k+1} + \eta_{k+1}) + (u_k^T + \xi_k^T) B_k (u_k + \xi_k) \\
&\qquad + 2(x_{k+1}^T + \eta_{k+1}^T) b_k - p_k^T P_k^{-1} p_k \} \\
&\quad + \begin{pmatrix} -x_0 - \eta_0 \\ x_{N+1} + \eta_{N+1} \end{pmatrix}^T \left\{ S \begin{pmatrix} -x_0 - \eta_0 \\ x_{N+1} + \eta_{N+1} \end{pmatrix} + 2s \right\}
\end{aligned}$$

$$
\begin{aligned}
& -\sum_{k=0}^{N}\left\{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k + 2x_{k+1}^T b_k - p_k^T P_k^{-1} p_k\right\} \\
& -\begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T \left\{ S \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} + 2s \right\} \\
= & \sum_{k=0}^{N}\left\{\eta_{k+1}^T C_k \eta_{k+1} + 2\eta_{k+1}^T C_k x_{k+1} + 2u_k^T B_k \xi_k + \xi_k^T B_k \xi_k + 2\eta_{k+1}^T b_k\right\} \\
& + \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T S \begin{pmatrix} -\eta_0 \\ \eta_{N+1} \end{pmatrix} + \begin{pmatrix} -\eta_0 \\ \eta_{N+1} \end{pmatrix}^T \left\{ S \begin{pmatrix} -x_0 - \eta_0 \\ x_{N+1} + \eta_{N+1} \end{pmatrix} + 2s \right\} \\
= & \; 2\sum_{k=0}^{N}\left\{\eta_{k+1}^T (C_k x_{k+1} + b_k) + u_k^T B_k \xi_k\right\} + \sum_{k=0}^{N}\left\{\eta_{k+1}^T C_k \eta_{k+1} + \xi_k^T B_k \xi_k\right\} \\
& + 2\begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T S \begin{pmatrix} -\eta_0 \\ \eta_{N+1} \end{pmatrix} + \begin{pmatrix} -\eta_0 \\ \eta_{N+1} \end{pmatrix}^T S \begin{pmatrix} -\eta_0 \\ \eta_{N+1} \end{pmatrix} + 2\begin{pmatrix} -\eta_0 \\ \eta_{N+1} \end{pmatrix}^T s \\
= & \; 2\sum_{k=0}^{N}\left\{\eta_{k+1}^T (C_k x_{k+1} + b_k) + u_k^T (\Delta \eta_k - A_k \eta_{k+1})\right\} \\
& + 2\begin{pmatrix} -\eta_0 \\ \eta_{N+1} \end{pmatrix}^T \left\{ S \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} + s \right\} + \mathcal{F}_2(\eta, \xi) \\
= & \; 2\sum_{k=0}^{N}\left\{\eta_{k+1}^T (C_k x_{k+1} - A_k^T u_k + b_k) + \Delta(u_k^T \eta_k) - (\Delta u_k^T)\eta_{k+1}\right\} \\
& + 2\begin{pmatrix} -\eta_0 \\ \eta_{N+1} \end{pmatrix}^T \left\{ S \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} + s \right\} + \mathcal{F}_2(\eta, \xi) \\
= & \; 2\sum_{k=0}^{N} \eta_{k+1}^T (C_k x_{k+1} - A_k^T u_k + b_k - \Delta u_k) \\
& + 2\begin{pmatrix} -\eta_0 \\ \eta_{N+1} \end{pmatrix}^T \left\{ \begin{pmatrix} u_0 \\ u_{N+1} \end{pmatrix} + S \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} + s \right\} + \mathcal{F}_2(\eta, \xi) \\
= & \; 2\mathcal{F}_1(x, u, \eta, \xi) + \mathcal{F}_2(\eta, \xi)
\end{aligned}
$$

holds, where we also used summation by parts, i.e., the fact that

$$\Delta(u_k^T \eta_k) = (\Delta u_k^T)\eta_{k+1} + u_k^T(\Delta \eta_k)$$

holds for all $k \in J$. ■

5. THE PROOFS OF THE MAIN RESULTS.

With the above auxiliary lemmas we can now provide the proofs of Theorem 1 and Theorem 2 from Section 3.

Proof of Theorem 1. Let $(\hat{x}, \hat{u})$ solve (BP), i.e.,

$$\begin{cases} \Delta \hat{x}_k = A_k \hat{x}_{k+1} + B_k \hat{u}_k + a_k,\ \Delta \hat{u}_k = C_k \hat{x}_{k+1} - A_k^T \hat{u}_k + b_k \quad (k \in J), \\ (\hat{x}, \hat{u}) \in \mathcal{R}, \end{cases}$$

assume $\mathcal{F}_2(\eta, \xi) \geq 0$ for all pairs (η, ξ) with $\Delta \eta_k = A_k \eta_{k+1} + B_k \xi_k$ $(k \in J)$ and $\binom{-\eta_0}{\eta_{N+1}} \in \mathrm{Im} R^T$, and define

$$\hat{v}_k := P_k^{-1}(\mathcal{B}_k^T \hat{u}_k - \tilde{Q}_k^T \hat{x}_{k+1} - p_k) \quad \text{for all} \quad k \in J.$$

By Lemma 1, $(\hat{x}, \hat{v})$ is admissible. Now, $(\hat{x}, \hat{u}) \in \mathcal{R}$ means

$$(RS + S^*) \begin{pmatrix} -\hat{x}_0 \\ \hat{x}_{N+1} \end{pmatrix} + R \begin{pmatrix} \hat{u}_0 \\ \hat{u}_{N+1} \end{pmatrix} = s^* - Rs$$

so that $\alpha \in \mathrm{Im} S^* \cap \mathrm{Im} R$ for

$$\alpha := R \left\{ \begin{pmatrix} \hat{u}_0 \\ \hat{u}_{N+1} \end{pmatrix} + S \begin{pmatrix} -\hat{x}_0 \\ \hat{x}_{N+1} \end{pmatrix} + s \right\} = s^* - S^* \begin{pmatrix} -\hat{x}_0 \\ \hat{x}_{N+1} \end{pmatrix}.$$

However, we have

$$\begin{aligned} \dim(\mathrm{Im} S^* \cap \mathrm{Im} R) &= \dim \mathrm{Im} S^* + \dim \mathrm{Im} R - \dim(\mathrm{Im} S^* + \mathrm{Im} R) \\ &= \dim \mathrm{Im} S^* + \dim \mathrm{Im} R^T - \dim \mathrm{Im}(S^* \ \ R) \\ &= \dim \mathrm{Im} S^* + \dim \mathrm{Ker} S^* - 2n = 2n - 2n = 0 \end{aligned}$$

so that $\alpha = 0$, hence $s^* = S^* \binom{-\hat{x}_0}{\hat{x}_{N+1}}$, and therefore $\hat{x} \in \tilde{\mathcal{R}}$ (see also [5, Lemma 2.3.2]). We proceed by letting (x, v) be an arbitrary admissible pair with $x \in \tilde{\mathcal{R}}$. The proof is done once we show $\mathcal{F}(\hat{x}, \hat{v}) \leq \mathcal{F}(x, v)$. To achieve this goal we define

$$u_k := \mathcal{B}_k \tilde{\mathcal{B}}_k^{-1}(P_k v_k + \tilde{Q}_k^T x_{k+1} + p_k) \quad \text{for all} \quad k \in J.$$

Then $v_k = P_k^{-1}(\mathcal{B}_k^T u_k - \tilde{Q}_k^T x_{k+1} - p_k)$ for all $k \in J$. We now put

$$\eta := x - \hat{x}, \quad \xi := u - \hat{u}, \quad \zeta := v - \hat{v}.$$

Then Lemma 1 implies that

$$\begin{aligned} A_k\eta_{k+1}+B_k\xi_k &= A_kx_{k+1}+B_ku_k+a_k-(A_k\hat{x}_{k+1}+B_k\hat{u}_k+a_k) \\ &= \mathcal{A}_kx_{k+1}+\mathcal{B}_kv_k+c_k-(\mathcal{A}_k\hat{x}_{k+1}+\mathcal{B}_k\hat{v}_k+c_k)=\Delta\eta_k \end{aligned}$$

holds on J. This together with $\zeta_k = P_k^{-1}(\mathcal{B}_k^T\xi_k - \tilde{Q}_k^T\eta_{k+1})$ on J yields

$$\mathcal{F}(x,v)-\mathcal{F}(\hat{x},\hat{v}) = 2\mathcal{F}_1(\hat{x},\hat{u},\eta,\xi)+\mathcal{F}_2(\eta,\xi)$$

by Lemma 2. Now, $x\in\tilde{\mathcal{R}}$ and $\hat{x}\in\tilde{\mathcal{R}}$ show

$$S^*\begin{pmatrix}-\eta_0\\ \eta_{N+1}\end{pmatrix} = S^*\begin{pmatrix}-x_0\\ x_{N+1}\end{pmatrix} - S^*\begin{pmatrix}-\hat{x}_0\\ \hat{x}_{N+1}\end{pmatrix} = s^*-s^*=0,$$

i.e., $\binom{-\eta_0}{\eta_{N+1}} \in \mathrm{Ker} S^* = \mathrm{Im} R^T$ so that $\mathcal{F}_2(\eta,\xi)\geq 0$ because of the positive semidefiniteness of $\mathcal{F}_2$. Finally we have (with $\binom{-\eta_0}{\eta_{N+1}} = R^T\beta$ for some $\beta\in\mathbb{R}^{2n}$)

$$\begin{aligned} \mathcal{F}_1(\hat{x},\hat{u},\eta,\xi) &= \sum_{k=0}^{N}\eta_{k+1}^T\left\{C_k\hat{x}_{k+1}-A_k^T\hat{u}_k+b_k-\Delta\hat{u}_k\right\} \\ &\quad +\begin{pmatrix}-\eta_0\\ \eta_{N+1}\end{pmatrix}^T\left\{\begin{pmatrix}\hat{u}_0\\ \hat{u}_{N+1}\end{pmatrix}+S\begin{pmatrix}-\hat{x}_0\\ \hat{x}_{N+1}\end{pmatrix}+s\right\} \\ &= 0+\beta^T\alpha = 0 \end{aligned}$$

so that $\mathcal{F}(x,v)-\mathcal{F}(\hat{x},\hat{v}) = \mathcal{F}_2(\eta,\xi)\geq 0$ and $\mathcal{F}(\hat{x},\hat{v})\leq\mathcal{F}(x,v)$. ■

Proof of Theorem 2. Let $(\hat{x},\hat{v})$ solve (VP), i.e.,

$$\begin{cases} \Delta\hat{x}_k = \mathcal{A}_k\hat{x}_{k+1}+\mathcal{B}_k\hat{v}_k+c_k \ (k\in J), \quad \hat{x}\in\tilde{\mathcal{R}}, \\ \mathcal{F}(\hat{x},\hat{v})\leq\mathcal{F}(x,v) \text{ for all admissible } (x,v) \text{ with } x\in\tilde{\mathcal{R}}, \end{cases}$$

and define

$$\hat{u}_k := \mathcal{B}_k\tilde{\mathcal{B}}_k^{-1}(P_k\hat{v}_k+\tilde{Q}_k^T\hat{x}_{k+1}+p_k) \quad \text{for all} \quad k\in J$$

so that $\hat{v}_k = P_k^{-1}(\mathcal{B}_k^T\hat{u}_k-\tilde{Q}_k^T\hat{x}_{k+1}-p_k)$ holds on J. Let (η,ξ) satisfy $\Delta\eta_k = A_k\eta_{k+1}+B_k\xi_k$ $(k\in J)$ and $\binom{-\eta_0}{\eta_{N+1}}\in\mathrm{Im}R^T$. We have to show that both $\mathcal{F}_1(\hat{x},\hat{u},\eta,\xi)=0$ and $\mathcal{F}_2(\eta,\xi)\geq 0$ hold. To do so we put

$$\eta_k^\alpha := \alpha\eta_k, \quad \xi_k^\alpha := \alpha\xi_k, \quad \zeta_k^\alpha := P_k^{-1}(\mathcal{B}_k^T\xi_k^\alpha - \tilde{Q}_k^T\eta_{k+1}^\alpha)$$

for all $k \in J$ and all $\alpha \in I\!R$. Since

$$A_k \eta_{k+1}^\alpha + B_k \xi_k^\alpha = \alpha(A_k \eta_{k+1} + B_k \xi_k) = \alpha \Delta \eta_k = \Delta \eta_k^\alpha$$

holds for all $k \in J$ and $\alpha \in I\!R$, we may apply again Lemma 2 to obtain

$$\mathcal{F}(\hat{x} + \eta^\alpha, \hat{v} + \zeta^\alpha) - \mathcal{F}(\hat{x}, \hat{v}) = 2\mathcal{F}_1(\hat{x}, \hat{u}, \eta^\alpha, \xi^\alpha) + \mathcal{F}_2(\eta^\alpha, \xi^\alpha)$$

for all $\alpha \in I\!R$. Of course, $\hat{x} + \eta^\alpha \in \tilde{\mathcal{R}}$ because of

$$S^* \begin{pmatrix} -\hat{x}_0 - \eta_0^\alpha \\ \hat{x}_{N+1} + \eta_{N+1}^\alpha \end{pmatrix} = s^* + \alpha S^* \begin{pmatrix} -\eta_0 \\ \eta_{N+1} \end{pmatrix} = s^*,$$

and Lemma 1 yields the admissibility of $(\hat{x} + \eta^\alpha, \hat{v} + \zeta^\alpha)$. Hence

$$0 \le 2\mathcal{F}_1(\hat{x}, \hat{u}, \eta^\alpha, \xi^\alpha) + \mathcal{F}_2(\eta, \xi) = 2\alpha \mathcal{F}_1(\hat{x}, \hat{u}, \eta, \xi) + \alpha^2 \mathcal{F}_2(\eta, \xi)$$

for all $\alpha \in I\!R$, i.e.,

$$\mathcal{F}_2(\eta, \xi) + \frac{2\mathcal{F}_1(\hat{x}, \hat{u}, \eta, \xi)}{\alpha} \ge 0 \quad \text{for all} \quad \alpha \in I\!R \setminus \{0\}.$$

This immediately yields $\mathcal{F}_1(\hat{x}, \hat{u}, \eta, \xi) = 0$ and $\mathcal{F}_2(\eta, \xi) \ge 0$. ■

Remark. From the proofs of Theorem 1 and Theorem 2 we obtain the following corollary. If $(\hat{x}, \hat{u})$ solves the boundary value problem (BP) and if $\mathcal{F}_2$ is positive definite (in the sense given in the introductory Section 1), then the solution $(\hat{x}, \hat{v})$ of the variational problem (VP) from Theorem 1 is unique in the sense that

$$\mathcal{F}(\hat{x}, \hat{v}) < \mathcal{F}(x, v) \text{ for all admissible } (x, v) \text{ with } x \in \tilde{\mathcal{R}} \text{ and } x \ne \hat{x}$$

holds. Conversely, the existence of a solution of (VP) which is unique in the above sense implies the positive definiteness of $\mathcal{F}_2$.

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