

Kneser's theorem in q -calculus

Martin Bohner¹ and Mehmet Ünal²

¹ Department of Mathematics and Statistics, University of Missouri–Rolla, Rolla, MO 65401, USA

² Department of Mathematics and Computer Science, University of Bahçeşehir, TR-34538 Bahçeşehir, Istanbul, Turkey

E-mail: bohner@umr.edu and munal@bahcesehir.edu.tr

Received 19 February 2005, in final form 7 June 2005

Published 13 July 2005

Online at stacks.iop.org/JPhysA/38/6729

Abstract

While difference equations deal with discrete calculus and differential equations with continuous calculus, so-called q -difference equations are considered when studying q -calculus. In this paper, we obtain certain oscillation criteria for second-order q -difference equations, among them a q -calculus version of the famous Kneser theorem.

PACS numbers: 02.30.Hq, 02.30.Gp, 02.30.Ks, 02.30.Lt, 02.20.Uw

Mathematics Subject Classification: 39A13, 39A12, 34C10, 34K11

1. Introduction

We shall be interested in obtaining Kneser-type oscillation criteria for second-order q -difference equations of the form

$$D_q^2 x + r(t)x^\sigma = 0, \quad t \in \mathbb{T} := q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\} \quad \text{with } q > 1, \quad (1.1)$$

where for $t \in \mathbb{T}$,

$$D_q x(t) = (x^\sigma(t) - x(t))/\mu(t), \quad x^\sigma(t) = x(\sigma(t)), \quad \sigma(t) = qt, \quad \mu(t) = \sigma(t) - t.$$

Before we give the precise formulation of our Kneser-type oscillation criterion for q -difference equations, we mention a few background details which serve to motivate the results of this paper. Equations of the type

$$x'' + r(t)x = 0, \quad t \in \mathbb{R} \quad (1.2)$$

and

$$\Delta^2 x + r(t)x^\sigma = 0, \quad t \in \mathbb{Z}, \quad (1.3)$$

where $\Delta x = x^\sigma - x$ and $x^\sigma(t) = x(t+1)$ for $t \in \mathbb{Z}$ have been studied in the continuous and discrete settings, respectively.

Among the many topics related to the study of equation (1.2), one which has held the interest of mathematicians for over a century is the study of the existence and location of the number of zeros of its solutions. In particular, the search for comparison-type oscillation and nonoscillation criteria, that is to say, necessary and/or sufficient conditions on the function for real-valued solutions of (1.2) to have (or not to have) an infinite number of zeros in the interval has long been studied in both the continuous and discrete settings.

Historically, the development of such comparison-type oscillation criteria for equation (1.2) has its foundations in the classic 1836 memoir of Sturm [30], where he also developed a first comparison criterion as

$$r(t) \geq r_0 > 0 \quad \text{for oscillation}$$

and

$$r(t) \leq 0 \quad \text{for nonoscillation.}$$

However, the general importance and usefulness of Sturm's work was not properly recognized until much later when it was extended in a variety of ways in a series of papers by Bôcher [7–11]. The discovery of another famous comparison-type criterion is due to Kneser [24] in 1893 who established that

$$t^2 r(t) \geq \frac{1 + \varepsilon}{4} \quad \text{for some } \varepsilon > 0 \text{ implies oscillation}$$

while

$$t^2 r(t) \leq \frac{1}{4} \quad \text{implies nonoscillation.}$$

Later Fite [17] considered the conditions

$$r(t) > 0 \quad \text{and} \quad \int r(s) \, ds = \lim_{t \rightarrow \infty} \int r(s) \, ds = \infty$$

and showed that they yield oscillation. In [21], Hille assumed

$$r(t) > 0 \quad \text{and} \quad \int r(s) \, ds = \lim_{t \rightarrow \infty} \int r(s) \, ds \quad \text{exists,}$$

and gave a generalized version of Kneser's result as

$$tP(t) \geq \frac{1 + \varepsilon}{4} \quad \text{implies oscillation}$$

while

$$tP(t) \leq \frac{1}{4} \quad \text{implies nonoscillation,}$$

where

$$P(t) = \int_t^\infty r(s) \, ds.$$

The work of Fite and Hille generated a great deal of activity in the search for further oscillation criteria. For example, the condition $r(t) > 0$ in Fite's result was removed by Wintner [34], and in the 'oscillatory part' of Hille's result, Moore [27] replaced $r(t) > 0$ by the assumption that $tP(t)$ is bounded. For further extensions of these results we refer to Wintner [32, 33], Reid [28, chapter 2] and [29], Hartman [19, 20] and Gesztesy and Ünal [18]. For results connected to the discrete equation (1.3), we refer to [1, 26].

Most of the above-described results are based on the following two theorems that we are citing here for the convenience of the reader. In theorem 1 (differential equations), $\sigma(t) = t$,

in theorem 2 (difference equations), $\sigma(t) = t + 1$, and in theorem 3 (q -difference equations), $\sigma(t) = qt$. We also put $x^\sigma = x \circ \sigma$.

Theorem 1. *The differential equation*

$$x'' + \frac{b}{t\sigma(t)}x^\sigma = 0 \quad \text{is oscillatory iff } b > \frac{1}{4}.$$

Theorem 2. *The difference equation*

$$\Delta^2 x + \frac{b}{t\sigma(t)}x^\sigma = 0 \quad \text{is oscillatory iff } b > \frac{1}{4}.$$

In this paper we prove the following theorem which is the basis for our Kneser-type oscillation criteria for (1.1).

Theorem 3. *The q -difference equation*

$$D_q^2 x + \frac{b}{t\sigma(t)}x^\sigma = 0 \quad \text{is oscillatory iff } b > \frac{1}{(\sqrt{q} + 1)^2}.$$

Note that the critical, and in the continuous and discrete cases well-known, constant $1/4$ becomes $1/(\sqrt{q} + 1)^2$ in q -calculus. Note also how theorem 3 nicely resembles the continuous result of theorem 1 as $q \rightarrow 1$. Note finally that theorem 3 solves *an open problem* connected to the theory of dynamic equations on time scales: it has been shown in [15, example 4.6] that

$$x^{\Delta\Delta} + \frac{b}{t\sigma(t)}x^\sigma = 0 \quad \text{is oscillatory if } b > \frac{1}{4}$$

for every so-called dynamic equation on any so-called time scale [12, 14]. (Note that $x^\Delta = x'$, $x^\Delta = \Delta x$ and $x^\Delta = D_q x$ if the 'time scale' \mathbb{T} is equal to \mathbb{R} , \mathbb{Z} , and $q^{\mathbb{N}_0}$, respectively.) It has also been conjectured that the 'if' can be replaced by 'iff' for every dynamic equation on any time scale. Our results show that this conjecture is *wrong*. In fact, e.g., if $q = 4$, then the critical constant is $1/9$, while if $q = 100$, the critical constant is $1/121$.

We will prove theorem 3 in section 3, after recalling some preliminaries about q -calculus and Euler–Cauchy q -difference equations in section 2. Section 3 also contains Kneser's theorem for q -difference equations, which is essentially a simple consequence of theorem 3. Finally, in section 4 we provide a generalization of theorem 3. On the basis of this generalization, further Kneser-type oscillation criteria are given.

2. Preliminaries on q -calculus

The purpose of this section is to outline some of the basic definitions and concepts of q -difference equations. Some of the material in this section is contained in the excellent monographs by Gaspard Bangerezako [3] and Viktor Kac and Pokman Cheung [23] and in the books about dynamic equations on time scales by Martin Bohner and Allan Peterson [12, 14] with slight modifications (see also [13]). For other results related to q -difference equations see [2, 4–6, 16, 25, 31].

The expression

$$D_q f(t) = \frac{f(qt) - f(t)}{(q-1)t} \quad (2.1)$$

is called the q -derivative (or *Jackson derivative* [3, formula (1.7)] or [22, 23]) of the function $f : \mathbb{T} \rightarrow \mathbb{R}$. The q -derivatives of the product and the quotient of $f, g : \mathbb{T} \rightarrow \mathbb{R}$ are given on \mathbb{T} by

$$D_q(fg) = (D_q f)g + f^\sigma(D_q g) = f(D_q g) + (D_q f)g^\sigma \quad (2.2)$$

and

$$D_q\left(\frac{f}{g}\right) = \frac{(D_q f)g - f(D_q g)}{gg^\sigma} = \frac{(D_q f)g^\sigma - f^\sigma(D_q g)}{gg^\sigma}, \quad (2.3)$$

and it follows from (2.1) that the q -derivative of f satisfies

$$f^\sigma(t) = f(qt) = f(t) + (q-1)tD_q f(t) \quad \text{for } t \in \mathbb{T}. \quad (2.4)$$

Example 1. The q -derivative of t^2 is $(q+1)t$, the q -derivative of $\frac{1}{t}$ is $-\frac{1}{qt^2}$, and the q -derivative of $\ln t$ is $\frac{\ln q}{(q-1)t}$.

In order to prove theorem 3, we will use (2.4) to rewrite (1.1) as a special case of an Euler–Cauchy equation. With this in mind, we now introduce and study the general form of the *Euler–Cauchy q -difference equation* on \mathbb{T} as

$$t\sigma(t)D_q^2 x + atD_q x + bx = 0, \quad \text{where } a, b \in \mathbb{R}, \quad (2.5)$$

subject to the condition

$$q - a(q-1) + b(q-1)^2 \neq 0. \quad (2.6)$$

By using the formulae (2.1) and

$$D_q^2 x(t) = \frac{x(q^2 t) - (q+1)x(qt) + qx(t)}{q(q-1)^2 t^2},$$

we can rewrite (2.5) as

$$x(q^2 t) - 2rx(qt) + (r^2 - d)x(t) = 0, \quad (2.7)$$

where

$$r = \frac{q+1-a(q-1)}{2} \quad \text{and} \quad d = \left[\left(\frac{a-1}{2} \right)^2 - b \right] (q-1)^2. \quad (2.8)$$

When rewriting (2.5) as (2.7), the relations

$$r = 1 - \frac{(a-1)(q-1)}{2} \quad \text{and} \quad r^2 - d = q - a(q-1) + b(q-1)^2 \quad (2.9)$$

are useful and easy to check.

Lemma 1. Suppose r and d are defined by (2.8). If

$$\alpha^2 - 2r\alpha + r^2 - d = 0, \quad (2.10)$$

then

$$x_\alpha(t) := \alpha^{\log_q t}, \quad t \in \mathbb{T}$$

solves the Euler–Cauchy equation (2.5).

Proof. Since, for $x = x_\alpha$,

$$x(qt) = \alpha^{\log_q(qt)} = \alpha^{\log_q q + \log_q t} = \alpha^{1 + \log_q t} = \alpha \alpha^{\log_q t} = \alpha x(t)$$

and

$$x(q^2 t) - 2rx(qt) + (r^2 - d)x(t) = (\alpha^2 - 2r\alpha + r^2 - d)x(t) = 0,$$

the claim follows. \square

Note also that, since $\alpha \neq 0$ by (2.6), (2.9) and (2.10), we can rewrite x_α as

$$x_\alpha(t) = \alpha^{\log_q t} = (\operatorname{sgn} \alpha)^{\log_q t} t^{\log_q |\alpha|}.$$

Now we can give the general solution of (2.5) depending on the sign of d .

Theorem 4. Assume that (2.6) holds. Suppose r and d are defined by (2.8). Then the general solution of (2.5) is given, where $c_1, c_2 \in \mathbb{R}$,

(i) if $d > 0$, putting $\alpha_1 = r + \sqrt{d}$ and $\alpha_2 = r - \sqrt{d}$, by

$$x(t) = c_1 \alpha_1^{\log_q t} + c_2 \alpha_2^{\log_q t},$$

(ii) if $d = 0$, putting $\alpha = r$, by

$$x(t) = (c_1 \ln t + c_2) \alpha^{\log_q t},$$

(iii) and if $d < 0$, putting $\alpha = r + i\sqrt{-d}$, by

$$x(t) = |\alpha|^{\log_q t} (c_1 \cos(\theta \log_q t) + c_2 \sin(\theta \log_q t)), \text{ where } \theta = \cos^{-1} \frac{\operatorname{Re} \alpha}{|\alpha|}.$$

Proof. If $d > 0$, since α_1 and α_2 are solutions of (2.10), lemma 1 yields that x_{α_1} and x_{α_2} are two solutions of (2.5). Next, if $d = 0$, since α is a solution of (2.10), lemma 1 yields that x_α solves (2.5). Now define $x(t) = x_\alpha(t) \ln t$. Then

$$x(qt) = \alpha x_\alpha(t) [\ln q + \ln t] = \alpha [x(t) + x_\alpha(t) \ln q]$$

and

$$\begin{aligned} x(q^2t) - 2rx(qt) + (r^2 - d)x(t) &= x(q^2t) - 2rx(qt) + r^2x(t) \\ &= \alpha x(qt) + \alpha x_\alpha(qt) \ln q - 2rx(qt) + r^2x(t) \\ &= (\alpha - 2r)\alpha [x(t) + x_\alpha(t) \ln q] + \alpha^2 x_\alpha(t) \ln q + r^2x(t) \\ &= (\alpha^2 - 2r\alpha + r^2)x(t) + 2\alpha(\alpha - r)x_\alpha(t) \ln q \\ &= 0 \end{aligned}$$

yield that x also solves (2.5). Finally, assume $d < 0$. Note that $\operatorname{Re} \alpha / |\alpha| \in (-1, 1)$ so that there exists $\theta \in (0, \pi)$ with $\cos \theta = \operatorname{Re} \alpha / |\alpha|$. We put

$$f(t) = \cos(\theta \log_q t), \quad g(t) = \sin(\theta \log_q t) \quad \text{and} \quad x = x_{|\alpha|} f, \quad y = x_{|\alpha|} g.$$

Now

$$\begin{aligned} f(q^2t) &= f(qt) \cos \theta - g(qt) \sin \theta, & f(t) &= f(qt) \cos \theta + g(qt) \sin \theta, \\ g(q^2t) &= g(qt) \cos \theta + f(qt) \sin \theta, & g(t) &= g(qt) \cos \theta - f(qt) \sin \theta \end{aligned}$$

so that

$$\begin{aligned} x(q^2t) - 2rx(qt) + (r^2 - d)x(t) &= x(q^2t) - 2rx(qt) + |\alpha|^2 x(t) \\ &= |\alpha| [x(qt) \cos \theta - y(qt) \sin \theta] - 2rx(qt) + |\alpha| [x(qt) \cos \theta + y(qt) \sin \theta] \\ &= 2|\alpha| x(qt) \cos \theta - 2rx(qt) \\ &= 2[|\alpha| \cos \theta - \operatorname{Re} \alpha] x(qt) \\ &= 0 \end{aligned}$$

and similarly

$$y(q^2t) - 2ry(qt) + (r^2 - d)y(t) = 2[|\alpha| \cos \theta - \operatorname{Re} \alpha] y(qt) = 0,$$

and hence x and y are solutions of (2.5).

We calculate $\mu W(x, y)$ between each two solutions, where the *Wronskian* [12, definition 3.5] is defined by $W(x, y) = x(D_q y) - (D_q x)y$, as

$$2\sqrt{d}(r^2 - d)^{\log_q t}, \quad r(r^2 - d)^{\log_q t} \ln q, \quad \text{and} \quad \sqrt{r^2 + d}(r^2 - d)^{\log_q t} \sin \theta,$$

respectively. Condition (2.6) together with (2.9) now ensures that none of these Wronskians in their respective cases is ever zero. Thus (see [12, theorem 3.7]), each of the three pairs given above form indeed fundamental sets of solutions in their respective cases, more precisely: in each of the three cases, the solution z of the initial value problem

$$D_q^2 z + r(t)z^\sigma = 0, \quad z(t_0) = z_0, \quad D_q z(t_0) = \tilde{z}_0,$$

where $t_0 \in \mathbb{T}$ is fixed, is given (and this can be checked easily) by

$$z(t) = \frac{D_q y(t_0)z_0 - y(t_0)\tilde{z}_0}{W(x, y)(t_0)}x(t) + \frac{x(t_0)\tilde{z}_0 - D_q x(t_0)z_0}{W(x, y)(t_0)}y(t).$$

This completes the proof. \square

3. Kneser's theorem

We recall that a solution x of (1.1) has a *generalized zero* at t in the case $x(t) = 0$. We say that x has a *generalized zero* in the interval $(t, \sigma(t))$ in the case $x(t)x(\sigma(t)) < 0$. We say that (1.1) is *disconjugate* on the interval $[c, d]$, if there is no nontrivial solution of (1.1) with two (or more) generalized zeros in $[c, d]$. Equation (1.1) is said to be *nonoscillatory* on $[\tau, \infty)$ if there exists $c \in [\tau, \infty)$ such that this equation is disconjugate on $[c, d]$ for every $c < d$. In the opposite case (1.1) is said to be *oscillatory* on $[\tau, \infty)$. Oscillation of (1.1) may equivalently be defined as follows: a nontrivial solution x of (1.1) is called *oscillatory* if it has infinitely many (isolated) generalized zeros in $[\tau, \infty)$. By the Sturm-type separation theorem [12, theorem 5.59], one solution of (1.1) is (non)oscillatory iff every solution of (1.1) is (non)oscillatory. The proof is easy: suppose x is a nonoscillatory solution of (1.1), i.e., $xx^\sigma > 0$ on $[T, \infty)$ for some $T > 0$. Let y be any solution of (1.1) such that x and y are linearly independent. Then $D_q(y/x) = W(x, y)/(xx^\sigma)$ by the quotient rule (2.3), where the Wronskian $W(x, y)$ actually is equal to a nonzero constant (use the product rule (2.2) to verify this). Hence y/x is eventually strictly monotone, and therefore it is eventually of one sign. Thus $(yy^\sigma)/(xx^\sigma) = (y/x)(y^\sigma/x^\sigma)$ is eventually positive, and hence $yy^\sigma > 0$ eventually, meaning that y is nonoscillatory as well.

Proof (of theorem 3). In order to set the stage, we use (2.4) to rewrite the equation

$$D_q^2 x + \frac{b}{qt^2}x^\sigma = 0 \tag{3.1}$$

as the Euler–Cauchy q -difference equation

$$qt^2 D_q^2 x + b(q-1)t D_q x + bx = 0. \tag{3.2}$$

Note that (3.2) is of the form (2.5) with $a = (q-1)b$ and $b \in \mathbb{R}$. By (2.9),

$$\begin{aligned} r &= \frac{q+1 - a(q-1)}{2} = \frac{q+1 - b(q-1)^2}{2} \\ &= \sqrt{q} - \frac{(q-1)^2}{2} \left[b - \frac{1}{(\sqrt{q}+1)^2} \right] \\ &= -\sqrt{q} - \frac{(q-1)^2}{2} \left[b - \frac{1}{(\sqrt{q}-1)^2} \right] \end{aligned} \tag{3.3}$$

and

$$r^2 - d = q - a(q-1) + b(q-1)^2 = q \neq 0$$

so that (3.2) clearly satisfies (2.6). We calculate the crucial quantity d :

$$\begin{aligned} d &= (q-1)^2 \left[\left(\frac{a-1}{2} \right)^2 - b \right] \\ &= \frac{(q-1)^2}{4} [b^2(q-1)^2 - 2b(q-1) + 1 - 4b] \\ &= \frac{(q-1)^2}{4} [b^2(q-1)^2 - 2b(q+1) + 1] \\ &= \frac{(q-1)^4}{4} \left[b^2 - b \frac{2(q+1)}{(q-1)^2} + \frac{1}{(q-1)^2} \right] \\ &= \frac{(q-1)^4}{4} \left[b - \left(\frac{1}{\sqrt{q}+1} \right)^2 \right] \left[b - \left(\frac{1}{\sqrt{q}-1} \right)^2 \right]. \end{aligned}$$

First, $d = 0$ happens if and only if

$$b = \left(\frac{1}{\sqrt{q}+1} \right)^2 \quad \text{or} \quad b = \left(\frac{1}{\sqrt{q}-1} \right)^2.$$

If $b = 1/(\sqrt{q}+1)^2$, then $r = \sqrt{q}$ by (3.3), and, taking into account the second part of theorem 4, the two solutions

$$(\sqrt{q})^{\log_q t} = \sqrt{t} \quad \text{and} \quad \sqrt{t} \ln t \quad \text{are nonoscillatory,}$$

and hence (3.2) is nonoscillatory. If $b = 1/(\sqrt{q}-1)^2$, then $r = -\sqrt{q}$ by (3.3), and, again taking into account the second part of theorem 4, the two solutions

$$(-\sqrt{q})^{\log_q t} = (-1)^{\log_q t} \sqrt{t} \quad \text{and} \quad (-1)^{\log_q t} \sqrt{t} \ln t \quad \text{are oscillatory,}$$

and hence (3.2) is oscillatory. Next, $d > 0$ happens if and only if

$$b < \left(\frac{1}{\sqrt{q}+1} \right)^2 \quad \text{or} \quad b > \left(\frac{1}{\sqrt{q}-1} \right)^2.$$

If $b < 1/(\sqrt{q}+1)^2$, then $r > \sqrt{q}$ by (3.3), and, taking into account the first part of theorem 4, the solution

$$(r + \sqrt{d})^{\log_q t} = t^{\log_q(r + \sqrt{d})} \quad \text{is nonoscillatory,}$$

and hence (3.2) is nonoscillatory. If $b > 1/(\sqrt{q}-1)^2$, then $r < -\sqrt{q}$ by (3.3), and, again taking into account the first part of theorem 4, the solution

$$(r - \sqrt{d})^{\log_q t} = (-1)^{\log_q t} t^{\log_q(\sqrt{d}-r)} \quad \text{is oscillatory,}$$

and hence (3.2) is oscillatory. Finally, $d < 0$ happens if and only if

$$\left(\frac{1}{\sqrt{q}+1} \right)^2 < b < \left(\frac{1}{\sqrt{q}-1} \right)^2.$$

Then, in this case, with the notation from the third part of theorem 4,

$$r \in (-\sqrt{q}, \sqrt{q}), \quad |\alpha| = \sqrt{q}, \quad \text{and} \quad \cos \theta = \frac{r}{\sqrt{q}} = \frac{q+1-b(q-1)^2}{2\sqrt{q}},$$

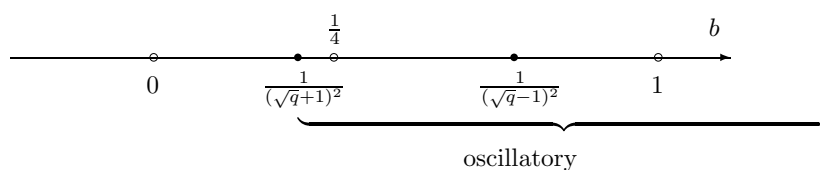


Figure 1. Oscillation and nonoscillation.

we have that the two solutions

$$\sqrt{t} \cos(\theta \log_q t) \quad \text{and} \quad \sqrt{t} \sin(\theta \log_q t) \quad \text{are oscillatory,}$$

and hence (3.2) is oscillatory. Altogether we have shown that (3.2), and hence (3.1),

$$\text{is oscillatory if and only if } b > \frac{1}{(\sqrt{q} + 1)^2}.$$

Figure 1 illustrates this fact and the proof above, which is now complete. \square

The basic statement of Sturm's comparison theorem [12, theorem 5.60] can be formulated as follows.

Theorem 5 (Sturm's comparison theorem). *Consider the equation*

$$D_q^2 x + r_1(t)x^\sigma = 0 \quad \text{on } \mathbb{T} = q^{\mathbb{N}_0}. \quad (3.4)$$

Suppose

$$r(t) \leq r_1(t) \quad \text{for all } t \in \mathbb{T}.$$

Then, if (3.4) is nonoscillatory on \mathbb{T} , then so is (1.1).

Our q -calculus version of Kneser's theorem now reads as follows.

Theorem 6 (Kneser's theorem). *The following statements hold:*

(i) If

$$\limsup_{t \rightarrow \infty} \{t\sigma(t)r(t)\} < \frac{1}{(\sqrt{q} + 1)^2},$$

then (1.1) is nonoscillatory on $q^{\mathbb{N}_0}$.

(ii) If

$$\liminf_{t \rightarrow \infty} \{t\sigma(t)r(t)\} > \frac{1}{(\sqrt{q} + 1)^2},$$

then (1.1) is oscillatory on $q^{\mathbb{N}_0}$.

Proof. By the Sturm comparison theorem, theorem 5, in order to show the first part, it suffices to show that for $b < 1/(\sqrt{q} + 1)^2$, the q -difference equation (3.1) is nonoscillatory and, in order to show the second part, that for $b > 1/(\sqrt{q} + 1)^2$, (3.1) is oscillatory. However, both of these facts have been shown in theorem 3, and therefore the proof is complete. \square

4. Extensions

We are now going to generalize theorem 3. This time we let $\gamma > 0$ and consider the equation

$$D_q \left(\frac{D_q x}{x_\gamma(t)} \right) + \frac{b}{t\sigma(t)x_\gamma(\sigma(t))} x^\sigma = 0. \quad (4.1)$$

Using the quotient rule (2.3) and formula (2.4), we can rewrite the left-hand side of (4.1) as

$$\frac{1}{t\sigma(t)x_\gamma(\sigma(t))} \left\{ qt^2 D_q^2 x + \left[b(q-1) - \frac{q(\gamma-1)}{q-1} \right] D_q x + bx \right\},$$

i.e., x solves (4.1) if and only if x solves the Euler–Cauchy q -difference equation

$$qt^2 D_q^2 x + \left[b(q-1) - \frac{q(\gamma-1)}{q-1} \right] D_q x + bx = 0. \tag{4.2}$$

Note that (4.2) is of the form (2.5) with $a = (q-1)b - q(\gamma-1)/(q-1)$ and $b \in \mathbb{R}$. Thus by (2.9),

$$r = \frac{q\gamma + 1 - b(q-1)^2}{2} \quad \text{and} \quad r^2 - d = q\gamma \neq 0$$

so that (4.2) clearly satisfies (2.6).

We remark that (4.2) is the same as (3.2) in the case of $\gamma = 1$. The strategy of the proof of our generalization of theorem 3 will be similar as in section 3. Hence we will not repeat the details from the proof of theorem 3 as given in section 3 but rather just supply the following calculation for d :

$$\begin{aligned} \frac{4}{(q-1)^2}d &= (a-1)^2 - 4b = \left[(q-1)b - \frac{\gamma q - 1}{q-1} \right]^2 - 4b \\ &= (q-1)^2 b^2 - 2(q-1) \left(\frac{\gamma q - 1}{q-1} \right) b + \left(\frac{\gamma q - 1}{q-1} \right)^2 - 4b \\ &= (q-1)^2 b^2 + (1+cq)^2 - 2b(q\gamma + 1) \\ &= (q-1)^2 b^2 + \left(\frac{q\gamma - 1}{q-1} \right)^2 - 2b(q\gamma + 1) \\ &= \frac{(q-1)^4 b^2 + (q\gamma - 1)^2 - 2b(q-1)^2(q\gamma + 1)}{(q-1)^2} \\ &= \frac{(q-1)^4 b^2 + (\sqrt{q\gamma} - 1)^2(\sqrt{q\gamma} + 1)^2 - b(q-1)^2[(\sqrt{q\gamma} - 1)^2 + (\sqrt{q\gamma} + 1)^2]}{(q-1)^2} \\ &= \frac{[(q-1)^2 b - (\sqrt{q\gamma} + 1)^2][(q-1)^2 b - (\sqrt{q\gamma} - 1)^2]}{(q-1)^2} \\ &= \left[b - \left(\frac{\sqrt{q\gamma} + 1}{q-1} \right)^2 \right] \left[b - \left(\frac{\sqrt{q\gamma} - 1}{q-1} \right)^2 \right] (q-1)^2. \end{aligned}$$

Therefore we arrive at the following result.

Theorem 7. *Let $\gamma > 0$. The q -difference equation*

$$D_q \left(\frac{D_q x}{\gamma^{\log_q t}} \right) + \frac{b}{\gamma t \sigma(t) \gamma^{\log_q t}} x^\sigma = 0 \quad \text{is oscillatory iff} \quad b > \left(\frac{\sqrt{q\gamma} - 1}{q-1} \right)^2.$$

Note how theorem 3 is a special case of theorem 7 by letting $\gamma = 1$.

As in section 3, it is now easy to obtain Kneser-type oscillation criteria for q -difference equations appearing as

$$D_q(p(t)D_q x) + r(t)x^\sigma = 0. \tag{4.3}$$

Theorem 8 (Kneser's theorem). *Let $\gamma > 0$. Then we have the following:*

(i) *If*

$$\limsup_{t \rightarrow \infty} \{\gamma t \sigma(t) \gamma^{\log_q t} r(t)\} < \left(\frac{\sqrt{q\gamma} - 1}{q - 1} \right)^2 \quad \text{and} \quad \liminf_{t \rightarrow \infty} \{p(t) \gamma^{\log_q t}\} > 1,$$

then (4.3) is nonoscillatory on $q^{\mathbb{N}_0}$.

(ii) *If*

$$\liminf_{t \rightarrow \infty} \{\gamma t \sigma(t) \gamma^{\log_q t} r(t)\} > \left(\frac{\sqrt{q\gamma} - 1}{q - 1} \right)^2 \quad \text{and} \quad \limsup_{t \rightarrow \infty} \{p(t) \gamma^{\log_q t}\} < 1,$$

then (1.1) is oscillatory on $q^{\mathbb{N}_0}$.

Another special case of theorem 7 is presented next. To state this corollary, we recall the notation (see e.g., [23, formula (3.8)])

$$[y]_q := \frac{q^y - 1}{q - 1} \quad \text{for } y \in \mathbb{R}.$$

Corollary 1. *Let $\beta \in \mathbb{R}$. The q -difference equation*

$$D_q(t^\beta D_q x) + \frac{b}{t(\sigma(t))^{1-\beta}} x^\sigma = 0 \quad \text{is oscillatory iff } b > \left[\frac{1 - \beta}{2} \right]_q^2.$$

Proof. In theorem 7, let $\gamma = 1 + (q - 1)[- \beta]_q$. Then

$$\gamma = 1 + (q - 1) \frac{q^{-\beta} - 1}{q - 1} = 1 + q^{-\beta} - 1 = q^{-\beta}$$

and

$$\frac{\sqrt{q\gamma} - 1}{q - 1} = \frac{\sqrt{q^{1-\beta}} - 1}{q - 1} = \frac{q^{(1-\beta)/2} - 1}{q - 1} = \left[\frac{1 - \beta}{2} \right]_q.$$

The proof is now complete by applying theorem 3. \square

Again note how theorem 3 is a special case of corollary 1 by letting $\beta = 0$ and observing the ' q -analogue of $1/2$ ' as $[1/2]_q = 1/(\sqrt{q} + 1)$.

Acknowledgments

We are indebted to the referees for their very detailed and useful comments that resulted in a considerable improvement of the paper. MB acknowledges support by a University of Missouri Research Board award. MÜ would like to thank the Department of Mathematics and Statistics at University of Missouri–Rolla for the hospitality extended to him during his stay when this work was initiated.

References

- [1] Agarwal R P 1992 *Difference Equations and Inequalities* 2nd edn (New York: Dekker)
- [2] Bangerezako G 2004 Variational q -calculus *J. Math. Anal. Appl.* **289** 650–65
- [3] Bangerezako G 2004 An introduction to q -difference equations <http://www.math.ucl.ac.be/membres/bangerezako/index.htm>
- [4] Bangerezako G 2005 Variational calculus on q -nonuniform lattices *J. Math. Anal. Appl.* **306** 161–79
- [5] Bangerezako G and Hounkonnou M N 2003 The factorization method for the general second-order q -difference equation and the Laguerre–Hahn polynomials on the general q -lattice *J. Phys. A: Math. Gen.* **36** 765–73

- [6] Bézivin J P 1993 Sur les équations fonctionnelles aux q -différences *Aequationes Math.* **43** 159–76
- [7] Bôcher M 1897 The theorems of oscillation of Sturm and Klein: I *Bull. Am. Math. Soc.* **4** 295–313
- [8] Bôcher M 1897 The theorems of oscillation of Sturm and Klein: II *Bull. Am. Math. Soc.* **4** 365–76
- [9] Bôcher M 1900 Application of a method of d'Alembert to the proof of Sturm's theorems of comparison *Trans. Am. Math. Soc.* **1** 414–20
- [10] Bôcher M 1901 An elementary proof of a theorem of Sturm *Trans. Am. Math. Soc.* **2** 150–1
- [11] Bôcher M 1901 Nonoscillatory linear differential equations of second order *Bull. Am. Math. Soc.* **7** 333–40
- [12] Bohner M and Peterson A 2001 *Dynamic Equations on Time Scales: An Introduction with Applications* (Boston, MA: Birkhäuser)
- [13] Bohner M and Peterson A 2001 First and second order linear dynamic equations on time scales *J. Diff. Eq. Appl.* **7** 767–92 (on the occasion of the 60th birthday of Calvin Ahlbrandt)
- [14] Bohner M and Peterson A 2003 *Advances in Dynamic Equations on Time Scales* (Boston, MA: Birkhäuser)
- [15] Bohner M and Saker S H 2004 Oscillation of second order nonlinear dynamic equations on time scales *Rocky Mountain J. Math.* **34** 1239–54
- [16] Di Vizio L, Ramis J-P, Sauloy J and Zhang C 2003 Équations aux q -différences *Gaz. Math.* **96** 20–49
- [17] Fite W B 1918 Concerning the zeros of the solutions of certain differential equations *Trans. Am. Math. Soc.* **19** 341–52
- [18] Gsztesy F and Ünal M 1998 Perturbative oscillation criteria and Hardy-type inequalities *Math. Nachr.* **189** 121–44
- [19] Hartman P 1952 On non-oscillatory linear differential equations of second order *Am. J. Math.* **74** 389–400
- [20] Hartman P 1964 *Ordinary Differential Equations* (New York: Wiley)
- [21] Hille E 1948 Non-oscillation theorems *Trans. Am. Math. Soc.* **64** 234–52
- [22] Jackson F H 1910 q -Difference equations *J. Math. Anal. Appl.* **32** 305–14
- [23] Kac V and Cheung P 2002 *Quantum Calculus* (New York: Springer)
- [24] Kneser A 1893 Untersuchungen über die reellen Nullstellen der Integrale linearer Differentialgleichungen *Math. Ann.* **42** 409–35
- [25] Levi D, Negro J and Olmo M A del 2004 Discrete q -derivatives and symmetries of q -difference equations *J. Phys. A: Math. Gen.* **37** 3459–73
- [26] Luef F and Teschl G 2004 On the finiteness of the number of eigenvalues of Jacobi operators below the essential spectrum *J. Difference Equ. Appl.* **10** 299–307
- [27] Moore R A 1955 The behavior of solutions of a linear differential equation of second order *Pacific J. Math.* **5** 125–45
- [28] Reid W T 1972 *Riccati Differential Equations* (New York: Academic)
- [29] Reid W T 1980 *Sturmian Theory for Ordinary Differential Equations* (New York: Springer)
- [30] Sturm C 1836 Sur les équations différentielles linéaires du second ordre *J. Math. Pures Appl.* **1** 106–86
- [31] Trijtzinsky W J 1933 Analytic theory of linear q -difference equations *Acta Math.* **61** 1–38
- [32] Wintner A 1949 A criterion of oscillatory stability *Q. Appl. Math.* **7** 115–7
- [33] Wintner A 1951 On the non-existence of conjugate points *Am. J. Math.* **73** 368–80
- [34] Wintner A 1957 On the comparison theorem of Kneser–Hille *Math. Scand.* **5** 255–60