LINEAR AND NONLINEAR NONLOCAL BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL-OPERATOR EQUATIONS

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Abstract. This study focuses on nonlocal boundary value problems (BVP) for linear and nonlinear elliptic differential-operator equations (DOE) that are defined in Banach-valued function spaces. The considered domain is a region with varying bound and depends on a certain parameter. Some conditions that guarantee the maximal L_p-regularity and Fredholmness of linear BVP, uniformly with respect to this parameter, are presented. This fact implies that the appropriate differential operator is a generator of an analytic semigroup. Then, by using these results, the existence, uniqueness, and maximal smoothness of solutions of nonlocal BVP for nonlinear DOE are shown. These results are applied to nonlocal boundary value problems for regular elliptic partial differential equations, finite and infinite systems of differential equations on cylindrical domains, in order to obtain the algebraic conditions that guarantee the same properties.

1. Introduction and Notation

BVPs for DOE have been studied in detail in [3, 7, 12]. A comprehensive introduction to DOE and historical references may be found in [2, 7]. The maximal L_p-regularity for differential operator equations has been discussed in e.g., [1, 5, 11]. The main objective of the present paper is to discuss the maximal regularity of nonlocal BVPs for linear DOEs in Banach-valued L_p-spaces and the existence and uniqueness of solutions of nonlocal BVPs for nonlinear elliptic DOEs. In this work

(1) BVPs for DOEs are considered in Banach-valued function spaces;
(2) boundary conditions are, in general, nonlocal;
(3) operators used in equations and in boundary conditions are, in general, unbounded;
(4) the considered domain is a region with varying bound and depends on a certain parameter.

The maximal L_p-regularity and Fredholmness of the linear problems, uniformly with respect to this parameter, are shown, and the existence, uniqueness and maximal smoothness in terms of Sobolev spaces of solutions of nonlinear BVPs are established. These

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results are applied to nonlocal boundary value problems for elliptic and quasi-elliptic partial differential equations and their finite or infinite systems on cylindrical domains with varying bounds.

Let E be a Banach space. Let $L_p(\Omega;E)$ denote the space of strongly measurable E-valued functions that are defined on Ω with the norm

$$\|f\|_{L_p} = \|f\|_{L_p(\Omega;E)} = \left(\int \|f(x)\|_E^p \, dx\right)^{\frac{1}{p}}, \quad 1 \le p < \infty.$$

By $L_{\mathbf{p}}(\Omega)$ and $W_{\mathbf{p}}^l(\Omega)$, $\mathbf{p} = (p_1, p_2)$, we will denote a scalar-valued $\mathbf{p}$-summable function space and Sobolev space with mixed norm, respectively [4]. Let $\mathbb{C}$ be the set of complex numbers and

$$S_\varphi = \{\lambda : \ \lambda \in \mathbb{C}, \ |\arg \lambda - \pi| \le \pi - \varphi\} \cup \{0\}, \quad 0 < \varphi \le \pi.$$

A linear operator A is said to be φ-positive in a Banach space E with bound $M > 0$ if $D(A)$ is dense on E and

$$\left\|(A - \lambda I)^{-1}\right\|_{L(E)} \le M \left(1 + |\lambda|\right)^{-1}$$

with $\lambda \in S_\varphi$, $\varphi \in (0, \pi]$, I is the identity operator in E, and $L(E)$ is the space of bounded linear operators in E. Sometimes instead of $A + \lambda I$ will be written $A + \lambda$ or A_λ. It is known [10, §1.15.1] that there exist fractional powers A^θ of a positive operator A. Let $E(A^\theta)$ denote the space $D(A^\theta)$ with graphical norm.

Let $E_0 \subset E$ be two Banach spaces. Let $(E_0, E)_{\theta,p}$, $0 < \theta < 1$, $1 \le p \le \infty$, denote interpolation spaces for $\{E_0, E\}$ by the K-method [10, §1.3.1]. A function

$$\Psi \in C^{(l)}\left(\mathbb{R}^n \setminus \{0\}; B(E_0, E)\right), \quad \text{where} \quad l \in \mathbb{N}$$

is called a multiplier from $L_p(\mathbb{R}^n; E_0)$ to $L_q(\mathbb{R}^n; E)$ if there exists a constant $C > 0$ such that

$$\left\|F^{-1}\Psi(\xi)Fu\right\|_{L_q(\mathbb{R}^n;E)} \le C \|u\|_{L_p(\mathbb{R}^n;E_0)}$$

for all $u \in L_p(\mathbb{R}^n; E_0)$, where $\hat{u} = Fu$ is the Fourier transformation of u. We denote the set of all multipliers from $L_p(\mathbb{R}^n; E_0)$ to $L_q(\mathbb{R}^n; E)$ by $M_p^q(E_0, E)$.

A set $K \subset B(E_0, E)$ is called R-bounded if there is a constant C such that for all $T_1, T_2, \ldots, T_m \in K$ and $u_1, u_2, \ldots, u_m \in E_0$, $m \in \mathbb{N}$,

$$\int_0^1 \left\|\sum_{j=1}^m r_j(y) T_j u_j\right\|_E dy \le C \int_0^1 \left\|\sum_{j=1}^m r_j(y) u_j\right\|_{E_0} dy,$$

where $\{r_j\}$ is a sequence of independent symmetric $[-1, 1]$-valued random variables on $[0, 1]$ (see [5]). Let

$$U_n = \{\beta = (\beta_1, \beta_2, \ldots, \beta_n) : \ |\beta| \le n\}, \quad \xi^\beta = \xi_1^{\beta_1} \xi_2^{\beta_2} \cdots \xi_n^{\beta_n}.$$

Definition 1.1. A Banach space E is said to be a space satisfying the multiplier condition with respect to $p \in (1,\infty)$ when for every $\Psi \in C^{(n)}(\mathbb{R}^n \setminus \{0\}; B(E))$ if the set

$$\left\{\xi^\beta D_\xi^\beta \Psi(\xi) :\ \xi \in \mathbb{R}^n \setminus \{0\},\ \beta \in U_n\right\} \tag{1.1}$$

is R-bounded, then $\Psi \in M_p^p(E)$. Moreover, if K is a subset of $C^{(n)}(\mathbb{R}^n \setminus \{0\}; B(E))$ such that the set (1.1) is uniformly R-bounded for $\Psi \in K$, then K is called a uniform collection of multipliers.

Remark 1.2. If E is a UMD space and $n = 1$ or if E is a UMD space with property (α) [6] and $n > 1$, then E satisfies the multiplier condition (see also [5, 11]).

Definition 1.3. The positive operator A is said to be R-positive in the Banach space E if there exists $\varphi \in (0,\pi]$ such that the set

$$L_A = \left\{(1+|\xi|)\,(A-\xi I)^{-1} :\ \xi \in S_\varphi\right\}$$

is R-bounded.

Note that in Hilbert spaces every norm-bounded set is R-bounded.

Let $\sigma_\infty(E)$ denote the space of compact operators acting in E. Let us consider the space $W_p^l(\Omega; E_0, E)$, $\Omega \subset \mathbb{R}^n$, $E_0 \subset E$, $l = (l_1, \ldots, l_n)$, that consists of functions $u \in L_p(\Omega; E_0)$ that have the generalized derivatives $D_k^{l_k} u = \frac{\partial^{l_k}}{\partial x_k^{l_k}} u \in L_p(\Omega; E)$ with the norm

$$\|u\|_{W_p^l} = \|u\|_{W_p^l(\Omega;E_0,E)} = \|u\|_{L_p(\Omega;E_0)} + \sum_{k=1}^n \left\|D_k^{l_k} u\right\|_{L_p(\Omega;E)} < \infty.$$

For $n = 1$, $l_1 = l$, $\Omega = (a,b) \subset \mathbb{R}$, the space $W_p^l(\Omega; E_0, E)$ will be denoted by $W_p^l(a,b; E_0, E)$.

2. Statement of the Problem

Let $b = b(t)$ be a positive continuous function on $[c,d]$. Consider a nonlocal linear BVP on $(0,b)$

$$Lu := -u''(x) + Au(x) + B_1(x)u'(x) + B_2(x)u(x) = f(x), \quad x \in (0, b(t)), \tag{2.1}$$

$$L_{tk}u := \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(b) + \sum_{j=1}^{N_k} \delta_{kj} u^{(m_k)}(x_{tkj}) + \sum_{j=1}^{M_k} T_{kj} u(x_{tkj0}) = f_k, \quad k \in \{1,2\}, \tag{2.2}$$

where $f_k \in E_k = (E(A), E)_{\theta_k,p}$, $\theta_k = \frac{m_k}{2} + \frac{1}{2p}$, $p \in (1,\infty)$, $m_k \in \{0,1\}$; $\alpha_k, \beta_k, \delta_{kj}$ are, in general, complex-valued continuous functions of t, $x_{tkj} \in (0,b)$, $x_{tkj0} \in [0,b]$, x_{tkj} and x_{tkj0} are continuous in t; $A, B_k(x)$ for $x \in [0,b]$, and $T_{kj} = T_{kj}(t)$ are, generally speaking, unbounded operators in E.

Let us also consider the nonlinear BVP

$$Lu := -u''(x) + A(x,u,u')u + \lambda u = F(x,u,u'), \quad x \in (0,a), \tag{2.3}$$

$$L_k u := \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(a) + \sum_{j=1}^{N_k} \delta_{kj} u^{(m_k)}(x_{kj}) = f_k, \quad k \in \{1,2\}, \tag{2.4}$$

where $\alpha_k, \beta_k, \delta_{kj}$ are, in general, complex numbers and $0 < a \le a_0$, $x_{kj} \in (0,a)$, $f_k \in E_k$, and $A = A(0,0,0)$ is a positive operator in E.

The problem (2.1)–(2.2) is said to be maximal L_p-regular if the problem (2.1)–(2.2) for all $f \in L_p(0,b;E)$ has a unique solution $u \in W_p^2(0,b;E(A),E)$ satisfying

$$\|Au\|_{L_p(0,b;E)} + \|u''\|_{L_p(0,b;E)} \le \|f\|_{L_p(0,b;E)}.$$

3. Background Material

From [1, 9] we obtain the following background material:

Theorem 3.1. *Suppose*

(1) *E is a Banach space satisfying the multiplier condition;*
(2) *$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $l = (l_1, l_2, \ldots, l_n)$ are n-tuples of nonnegative integers such that*

$$\kappa = |\alpha : l| = \sum_{k=1}^{n} \frac{\alpha_k}{l_k} \le 1, \quad 0 \le \mu \le 1-\kappa;$$

(3) *A is an R-positive operator in E for $\varphi \in (0,\pi]$;*
(4) *$\Omega \subset \mathbb{R}^n$ is a region such that there exists a bounded linear extension operator acting from $L_p(\Omega;E)$ to $L_p(\mathbb{R}^n;E)$ and also from $W_p^l(\Omega;E(A),E)$ to $W_p^l(\mathbb{R}^n;E(A),E)$.*

Then the embedding

$$D^\alpha W_p^l(\Omega;E(A),E) \subset L_p\left(\Omega; E(A^{1-\kappa-\mu})\right) \tag{3.1}$$

is continuous and there exists a positive constant C_μ such that

$$\|D^\alpha u\|_{L_p(\Omega;E(A^{1-\kappa-\mu}))} \le C_\mu \left[h^\mu \|u\|_{W_p^l(\Omega;E(A),E)} + h^{-(1-\mu)} \|u\|_{L_p(\Omega;E)}\right]$$

for all $u \in W_p^l(\Omega;E(A),E)$ and $0 < h \le h_0 < \infty$.

Theorem 3.2. *Suppose all conditions of Theorem 3.1 are satisfied and suppose Ω is a bounded region in $\mathbb{R}^n$ and $A^{-1} \in \sigma_\infty(E)$. Then for $0 < \mu \le 1-\kappa$ the embedding* (3.1) *is compact.*

Theorem 3.3. *Suppose all conditions of Theorem 3.1 are satisfied. Then the embedding*

$$D^\alpha W_p^l(\Omega;E(A),E) \subset L_p\left(\Omega; (E(A),E)_{\kappa,1}\right)$$

is continuous and there exists a positive constant C such that

$$\|D^\alpha u\|_{L_p(\Omega;(E(A),E)_{\kappa,1})} \le C \|u\|_{W_p^l(\Omega;E(A),E)}$$

for all $u \in W_p^l(\Omega;E(A),E)$.

Theorem 3.4. *Let E be a Banach space and A be a positive operator in E of type φ. Let $m \in \mathbb{N}$, $1 \leq p < \infty$, and $\frac{1}{2p} < \alpha < m + \frac{1}{2p}$. Let $0 \leq \gamma < 2p\alpha - 1$. Then for $\lambda \in S(\varphi)$, the operator $-A_\lambda^{1/2}$ generates a semigroup $e^{-A_\lambda^{1/2}x}$, which is holomorphic for $x > 0$ and strongly continuous for $x \geq 0$. Moreover, there exists a constant $C > 0$ such that*

$$\int_0^\infty \left\| A_\lambda^\alpha e^{-xA_\lambda^{1/2}} u \right\|_E^p x^\gamma dx \leq C \left(\|u\|^p_{(E,E(A^m))_{\frac{\alpha}{m} - \frac{1+\gamma}{2mp},p}} + |\lambda|^{p\alpha - \frac{1+\gamma}{2}} \|u\|_E^p \right)$$

for every $u \in (E, E(A^m))_{\frac{\alpha}{m} - \frac{1+\gamma}{2pm},p}$ and $\lambda \in S(\varphi)$.

By using a similar technique as in [8] (or [10, §1.8.1]) we obtain the following.

Theorem 3.5. *Let $l, s \in \mathbb{N}_0$ with $0 \leq s \leq l - 1$, $\theta = \frac{ps+1}{pl}$, $x_0 \in [0, b]$, $0 < h \leq h_0$. Then the mapping $u \to u^{(s)}(x_0)$ is continuous from $W_p^l(0, b; E_0, E)$ onto $(E_0, E)_{\theta,p}$,*

$$\left\| u^{(s)}(x_0) \right\|_{(E_0,E)_{\theta,p}} \leq C \|u\|_{W_p^l(0,b;E_0,E)},$$

and

$$\left\| u^{(s)}(x_0) \right\|_E \leq C \left[h^{1-\theta} \left\| u^{(l)} \right\|_{L_p(0,b;E)} + h^{-\theta} \|u\|_{L_p(0,b;E)} \right].$$

4. Homogeneous Equations

Let us first consider the problem

$$L_0(\lambda)u := -u''(x) + (A + \lambda)u(x) = 0 \tag{4.1}$$

$$L_{tk0}u := \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(b) + \sum_{j=1}^{N_k} \delta_{kj} u^{(m_k)}(x_{tkj}) = f_k, \quad k \in \{1, 2\}, \tag{4.2}$$

Theorem 4.1. *Let the following conditions be satisfied:*

(1) *A is a φ-positive operator in a Banach space E for $\varphi \in (0, \pi]$;*
(2) *$\theta = \theta(t) = (-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \neq 0$ and $\theta_k = m_k/2 + 1/(2p)$ for $k \in \{1, 2\}$.*

Then the problem (4.1)–(4.2) for $f_k \in E_k$, $|\arg \lambda| \leq \pi - \varphi$ and sufficiently large $|\lambda|$ has a unique solution that belongs to the space $W_p^2(0, b; E(A), E)$, and coercive uniformity with respect to the parameters t and λ holds, i.e.,

$$\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \left\| u^{(i)} \right\|_{L_p} + \|Au\|_{L_p} \leq M \sum_{k=1}^{2} \left(\|f_k\|_{E_k} + |\lambda|^{1-\theta_k} \|f_k\|_E \right) \tag{4.3}$$

is satisfied for the solution of the problem (4.1)–(4.2).

Proof. By condition (1) and by virtue of [10, §1.14] for $|\arg \lambda| \leq \pi - \varphi$, there exists the holomorphic (for $x > 0$) and strongly continuous (for $x \geq 0$) semigroup $e^{-xA_\lambda^{1/2}}$. In a similar way as in [12, Lemma 5.3.2/1], we obtain that an arbitrary solution of the equation (4.1), for $|\arg \lambda| \leq \pi - \varphi$, belonging to the space $W_p^2(0, b; E(A), E)$, has the form

$$u(x) = V_{\lambda 1} g_1 + V_{\lambda 2} g_2,$$

where

$$g_k \in (E(A), E)_{\frac{1}{2p},p}, \quad V_{\lambda 1} = e^{-xA_\lambda^{1/2}}, \quad V_{\lambda 2} = e^{-(b-x)A_\lambda^{1/2}}.$$

Now taking into account the boundary conditions (4.2), we obtain a representation of the solution of the problem (4.1)–(4.2) as

$$u(x) = Q(\lambda, t)\left[\sum_{k=1}^{2}\sum_{i=1}^{2} C_{ki}(\lambda, t) V_{\lambda k}(x) A_\lambda^{-\nu_k} f_i\right], \tag{4.4}$$

where $Q(\lambda, t)$ is a uniformly (with respect to λ and t) bounded operator in E. By virtue of the properties of holomorphic semigroups [10, §1.14] and by using Theorem 3.4, from (4.4) we obtain the estimate (4.3). □

5. Nonhomogenous Equations

Now consider a boundary value problem for nonhomogenous equations with parameters on the region $(0, b)$:

$$L_0(\lambda)u := -u''(x) + (A + \lambda I)u(x) = f(x), \tag{5.1}$$

$$L_{tk0}u := \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(b) + \sum_{j=1}^{N_k} \delta_{kj} u^{(m_k)}(x_{tkj}) = f_k, \quad k \in \{1, 2\}. \tag{5.2}$$

Theorem 5.1. *Let all conditions of Theorem 4.1 be satisfied, let E be a Banach space satisfying the multiplier condition, and let A be an R-positive operator in E for $\varphi \in (0, \pi]$. Then the operator $u \to D_0(\lambda, t)u := \{L_0(\lambda)u, L_{t10}u, L_{t20}u\}$ for $|\arg\lambda| \le \pi - \varphi$, $0 < \varphi \le \pi$, and sufficiently large $|\lambda|$, is an isomorphism from $W_p^2(0, b; E(A), E)$ onto $L_p(0, b; E) \times E_1 \times E_2$. Moreover, coercive uniformity with respect to the parameters λ and t holds, i.e.,*

$$\sum_{j=0}^{2} |\lambda|^{1-\frac{j}{2}} \left\|u^{(j)}\right\|_{L_p} + \|Au\|_{L_p} \le C\left[\|f\|_{L_p} + \sum_{k=1}^{2}\left(\|f_k\|_{E_k} + |\lambda|^{1-\theta_k}\|f_k\|_E\right)\right] \tag{5.3}$$

is satisfied for the solution of the problem (5.1)–(5.2).

Proof. We have proved the uniqueness of the solution of problem (5.1)–(5.2) in Theorem 4.1. Let

$$\bar{f}_t(x) = \begin{cases} f(x) & \text{if} \quad x \in [0, b(t)] \\ 0 & \text{if} \quad x \notin [0, b(t)]. \end{cases}$$

We now show that a solution of the problem (5.1)–(5.2) which belongs to the space $W_p^2(0, b; E(A), E)$ can be represented as a sum $v(x) = u_1(x) + u_2(x)$, where u_1 is the restriction on $[0, b]$ of the solution u of the equation

$$L_0(\lambda)u = \bar{f}_t(x), \quad x \in \mathbb{R} = (-\infty, \infty) \tag{5.4}$$

and u_2 is a solution of the problem

$$L_0(\lambda)u = 0, \quad L_{tk0}u = f_k - L_{tk0}u_1. \tag{5.5}$$

A solution of equation (5.4) is given by the formula

$$u(x) = F^{-1}L_0^{-1}(\lambda, t, \xi)F\bar{f}_t = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}L_0^{-1}(\lambda, t, \xi)\left(F\bar{f}_t\right)(\xi)d\xi,$$

where $F\bar{f}$ is the Fourier transform of the function $\bar{f}$, and

$$L_0(\lambda, \xi) = \left(\xi^2 + \lambda\right) I + A.$$

Using R-positivity of A, we show that the operator functions

$$\Psi_\lambda(\xi) = AL_0^{-1}(\lambda, \xi) \quad \text{and} \quad \Psi_{\lambda,j}(\xi) = |\lambda|^{1-\frac{j}{2}}\xi^j L_0^{-1}(\lambda, \xi)$$

are Fourier multipliers in $L_p(\mathbb{R}; E)$. This implies that the problem (5.4) has a solution $u \in W_p^2(\mathbb{R}; E(A), E)$ and

$$\sum_{j=0}^{2} |\lambda|^{1-\frac{j}{2}} \left\|u^{(j)}\right\|_{L_p} + \|Au\|_{L_p} \leq C\|f\|_{L_p}. \tag{5.6}$$

Moreover, by using Theorem 4.1 and (the trace) Theorem 3.5, we obtain that the problem (5.5) has a solution $u_2 \in W_p^2(0, b; E(A), E)$ with the estimate (4.3). Then the estimates (4.3) and (5.6) imply (5.3). □

Remark 5.2. Let O be a realization operator of the problem (2.1)–(2.2) in $L_p(0, b; E)$, i.e.,

$$D(O) = \left\{W_p^2(0, b; E(A), E),\ L_1u = 0,\ L_2u = 0\right\}$$
$$Ou = -u''(x) + Au(x), \quad x \in (0, b(t)).$$

By Theorem 5.1, the differential operator O has a resolvent $(O - \lambda I)^{-1}$ for $\lambda \in S(\varphi)$, $0 < \varphi \leq \pi$, and uniformity with respect to the parameter t

$$\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \left\|D^i(O - \lambda I)^{-1}\right\|_{B(L_p(0,b;E))} + \left\|A(O - \lambda I)^{-1}\right\|_{B(L_p(0,b;E))} \leq C$$

holds. This estimate implies that the operator O is a generator of an analytic semigroup in $L_p(0, b; E)$ for $\varphi < \frac{\pi}{2}$.

6. Coerciveness on the Space Variable and Fredholmness

Consider the problem (2.1)–(2.2).

Theorem 6.1. *Let the following conditions be satisfied:*

1. *E is a Banach space satisfying the multiplier condition and A is an R-positive operator in E for $\varphi = \pi$ and $A^{-1} \in \sigma_\infty(E)$;*
2. $\theta(t) = (-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \neq 0$, $\theta_k = \frac{m_k}{2} + \frac{1}{2p}$;

(3) *for any $\varepsilon > 0$ there is $C(\varepsilon)$ such that for almost all $x \in [0,b]$,*

$$\|B_1(x)u\| \le \varepsilon \|u\|_{(E(A),E)_{1/2,1}} + C(\varepsilon)\|u\|, \quad u \in (E(A),E)_{1/2,1},$$
$$\|B_2(x)u\| \le \varepsilon \|Au\| + C(\varepsilon)\|u\|, \quad u \in D(A),$$

where the functions $B_1(x)u$ for $u \in (E(A),E)_{1/2,1}$ and $B_2(x)u$ for $u \in D(A)$ are measurable on $[0,1]$ in E;

(4) *if $m_k = 0$, then $T_{kj} = 0$; if $m_k = 1$, then for any $\varepsilon > 0$ there is $C(\varepsilon)$ such that for $u \in (E(A),E)_{\frac{1}{2p},p}$, $p > 1$,*

$$\|T_{kj}u\|_{(E(A),E)_{\frac{1}{2}+\frac{1}{2p},p}} \le \varepsilon \|u\|_{(E(A),E)_{\frac{1}{2p},p}} + C(\varepsilon)\|u\|.$$

Then for all $u \in W_p^2(0,b;E(A),E)$, coercive uniformity with respect to the parameters t and λ holds, i.e.,

$$\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \left\|u^{(i)}\right\|_{L_p} \le C\left[\|Lu\|_{L_p} + \sum_{k=1}^{2}\|L_{tk}u\|_{E_k} + \|u\|_{L_p}\right] \tag{6.1}$$

is satisfied for the solution of the problem (2.1)–(2.2). Moreover, the operator $u \to D(t)u = \{Lu, L_{t1}u, L_{t2}u\}$ from $W_p^2(0,b;E(A),E)$ into $L_p(0,1;E) \times E_1 \times E_2$ is bounded and Fredholm.

Proof. Assume that the condition (1) is satisfied for $\arg\lambda = \pi$. The general case is reduced to the latter if the operator $A+\lambda_0 I$, for some sufficiently large $\lambda_0 > 0$, is considered instead of the operator A, and the operator $B_2(x)-\lambda_0 I$ is considered instead of the operator $B_2(x)$. Let $u \in W_p^2(0,b;E(A),E)$ be a solution of the problem (2.1)–(2.2). Then u is a solution of the problem

$$-\frac{d^2}{dx^2}u(x) + (A+\lambda I)\,u(x) = f(x) + \lambda u(x) - B_1(x)\frac{d}{dx}u(x) - B_2(x)u(x),$$
$$L_{tk0}u = f_k - \sum_{j=1}^{M_k} T_{kj}u(x_{kj0}), \quad k \in \{1,2\},$$

where L_{tk0} are defined in (4.2). By Theorem 5.1, for sufficiently large $\lambda_0 > 0$, we have

$$\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \left\|u^{(i)}\right\|_{L_p} + \|Au\|_{L_p} \le C\left[\left\|f + \lambda_0 u - B_1 u^{(1)} - B_2 u\right\|_{L_p} \right.$$
$$\left. + \sum_{k=1}^{2}\left\|f_k - \sum_{j=1}^{M_k} T_{kj}u(x_{tkj0})\right\|_{E_k}\right]. \tag{6.2}$$

By Theorem 3.1, Theorem 3.3, and condition (3), for all $u \in W_p^2(0,b;E(A),E)$ we have

$$\left\|B_1 u^{(1)}\right\|_{L_p} \le \varepsilon \left\|u^{(1)}\right\|_{W_p^1(0,b;(E(A),E)_{1/2,1},E)} + C(\varepsilon)\left\|u^{(1)}\right\|_{L_p},$$

$$\|B_2u\|_{L_p} \leq \varepsilon \|u\|_{W_p^1(0,b;E(A),E)} + C(\varepsilon) \|u\|_{L_p}, \quad \varepsilon > 0.$$

From these two inequalities, we obtain for $u \in W_p^2(0,b;E(A);E)$ by using Theorem 3.1

$$\max\left\{\left\|B_1u^{(1)}\right\|_{L_p},\ \|B_2u\|_{L_p}\right\} \leq \varepsilon \|u\|_{W_p^2(0,b;E(A),E)} + C(\varepsilon) \|u\|_{L_p}. \tag{6.3}$$

By virtue of Theorem 3.5, the operator $u \to u(x_0)$ from $W_p^2(0,b;E(A),E)$ into $(E(A),E)_{\frac{1}{2p},p}$ is bounded and

$$\|u(x_0)\|_{(E(A),E)_{\frac{1}{2p},p}} \leq C \|u\|_{W_p^2(0,b;E(A),E)}. \tag{6.4}$$

Consequently, from condition (4) and the estimate (6.4), it follows for all $\varepsilon > 0$ and $u \in W_p^2(0,b;E(A),E)$ that

$$\|T_{kj}u(x_{kj0})\|_{E_k} \leq \varepsilon \|u\|_{W_p^2(0,b;E(A),E)} + C(\varepsilon) \|u\|_{L_p}. \tag{6.5}$$

Using the estimates (6.2), (6.3), and (6.5), we get (6.1).

Next, the operator D can be rewritten in the form

$$D = D_0(\lambda_0, t) + L_1, \quad \text{where} \quad D_0(\lambda_0,t)u = \{L_0(\lambda)u,\ L_{t10},\ L_{t20}\},$$

using the notation from (4.1)–(4.2), and

$$D_1(\lambda_0,t)u = \left\{-\lambda_0u(x) + B_1(x)u^{(1)}(x) + B_2(x)u(x),\ \sum_{j=1}^{M_1} T_{1j}u(x_{t1j}),\ \sum_{j=1}^{M_2} T_{2j}u(x_{t2j}),\right\}.$$

We can conclude from the first part of this theorem that the operator $D_0(\lambda_0,t)$ from W_p^2 onto $L_p \times E_1 \times E_2$ has an inverse. From the estimate (6.1) and in view of Theorem 3.1 and Theorem 3.2, it follows that the operator D_1 from W_p^2 into $L_p \times E_1 \times E_2$ is compact. Then by the perturbation theory for linear operators, it follows that the operator $D(t)$ from W_p^2 into $L_p \times E_1 \times E_2$ is a Fredholm operator. □

7. Nonlinear BVPs for DOEs

Consider the nonlinear BVP (2.3)–(2.4). Let

$$E_k = (E(A),E)_{\eta_k,p}, \quad \eta_k = \frac{k}{2} + \frac{1}{2p}, \quad k \in \{0,1\}, \quad F_0 = E_0 \times E_1$$

$$X = L_p(0,a;E), \quad Y = W_p^2((0,a);E(A),E), \quad 0 < a \leq a_0, \quad \theta_j = \frac{m_j}{2} + \frac{1}{2p}$$

$$Y_0 = \left\{u \in W_p^2((0,a);E(A),E) :\ L_1u = 0,\ L_2u = 0\right\}, \quad f(x) = F(x,0,0).$$

Remark 7.1. By Theorem 3.5, the embedding $D^kW_p^2(0,a;E(A),E) \subset E_k$ is continuous and there exist constants C_0 and C_1 such that for $w \in Y$, $W = (w_1,w_2)$, $w_k = D^{m_k}w$, we have

$$\left\|D^kw\right\|_{E_k,\infty} = \sup_{x\in[0,a]} \left\|D^kw(x)\right\|_{E_k} \leq C_1 \|w\|_Y,$$

$$\|W\|_{0,\infty} = \sup_{x\in[0,a]} \sum_{k=0}^{1} \|w_k\|_{E_k} \le C_0 \|w\|_Y .$$

By virtue of Theorem 3.5 and Theorem 5.1, both C_0 and C_1 clearly may be chosen independent of a.

In what follows, let us assume the following **condition**:

(C) Let E be a Banach space satisfying the multiplier condition with respect to $p > 1$. Suppose there exist $f_k \in E_k$ such that the operator $A(0, \Phi)$ for $\Phi = (f_1, f_2)$ is R-positive in E.

Theorem 7.2. *Assume* (C) *holds and let the following conditions be satisfied:*

(1) $(-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \neq 0$*;*

(2) $A(x,U)$ *for* $U = (u_0, u_1)$, $x \in [0,a]$, $u_k \in E_k$, *is a* φ*-positive operator in a Banach space* E *for* $0 < \varphi \le \pi$, *where the domain of definition* $D(A(x,U))$ *does not depend on* x, u_k, *and* $A : [0,a] \times F_0 \to B(E(A), E)$ *is continuous. Moreover, for each* $R > 0$ *there is a constant* $L(R) > 0$ *such that*

$$\left\| \left[A(x,U) - A(x,\bar U)\right] u\right\|_E \le L(R) \left\|U - \bar U\right\|_{F_0} \|Au\|_E$$

for $x \in [0,a]$, $U, \bar U \in F_0$, $\bar U = (\bar u_0, \bar u_1)$, *and* $\|U\|_{F_0} \le R$, $\left\|\bar U\right\|_{F_0} \le R$*;*

(3) *the function* $F : [0,a] \times F_0 \to E$ *is such that* $F(\cdot, U)$ *is measurable for each* $U \in F_0$ *and* $F(t,U)$ *is continuous for almost all* $x \in [0,a]$. *Moreover, for each* $R > 0$ *there is a function* $\varphi_R \in L_p(0,a)$ *such that*

$$\left\|F(x,U) - F(x,\bar U)\right\|_E \le \varphi_R(x) \left\|U - \bar U\right\|_{F_0}$$

for almost all $x \in [0,a]$, $U, \bar U \in F_0$, *and* $\|U\| \le R$, $\left\|\bar U\right\| \le R$.

Then there is $a \in (0, a_0)$ *such that the problem* (2.3)–(2.4) *for* $|\arg\lambda| \le \pi - \varphi$, $0 < \varphi \le \pi$ *and sufficiently large* $|\lambda|$ *has a unique solution belonging to the space* $W_p^2(0,a; E(A), E)$.

Proof. We want to solve the problem (2.3)–(2.4) locally by means of maximal regularity of the linear problem (2.1)–(2.2) via the contraction mapping theorem. By virtue of Theorem 5.1, for $|\arg\lambda| \le \pi - \varphi$, $0 < \varphi \le \pi$ and sufficiently large $|\lambda|$, the linear BVP

(7.1) $(L+\lambda)w = -w''(x) + A_\lambda(0,H)w(x) = f(x), \quad x \in (0,a), \quad L_1 w = f_1, \quad L_2 w = f_2,$

where $H = (w(x_0), w'(x_0))$, $x_0 \in (0,a)$, and L_k is defined in (2.4), is maximal regular in X and satisfies the estimate

$$\|w\|_Y \le C \left(\|f\|_X + \sum_{k=1}^{2} \|f_k\|_{(E(A),E)_{\theta_k,p}} \right),$$

where C does not depend on $a \in (0, a_0]$. Let w be a solution of the BVP (7.1). Consider a ball

$$B_r = \left\{ v \in X : \ v - w \in Y_0, \ \|v - w\|_Y \le r \right\}.$$

Given $v \in B_r$ and $V = (v, v')$, consider the linear BVP

(7.2)
$$-u''(x)+A(0,H)u(x)+\lambda u(x)=F(x,V)+[A(0,H)-A(x,V)]\,v(x),\quad x\in(0,a),$$
$$L_1u=f_1,\quad L_2u=f_2.$$

Define a map Q on B_r by $Qv=u$, where u is a solution of the problem (7.2). We want to show that $Q(B_r)\subset B_r$ and that L is a contraction operator in Y, provided a is sufficiently small and r is chosen properly. To this end, by using maximal regularity of the problem (7.1), we obtain from (7.2)

$$\begin{aligned}\|Qv-w\|_Y &= \|u-w\|_Y\\ &\leq C_0\left\{\|F(x,V)-F(x,0)\|_X+\|[A(0,H)-A(x,V)]\,v\|_X\right\}\end{aligned}$$

Then, by using the assumption (2), we obtain

$$\begin{aligned}&\|[A(0,H)-A(x,V)]\,v\|_X\\ &\leq \sup_{x\in[0,a]}\left[\|A(0,H)-A(x,H)\|_{B(F_0,E)}+\|A(x,H)-A(x,V)\|_{B(F_0,E)}\right]\|v\|_Y\\ &\leq \left[K_b+L(R)\,\|V-H\|_{0,\infty}\right][\|v-w\|_Y+\|w\|_Y]\\ &\leq \left[K_b+L(R)\left(C_1\,\|v-w\|_Y+\|W-H\|_{0,\infty}\right)\right][\|v-w\|_Y+\|w\|_Y]\\ &\leq \left[K_b+L(R)\left(C_1r+\|W-H\|_{0,\infty}\right)\right][r+\|w\|_Y],\end{aligned}$$

where $K_b=\sup_{x\in[0,a]}\|A(0,H)-A(x,H)\|_{B(F_0,E)}$ and $R=C_1+\|W\|_{0,\infty}$ is a fixed number. By assumption (3), we similarly obtain

$$\begin{aligned}\|F(x,V)-F(x,0)\|_X &\leq \|F(x,V)-F(x,H)\|_X+\|F(x,H)-F(x,0)\|_X\\ &\leq \|\varphi\|_p\left[\|V-W\|_{0,\infty}+\|W\|_{0,\infty}\right]\\ &\leq C_1\,\|\varphi\|_p\,\|v-w\|_Y+\|W\|_{0,\infty}\\ &\leq \|\varphi\|_p\left[C_1r+\|W\|_{0,\infty}\right].\end{aligned}$$

Since $r\leq 1$, we have from the above estimates that

$$\|Qv-w\|_Y\leq C_0\left\{\left[K_b+L(R)\left(C_1r+\|W-H\|_{0,\infty}\right)\right][r+\|w\|_Y]\right\}$$
$$+C_0\,\|\varphi\|_{L_p}\left[C_1r+\|W\|_{0,\infty}\right].$$

In a similar way, for $v,\bar{v}\in B_r$, $\bar{V}=(\bar{v},\bar{v}')$, we obtain

$$\begin{aligned}\|Qv-Q\bar{v}\|_Y &\leq C_0\left\{\left\|F(x,V)-F(x,\bar{V})\right\|_X+\left\|\left[A(0,H)-A(x,\bar{V})\right](v-\bar{v})\right\|_X\right.\\ &\quad\left.+\left\|\left[A(x,V)-A(x,\bar{V})\right]\bar{v}\right\|_X\right\}\\ &\leq C_0\left[\|\varphi\|_p\left\|V-\bar{V}\right\|_{0,\infty}+\left(K_b+L(R)\,\|H-V\|_{0,\infty}\right)\|v-\bar{v}\|_Y\right.\end{aligned}$$

$$+L(R)\left\|H-\bar{V}\right\|_{0,\infty}\|\bar{v}\|_Y\Big]$$

$$\leq C_0\left[C_1\|\varphi\|_p+K_b+L(R)\left(\|H-W\|_{0,\infty}+C_1r\right)+L(R)C_1\left(r+\|w\|_Y\right)\right]\|v-\bar{v}\|_Y.$$

Now by suitably choosing r, $a\in(0,a_0)$, f_k, and by using Remark 7.1 and condition (C), we obtain

$$\|Qv-Q\bar{v}\|_Y\leq\delta\|v-\bar{v}\|_Y,\quad \text{where}\quad \delta<1,$$

i.e., Q is a contraction operator. If a is chosen so small, then we have

$$Q(B_r)\subset B_r.$$

The contraction mapping principle therefore implies the existence of a unique fixed point of Q in B_r, which is the unique strong solution $u\in Y_0$. □

8. Nonlocal BVPs

8.1. **Elliptic Equations on a Domain with Varying Bound.** The Fredholm property of BVPs for elliptic equations in smooth domains was studied in e.g., [2, 5]. In this section we use Theorem 5.1 and Theorem 6.1 in order to establish the maximal $L_{\mathbf{p}}$-regularity for the solution of nonlocal elliptic BVPs on a cylindrical region with varying bound. Let $G\subset\mathbb{R}^m$, $m\geq 2$, be a bounded domain with an $(m-1)$-dimensional boundary ∂G which locally admits rectification. In the domain $\Omega_t=[0,b(t)]\times G$, $t\in[c,d]$, we consider a nonlocal BVP

$$(8.1)\quad (L+\lambda)u:=-D_x^2u(x,y)-\sum_{k,j=1}^m a_{kj}(y)D_kD_ju(x,y)+a(x,y)D_xu(x,y)$$
$$+\sum_{j=1}^m a_j(x,y)D_ju(x,y)+a_0(x,y)u(x,y)+\lambda u(x,y)=f(x,y),$$

$$(8.2)\quad L_{tk}u:=\alpha_k\frac{\partial^{m_k}}{\partial x^{m_k}}u(0,y)+\beta_k\frac{\partial^{m_k}}{\partial x^{m_k}}u(1,y)+\sum_{j=1}^{N_k}\delta_{kj}\frac{\partial^{m_k}}{\partial x^{m_k}}u(x_{kj},y)$$
$$+\sum_{j=-1}^{M_k}T_{kj}u(x_{kj0},y)=f_k(y),\quad k\in\{1,2\},\quad y\in G,$$

$$(8.3)\qquad L_0u:=\sum_{j=1}^m c_j(y')\frac{\partial}{\partial y_j}u(x,y')+c_0(y')u(x,y')=0,\quad x\in(0,1),\quad y'\in\partial G,$$

where $D_x=\frac{\partial}{\partial x}$, $D_j=-i\frac{\partial}{\partial y_j}$, $D_y=(D_1,\ldots,D_m)$, $m_k\in\{0,1\}$, $\alpha_k,\beta_k,\delta_{kj}$ are complex-valued continuous functions of $t\in[c,d]$, $r=\operatorname{ord}L_0$, $y=(y_1,\ldots,y_m)$, $x_{kj}\in(0,1)$, $x_{kj0}\in[0,1]$, and $T_{kj}=T_{kj}(t)$ are, in general, unbounded operators in $L_p(G)$.

Theorem 8.1. *Let the following conditions be satisfied:*

(1) *$a_{kj} \in C(\bar{G})$, $a, a_j, a_0 \in C^1(\bar{G})$, $c_0 \in C(\bar{G})$, $\partial G \in C^2$;*
(2) *$\sum_{j=1}^{m} c_j(y')\sigma_j \neq 0$, $y' \in \partial G$, $\sigma \in \mathbb{R}^m$ is normal to ∂G, $c_j, c_0 \in C^1(\bar{G})$ for $r = 1$ and $c_0(y') \neq 0$, $y' \in \partial G$ for $r = 0$;*
(3) *for $y \in G$, $\sigma \in \mathbb{R}^m$, $\arg\lambda = \pi$, $|\sigma| + |\lambda| \neq 0$, $\lambda + \sum_{k,j=1}^{m} a_{kj}(y)\sigma_k\sigma_j \neq 0$;*
(4) *for the tangent vector σ' and a normal vector σ to ∂G at the point $y' \in \partial G$, the boundary value problem*

$$\left[\lambda + \sum_{k,j=1}^{m} a_{kj}(y')\left(\sigma_k' - i\sigma_j \frac{d}{d\xi}\right)\left(\sigma_k' - i\sigma_j \frac{d}{d\xi}\right)\right] u(\xi) = 0, \quad \xi > 0, \quad \lambda \leq 0,$$

$$\sum_{j=1}^{m} c_j(y')\left(\sigma_k' - i\sigma_j \frac{d}{d\xi}\right) u(\xi)\Bigg|_{\xi=0} = h$$

(and for $r = 0$, the problem generated by the same equations with $u(0) = h$) has one and only one solution which, including all of its derivatives, tends to zero as $\xi \to \infty$ for any numbers $h \in \mathbb{C}$;
(5) *$(-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \neq 0$ for $t \in [c, d]$;*
(6) *if $m_k = 0$, then $T_{kj} = 0$; if $m_k = 1$, then for any $\varepsilon > 0$ there is $c(\varepsilon)$ such that for all $u \in B_{\mathbf{p}}^{2-\frac{1}{p_1}}(G; L_0 u = 0, r < 2 - m_k - 2/p_2)$,*

$$\|T_{kj}u\|_{B_{\mathbf{p}}^{1-\frac{1}{p_1}}(G)} \leq \varepsilon \|u\|_{B_{\mathbf{p}}^{2-\frac{1}{p_1}}(G)} + c(\varepsilon)\|u\|_{L_{p_1}(G)},$$

where $p_2 \in (1, \infty) \setminus \{2\}$, or $p_2 = 2$ and $r \neq 1$.

Then, for $f \in L_{\mathbf{p}}(\Omega_t)$, $\mathbf{p} = (p_1, p_2)$, $0 < p_1, p_2 < \infty$, $f_k \in B_{\mathbf{p}}^{2-m_k-\frac{1}{p_1}}(G)$ and $|\arg\lambda| = \pi$ and sufficiently large $|\lambda|$, the problem (8.1)–(8.3) *has a unique solution that belongs to the space $W_{\mathbf{p}}^2(\Omega_t)$, and coercive uniformity with respect to the parameters t and λ holds, i.e.,*

$$\|u\|_{W_{\mathbf{p}}^2(\Omega_t)} \leq C\left[\|(L+\lambda)u\|_{L_{\mathbf{p}}(\Omega_t)} + \sum_{k=1}^{2} \|L_k u\|_{B_{\mathbf{p}}^{2-m_k-\frac{1}{p_1}}(G)} + \|u\|_{L_{\mathbf{p}}(\Omega_t)}\right]$$

is satisfied for the solution of the problem (8.1)–(8.3)*. Moreover, the operator*

$$u \to Q(t)u = \{Lu,\ L_{t1}u,\ L_{t2}u\}$$

from $W_{\mathbf{p}}^2(\Omega_t; L_0 u = 0)$ into

$$L_{\mathbf{p}}(\Omega_t) \times \prod_{k=1}^{2} B_{\mathbf{p}}^{2-m_k-\frac{1}{p_1}}(G, L_0 u = 0, r < 2 - m_k - 2/p_2)$$

is bounded and Fredholm.

Proof. Let $E = L_{p_1}(G)$. Consider the operator A defined by

$$D(A) = W_{p_1}^2(G; L_0 u = 0), \quad Au = -\sum_{k,j=1}^{m} a_{kj}(y) D_k D_j u(x, y).$$

For $x \in [0,b]$, also consider the operators

$$B_1(x)u = a(x,y)u(x,y), \quad B_2(x)u = \sum_{j=1}^{m} a_j(x,y)D_j u(x,y) + a_0(x,y)u(x,y).$$

Let us apply Theorem 6.1 to the problem (8.1)–(8.3). Using [5, Theorem 8.2], the operator $A+\mu I$ is R-positive in L_{p_1} for sufficiently large $\mu \geq 0$. Moreover, it is known that the embedding $W^2_{p_1}(G) \subset L_{p_1}(G)$ is compact (see, e.g., Triebel [10, Theorem 3.2.5]). Then due to the positivity of $A+\mu I$ in $L_{p_1}(G)$, we obtain that $(A+\mu I)^{-1} \in \sigma_\infty(L_{p_1}(G))$. Therefore, condition (1) of Theorem 6.1 is fulfilled. Condition (2) of Theorem 6.1 coincides with condition (5). By virtue of condition (1) of Theorem 8.1, the operators $B_1(x)$ in $L_{p_1}(G)$ and $B_2(x)$ from $W^1_{p_1}(G)$ to $L_{p_1}(G)$ are bounded. Using [10, §4.3.3] and in view of the embedding between Sobolev and Besov spaces [4, §18, Theorem 18.9], we have

$$(E(A),E)_{\frac{1}{2},1} = \left(W^2_{p_1}(G,L_0), L_{p_1}(G)\right)_{\frac{1}{2},1} = B^1_{p_1,1}(G,L_0,r=0) \subset W^1_{p_1}(G).$$

Therefore, the operator $B_1(x)$ from $W^1_{p_1}(G)$ into $L_{p_1}(G)$ and, consequently, from $(E(A),E)_{\frac{1}{2},1}$ into $L_{p_1}(G)$, is compact. Then we obtain that the operator $B_1(x)$ satisfies condition (3) of Theorem 6.1. In a similar way, we prove that the operator $B_2(x)$ satisfies condition (3) of Theorem 6.1, too. Moreover, using interpolation properties of Sobolev spaces (see, e.g., [10, §4]), it is easy to see that condition (4) of Theorem 6.1 is also fulfilled. ☐

8.2. **Infinite Systems of Nonlinear Differential Equations.** Consider a BVP for an infinite system

$$-u_m''(x) + \left[A_m(x,u,u') + \lambda\right] u_m(x) = F_m(x,u,u'), \quad x \in (0,a), \quad m \in \mathbb{N}, \tag{8.4}$$

$$\alpha_k D^{m_k} u_m(0) + \beta_k D^{m_k} u_m(a) + \sum_{i=1}^{N_k} \delta_{ki} D^{m_k} u_m(\delta_{ki}) = f_{km}, \quad k \in \{1,2\}, \tag{8.5}$$

where $u = \{u_m\}$ and $u' = \{u_m'\}$. Let

$$A(x,u,u') = \{A_m(x,u,u')\}, \quad A = A(0,0,0), \quad u = \{u_m\} \in \ell_q,$$

$$D(A) = \ell_q^s := \left\{ u \in \ell_q : \|u\|_{l_q^s} := \left(\sum_{m=1}^{\infty} |2^{ms} u_m|^q \right)^{\frac{1}{q}} < \infty \right\}, \quad s>0, \quad q \in (1,\infty),$$

$$Au = \sum_{m=1}^{\infty} 2^{ms} u_m, \quad \eta_k = \frac{k+1/p}{2}, \quad F_0 = \prod_{k=0}^{1} \ell_p^{s(1-\eta_k)}.$$

From Theorem 7.2 by using [10, §1.18.2], we obtain the following result.

Theorem 8.2. *Let the following conditions be satisfied:*

(1) $(-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \neq 0$;

(2) *the functions* $A_m(x,U)$ *and* $F_m(x,U)$ *are continuously differentiable on* $[0,a_0]$ *for* $U \in F_0$.

Then there is $a \in (0,a_0)$ *such that the problem* (8.4)–(8.5) *for* $|\arg\lambda| \le \pi-\varphi$, $0<\varphi\le\pi$, *and sufficiently large* $|\lambda|$ *has a unique solution belonging to the space* $W_p^2(0,a;\ell_q^s,\ell_q)$.

Remark 8.3. It should be noted that the assertion of Theorem 8.2 is valid for all finite systems satisfying the conditions of Theorem 8.2.

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