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Linear Hamiltonian dynamic systems on time scales: Sturmian property of the principal solution

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Abstract

Basic results of the oscillation and transformation theories of linear Hamiltonian dynamic systems on time scales are presented. It is shown that these results incorporate, as particular cases, oscillation and transformation theories of linear Hamiltonian differential systems and symplectic difference systems. In the last part of the paper a Sturmian-type property of the principal solution of linear Hamiltonian dynamic systems is established.

Key words: linear Hamiltonian dynamic system, time scale, roundabout theorem, principal solution

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1 Introduction

The aim of this paper is to present basic results of oscillation and transformation theories of linear Hamiltonian dynamic systems on time scales and to establish a Sturmian property of principal solutions of these systems. Recall that a *time scale* $\mathbb{T}$ is any closed subset of the reals $\mathbb{R}$. An alternative terminology for this object is *measure chain* which is sometimes used in a more general context. The time scale calculus was established in [7], developed in the subsequent papers [8,9], and incorporates as particular cases the differential calculus if $\mathbb{T} = \mathbb{R}$ and the difference calculus if $\mathbb{T} = \mathbb{Z}$. Basic ideas of this calculus are given in the next section. At this place we recall only that if $f : \mathbb{T} \to \mathbb{X}$, where $\mathbb{X}$ is a Banach space, one can define the so-called *generalized derivative* (sometimes also called *delta derivative*) which reduces to the usual derivative f' if $\mathbb{T} = \mathbb{R}$ and to the forward difference Δf if $\mathbb{T} = \mathbb{Z}$.

A linear Hamiltonian dynamic system on a time scale $\mathbb{T}$ is the first order linear dynamic system

$$z^{\Delta} = \mathcal{H}(t)z, \tag{1}$$

where $\mathcal{H} : \mathbb{T} \to \mathbb{R}^{2n\times 2n}$ is a matrix of rd-continuous functions satisfying the identity

$$\mathcal{H}^T(t)\mathcal{J} + \mathcal{J}\mathcal{H}(t) + \mu(t)\mathcal{H}^T(t)\mathcal{J}\mathcal{H}(t) = 0, \tag{2}$$

μ being the *graininess* of the time scale and $\mathcal{J} = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. System (1) with the matrix $\mathcal{H}$ satisfying (2) was introduced and studied in [6,11] under the name *symplectic dynamic system.* The concept of a *Hamiltonian dynamic system* was introduced by Ahlbrandt *et al.* [1]. Here we adopt the latter terminology (Hamiltonian system) since it perhaps better characterize the essence of the problem. Note also that if $\mathbb{T} = \mathbb{R}$ then (1) reduces to the classical linear Hamiltonian differential system and to the symplectic difference system if $\mathbb{T} = \mathbb{Z}$.

The paper is organized as follows. In the next section we recall basic facts of the time scale calculus and oscillation theory of Hamiltonian dynamic systems. These systems are studied in more detail in Section 3, in particular, we formulate the so-called Roundabout theorem for Hamiltonian dynamic systems and recall also the concept of the principal solution and its properties as introduced in [5]. In the last section we establish a Sturmian-type property of principal solutions.

2 Auxiliary results

We start with basic facts of the time scale calculus, see [7], unifying the differential and difference calculus. A *time scale* $\mathbb{T}$ is any closed subset of the set of real numbers $\mathbb{R}$. On any time scale $\mathbb{T}$ we define the following operators and concepts:

$$\sigma(t) := \inf\{s \in \mathbb{T},\ s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T},\ s < t\}$$

are the *forward* and *backward jump operators*. A point $t \in \mathbb{T}$ is said to be *left-dense* (l-d) if $\rho(t) = t$, *right-dense* (r-d) if $\sigma(t) = t$, *left-scattered* (l-s) if $\rho(t) < t$, *right-scattered* (r-s) if $\sigma(t) > t$ and it is said to be *dense* if it is r-d or l-d. The *graininess* μ of a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$. For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f may be actually replaced by any Banach space) the *generalized derivative* is defined by

$$f^\Delta(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}, \quad \text{where } s \in \mathbb{T}\backslash\{\sigma(t)\}.$$

If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$, $\mu(t) = 0$ and $f^\Delta = f'$ is the usual derivative. In case $\mathbb{T} = \mathbb{Z}$, we have $\sigma(t) = t + 1$, $\mu(t) = 1$ and $f^\Delta = \Delta f$ is the forward difference operator. We will write f^σ for the composite function $f \circ \sigma$.

Directly one can verify the following basic rules of the differential calculus on time scales

$$\begin{gathered}
[f(t) \pm g(t)]^\Delta = f^\Delta(t) \pm g^\Delta(t), \quad f^\sigma(t) = f(t) + \mu(t) f^\Delta(t), \\
[f(t)g(t)]^\Delta = f^\Delta(t) g^\sigma(t) + f(t) g^\Delta(t) = f^\Delta(t) g(t) + f^\sigma(t) g^\Delta(t), \\
\left\{\frac{f(t)}{g(t)}\right\}^\Delta = \frac{f^\Delta(t) g(t) - f(t) g^\Delta(t)}{g(t) g^\sigma(t)}
\end{gathered}$$

For the investigation of solvability of *dynamic equations* on time scales (dynamic equation means an equation involving an unknown function together with its generalized derivatives) we need also the following concepts. Here the usual notation for an interval $[a, b]$ actually means the set $\{t \in \mathbb{T},\ t \in [a, b]\}$, open and half open intervals are defined in the same way, $[a, b]^\kappa = [a, b]$ if b is l-d and $[a, b]^\kappa = [a, b)$ if b is l-s. A function $f : [a, b] \to \mathbb{R}$ is said to be *rd-continuous* if it is continuous at each r-d point and there exists a finite left hand limit at all l-d points, and this function is said to be *rd-continuously differentiable* if its generalized derivative exists and is rd-continuous, these classes of functions will be denoted by C_{rd} and C^1_{rd}, respectively. For every rd-continuous function f there exists its *generalized antiderivative* - a function F such that $F^\Delta = f$. Using this antiderivative we define the integral $\int_a^b f(t)\,\Delta t := F(b) - F(a)$. A function f is said to be *regressive* if $1 + \mu(t)f(t) \neq 0$ (the mapping $\mathrm{id} + \mu(t)f(t)$ is invertible if the range of f is a Banach space). The initial value problem for the linear dynamic equation

$$z^\Delta = g(t)z, \quad z(t_0) = z_0$$

with a regressive and rd-continuous function g has a unique solution [7, Theorem 5.5].

Now we turn our attention to linear Hamiltonian dynamic systems. First observe that (2) implies $(I+\mu\mathcal{H})^T\mathcal{J}(I+\mu\mathcal{H}) = \mathcal{J}$, i.e., the matrix valued function $I + \mu\mathcal{H}$ is symplectic, i.e. nonsingular, and hence $\mathcal{H}$ is regressive. This means that if the matrix $\mathcal{H}$ consists of rd-continuous functions, the initial condition $z(t_0) = z_0$ determines a unique solution of (1). Let us write the matrix $\mathcal{H}$ in the form

$$\mathcal{H} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}. \tag{3}$$

If $\mathbb{T} = \mathbb{R}$, i.e. $\mu(t) \equiv 0$, then (2) reduces to the identity $\mathcal{H}^T(t)\mathcal{J} + \mathcal{J}\mathcal{H}(t) = 0$ which implies that the matrices $\mathcal{B}, \mathcal{C}$ are symmetric and $\mathcal{D} = -\mathcal{A}^T$. This means that (1) is the classical linear Hamiltonian differential system

$$x' = \mathcal{A}(t)x + \mathcal{B}(t)u, \quad u' = \mathcal{C}(t)x - \mathcal{A}^T(t)u, \tag{4}$$

whose oscillation theory is deeply developed, see e.g. [12–14] and the references given therein. If $\mathbb{T} = \mathbb{Z}$, then (1) takes the form

$$z_{k+1} = (I + \mathcal{H}_k)z_k$$

and (2) (with $\mu \equiv 1$) implies that (1) complies with the so-called symplectic difference system whose oscillatory properties are studied in [2,3].

The identity (2) is translated in terms of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ as

$$\begin{aligned} \mathcal{C} - \mathcal{C}^T + \mu(\mathcal{A}^T\mathcal{C} - \mathcal{C}^T\mathcal{A}) &= 0, \\ \mathcal{B}^T - \mathcal{B} + \mu(\mathcal{B}^T\mathcal{D} - \mathcal{D}^T\mathcal{B}) &= 0, \\ \mathcal{A}^T + \mathcal{D} + \mu(\mathcal{A}^T\mathcal{D} - \mathcal{C}^T\mathcal{B}) &= 0. \end{aligned} \tag{5}$$

In particular, we have $z^\sigma = (I + \mu(t)\mathcal{H}(t))z$. Since the matrix

$$I + \mu\mathcal{H} = \begin{pmatrix} I + \mu\mathcal{A} & \mu\mathcal{B} \\ \mu\mathcal{C} & I + \mu\mathcal{D} \end{pmatrix}$$

is symplectic (compare (5)), we have

$$\begin{pmatrix} I+\mu\mathcal{A} & \mu\mathcal{B} \\ \mu\mathcal{C} & I+\mu\mathcal{D} \end{pmatrix}^{-1} = \begin{pmatrix} I+\mu\mathcal{D}^T & -\mu\mathcal{B}^T \\ -\mu\mathcal{C}^T & I+\mu\mathcal{A}^T \end{pmatrix}.$$

Consequently, if we write $z = \binom{x}{u}$ with n-dimensional vectors x, u, we have

$$\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} I+\mu\mathcal{D}^T & -\mu\mathcal{B}^T \\ -\mu\mathcal{C}^T & I+\mu\mathcal{A}^T \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}^{\sigma}. \tag{6}$$

Let $Z = \binom{X}{U}$, $\tilde{Z} = \binom{\tilde{X}}{\tilde{U}}$ be $2n \times n$-matrix valued solutions of (1). By a direct computation (using (2)) one can easily verify that $(Z^T\mathcal{J}\tilde{Z})^\Delta = 0$, i.e. $Z^T(t)\mathcal{J}\tilde{Z}(t) \equiv M$, where M is a constant matrix. If $Z = \tilde{Z}$, $M = 0$ and rank $\binom{X}{U} = n$, the solution $\binom{X}{U}$ is called a *conjoined basis* of (1). Oscillatory properties of (1) are defined using the concept of a *focal point* introduced in [6,10]. A conjoined basis $Z = \binom{X}{U}$ has no *focal point* in an interval $(a, b]$ if $X(t)$ is nonsingular at all dense points $t \in (a, b]$ and both

$$\operatorname{Ker} X^\sigma(t) \subseteq \operatorname{Ker} X(t) \quad \text{and} \quad D(t) := X(t)(X^\sigma(t))^\dagger \mathcal{B}(t) \geq 0 \tag{7}$$

hold for $t \in (a, b]^\kappa$. Here Ker, † and $\geq$ stand for the kernel, Moore-Penrose generalized inverse and nonnegative definiteness of a matrix indicated, respectively. Note that if the previous kernel condition is satisfied then the matrix $D(t)$ is really symmetric as is shown in [6]. System (1) is said to be *disconjugate* on the interval $[a, b]$ if the conjoined basis $Z = \binom{X}{U}$ given by the initial condition $X(a) = 0, U(a) = I$ has no focal point in $(a, b]$. If a time scale $\mathbb{T}$ is not bounded above, system (1) is said to be nonoscillatory if there exists $T \in \mathbb{T}$ such that it is disconjugate on $[T, \infty)$, in the opposite case it is said to be *oscillatory*. Let us compare the above definition of the focal point with this definition in the continuous case $\mathbb{T} = \mathbb{R}$. There, the focal points are just singularities of the first component X of a conjoined basis $\binom{X}{U}$. The kernel condition in (7) is automatically satisfied since $X(t) = X^\sigma(t)$ in the continuous case. Concerning the "$D(t)$-condition", this condition reduces to $\mathcal{B}(t) \geq 0$ if $X(t)$ is nonsingular. However, this condition is known as the Legendre condition in the theory of linear Hamiltonian system (4), and the classical Sturmian theory extends to (4) only under this assumption, see [14]. Hence Legendre condition is *apriori* supposed in oscillation theory of (4) and the "time scale definition" of the focal point actually reduces to the usual definition if $\mathbb{T} = \mathbb{R}$. In the discrete case $\mathbb{T} = \mathbb{Z}$ there are no dense points, so X is allowed to be singular and (7) reduces to the concept of *no focal point* in $(t, t+1]$ introduced in [3].

Consider the transformation of (1)

$$z = \mathcal{R}(t)w, \tag{8}$$

where $\mathcal{R}$ is a $2n \times 2n$ symplectic matrix of C^1_{rd} functions. This transformation transforms (1) into the system

$$w^\Delta = \tilde{\mathcal{H}}(t)w, \quad \tilde{\mathcal{H}} = (R^\sigma)^{-1}\left[-\mathcal{R}^\Delta + \mathcal{H}\mathcal{R}\right] \tag{9}$$

and symplecticity of $\mathcal{R}$ implies that this system also has the Hamiltonian structure, i.e. $\tilde{\mathcal{H}}^T\mathcal{J} + \mathcal{J}\tilde{\mathcal{H}} + \mu\tilde{\mathcal{H}}^T\mathcal{J}\tilde{\mathcal{H}} = 0$. Moreover, if

$$\mathcal{R}(t) = \begin{pmatrix} H(t) & 0 \\ K(t) & (H^T(t))^{-1} \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} y \\ v \end{pmatrix},$$

where H, K are C^1_{rd} $n \times n$ matrix functions such that H is nonsingular and H^TK is symmetric, then (8) preserves oscillatory nature of transformed systems. Let us write the matrix $\tilde{\mathcal{H}}$ in the form

$$\tilde{\mathcal{H}}(t) = \begin{pmatrix} \tilde{\mathcal{A}}(t) & \tilde{\mathcal{B}}(t) \\ \tilde{\mathcal{C}}(t) & \tilde{\mathcal{D}}(t) \end{pmatrix}$$

with

$$\begin{aligned}
\tilde{\mathcal{A}} &= -(H^\sigma)^{-1}(H^\Delta - \mathcal{A}H - \mathcal{B}K), \\
\tilde{\mathcal{B}} &= (H^\sigma)^{-1}\mathcal{B}(H^T)^{-1}, \\
\tilde{\mathcal{C}} &= (K^\sigma)^T(H^\Delta - \mathcal{A}H - \mathcal{B}K) - (H^\sigma)^T(K^\Delta - \mathcal{C}H - \mathcal{D}K), \\
\tilde{\mathcal{D}} &= (H^\Delta + \mathcal{D}^TH^\sigma - \mathcal{B}^TK^\sigma)^T(H^T)^{-1}.
\end{aligned}$$

In particular, if $\binom{X}{U}$ is a conjoined basis of (1) such that $X(t)$ is nonsingular, then setting $H = X$, $K = U$ shows that we have $\tilde{\mathcal{A}} = 0$, $\tilde{\mathcal{B}} = (X^\sigma)^{-1}\mathcal{B}(X^T)^{-1}$, $\tilde{\mathcal{C}} = 0$ in (9). More precisely,

$$\begin{aligned}
\bar{X}(t) &= X(t)\int_{t_1}^{t}(X^\sigma(s))^{-1}\mathcal{B}(s)(X^T(s))^{-1}\,\Delta s, \\
\bar{U}(t) &= U(t)\int_{t_1}^{t}(X^\sigma(s))^{-1}\mathcal{B}(s)(X^T(s))^{-1}\,\Delta s + (X^T(t))^{-1}
\end{aligned}$$

is a conjoined basis of (1) for which $X^T\bar{U} - U^T\bar{X} = -I$.

We finish this section with the definition of the principal solution of (1). First recall some necessary concepts [5, Section 2], [6, Definition 6]. System (1) is said to be *dense normal* on an interval $[a, b]$ if for any dense point $s \in (a, b]$

the trivial solution $\binom{x}{u} \equiv \binom{0}{0}$ is the only solution of (1) for which $x(t) \equiv 0$ on $[a, s]$. System (1) is said to be *eventually dense normal* if there exists $T \in \mathbb{T}$ and $\ell \in \mathbb{N}$ such that this system is dense normal on $[T, s]$ for every dense $s > T$ and, if there is no dense point in (T, ∞), then for any $t \geq T$ $x^{\sigma^k}(t) = 0$, $k = 0, \ldots, \ell$, implies $\binom{x}{u} \equiv \binom{0}{0}$ on (t, ∞). Here $\sigma^k = \underbrace{\sigma \circ \cdots \circ \sigma}_{k-\text{times}}$, $\sigma^0(t) = t$. Observe that in the continuous case $\mathbb{T} = \mathbb{R}$ the dense normality is just usual controllability condition (sometimes also called identical normality), see [12–14].

A conjoined basis $\binom{\tilde{X}}{\tilde{U}}$ of (1) is said to be a *principal solution* of system (1) if $\tilde{X}(t)$ is nonsingular,

$$(\tilde{X}^\sigma(t))^{-1}\mathcal{B}(t)(\tilde{X}^T(t))^{-1} \geq 0,$$

both for large t, and

$$\lim_{t\to\infty} X^{-1}(t)\tilde{X}(t) = 0 \tag{10}$$

for any conjoined basis $\binom{X}{U}$ for which the (constant) matrix

$$L := X^T\tilde{U} - U^T\tilde{X} \quad \text{is nonsingular.} \tag{11}$$

Any conjoined basis $\binom{X}{U}$ for which (10) and (11) hold is said to be a *nonprincipal* solution. Note that in the theory of difference equations the concepts *recessive* and *dominant* solution is used instead of the concepts principal and nonprincipal solution, respectively.

3 Roundabout theorem and principal solution

In this section we present basic oscillatory properties of (1) and of its principal solution. As we have mentioned in the previous section, oscillatory properties of (1) are defined using the concept of the focal point of a conjoined basis. These properties can be equivalently characterized via the concept of a *generalized zero* of a vector solution of (1) which is defined as follows [11, Definition 1]. A point $t \in \mathbb{T}$ is a zero of a solution $z = \binom{x}{u}$ if t is dense and $x(t) = 0$. A point $\sigma(t)$ is a *σ-zero* of z if t is r-s, $x(t) \neq 0$, $x^\sigma(t) \in \operatorname{Im} \mu(t)\mathcal{B}(t)$ and $x^T(t)\mathcal{B}^\dagger(t)x^\sigma(t) = 0$. A point $t \in \mathcal{I}$ is a *node* of z if t is r-d, $x(t) \neq 0$ and $\mathcal{B}(t) \not\geq 0$. A point $\frac{t+\sigma(t)}{2} \in \mathcal{I}$ is a *σ-node* of z if t is r-s, $(x(t) \neq 0)$, $x^\sigma(t) \in \operatorname{Im} \mu(t)\mathcal{B}(t)$ and $x^T(t)\mathcal{B}^\dagger(t)x^\sigma(t) < 0$. A *generalized zero* of z is defined as its zero, or its σ-zero, or its node, or its σ-node.

Basic oscillatory properties of linear Hamiltonian dynamic systems are summarized in the next theorem, which is (based on the terminology introduced by Reid [13,14] for linear Hamiltonian differential systems (4)) usually referred as *Roundabout theorem.* This statement, proven in [11], shows that the classical Sturmian theory extends to (1) and that certain quadratic functional and Riccati-type matrix dynamic equation play a crucial role in the characterization of disconjugacy of (1).

Theorem 1. *Suppose* (1) *is dense-normal on* $\mathcal{I} := [a, b]$. *Then the following statements are equivalent:*

(i) *System* (1) *is disconjugate on* $\mathcal{I}$;
(ii) *The quadratic functional*

$$\mathcal{F}_0(z) \equiv \int_a^b z^T(t)\left\{\mathcal{H}^T(t)\mathcal{K} + \mathcal{K}\mathcal{H}(t) + \mu(t)\mathcal{H}^T(t)\mathcal{K}\mathcal{H}(t)\right\} z(t)\,\Delta t > 0,$$

where $\mathcal{K} := \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$, *for every* $z : \mathcal{I} \to \mathbb{R}^{2n}$ *such that* $\mathcal{K}z \in C^1_{rd}(\mathcal{I})$, $\mathcal{K}^T z \in C_{rd}(\mathcal{I})$, $\mathcal{K}z(a) = 0 = \mathcal{K}(b)$, *satisfying* $\mathcal{K}z^\Delta(t) = \mathcal{K}\mathcal{H}(t)z(t)$ *on* $\mathcal{I}^\kappa$;
(iii) *No solution* $z = \binom{x}{u}$ *of* (1) *with* $x(a) = 0$ *has any generalized zero in* $(a, b]$;
(iv) *There exists a symmetric solution* Q *on* $\mathcal{I}$ *of the Riccati matrix equation*

$$R[Q] := -Q^\Delta + \mathcal{C}(t) + \mathcal{D}(t)Q - Q^\sigma(\mathcal{A}(t) + \mathcal{B}(t)Q) = 0 \tag{12}$$

with $I + \mu(\mathcal{A} + \mathcal{B}Q)$ *nonsingular and* $[I + \mu(\mathcal{A} + \mathcal{B}Q)]^{-1}\mathcal{B} \geq 0$ *on* $\mathcal{I}^\kappa$;
(v) *There exists a conjoined basis* $Z = \binom{X}{U}$ *of* (1) *with no focal point in* $(a, b]$ *such that* X *is invertible on* $\mathcal{I}$.

Observe that the hypothesis of dense-normality is missing in the discrete case ($\mathbb{T} = \mathbb{Z}$) and that the Legendre condition is contained in the item (i). Thus, no apriori assumption like "$\mathcal{B}(t) \geq 0$" is needed here.

We finish this section with basic facts concerning principal solutions of (1). The statements given below were proved in [5].

Theorem 2. *Suppose that* (1) *is nonoscillatory and eventually dense normal. Then this system possesses a principal solution* $\tilde{Z} = \binom{\tilde{X}}{\tilde{U}}$. *This solution is determined uniquely in the following sense. If* $\hat{Z} = \binom{\hat{X}}{\hat{U}}$ *is another principal solution, then there exists a nonsingular* $n \times n$ *matrix* M *such that* $\binom{\tilde{X}}{\tilde{U}} = \binom{\hat{X}}{\hat{U}} M$.

In addition to the "minimality" of the first component of the principal solution with respect to first components of other conjoined bases of (1) involved already in the definition of a principal solution, this solution has another "minimality property".

Theorem 3. *Suppose that* (1) *is nonoscillatory and eventually dense normal. Let* $\binom{\tilde{X}}{\tilde{U}}$ *be its principal solution. Then the solution* $\tilde{Q} = \tilde{U}\tilde{X}^{-1}$ *of the associated Riccati equation* (12) *is eventually minimal in the sense that if* Q *is any solution of this equation which exists on some interval* $[a,\infty)$ *and* $[I + \mu(\mathcal{A} + \mathcal{B}Q)^{-1}]\mathcal{B} \geq 0$ *in this interval, then* $Q(t) - \tilde{Q}(t) \geq 0$ *for* $t \in [a,\infty)$.

4 A Sturmian property of the principal solution

In this section we present a Sturmian-type property of the principal solution of (1). This statement can be explained as follows. Consider the linear Hamiltonian differential system (4) with $\mathcal{B}(t) \geq 0$, which is a special case of (1). Let $\binom{X_b}{U_b}$ be the conjoined basis of this system given by the condition $X_b(b) = 0$, $U_b(b) = I$, and suppose that $X_b(t)$ is nonsingular in some left neighbourhood of b. Further suppose that there exists $a < b$ such that $X_b(a)$ is singular. Now, if $\binom{\hat{X}}{\hat{U}}$ is any other conjoined basis of (4), then by the classical Sturmian theorem there exists $t \in [a,b)$ such that $X(t)$ is singular. A similar situation we have if $b = \infty$ and the solution $\binom{X_b}{U_b}$ is replaced by the principal solution $\binom{\tilde{X}}{\tilde{U}}$; if $a \in \mathbb{R}$ is the largest singularity of $\tilde{X}$, then the first component $\hat{X}$ of any other conjoined basis $\binom{\hat{X}}{\hat{U}}$ is singular at some point of the interval $[a,\infty)$, see e.g. [14]. In [4] it was proved that recessive solutions of the linear Hamiltonian difference system

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A_k^T u_k,$$

have the same Sturmian-type property. The next theorem unifies these statements in the scope of the oscillation theory of Hamiltonian dynamic systems on time scales.

Theorem 4. *Suppose that* (1) *is nonoscillatory and eventually dense normal. Let* $\tilde{Z} = \binom{\tilde{X}}{\tilde{U}}$ *be a principal solution of* (1) *such that* $\tilde{X}(t)$ *is invertible and* $\tilde{D}(t) := \tilde{X}(t)(\tilde{X}^\sigma(t))^{-1}\mathcal{B}(t) \geq 0$ *for* $t \in (a,\infty)$. *If* $\tilde{X}(a)$ *is singular or* $\tilde{D}(a) \not\geq 0$, *then for any conjoined basis* $Z = \binom{X}{U}$ *of* (1) *either* $X(t)$ *is singular or* $D(t) := X(t)(X^\sigma(t))^{-1}\mathcal{B}(t) \not\geq 0$ *for some* $t \geq a$.

PROOF. Suppose, by contradiction, that there exists a conjoined basis $Z = \binom{X}{U}$ for which $X(t)$ is nonsingular and $D(t) \geq 0$ for $t \geq a$. If a is r-d, i.e. $\mu(a) = 0$, then $0 \leq D(a) = X(a)X^{-1}(a)\mathcal{B}(a) = \mathcal{B}(a)$ and if $\tilde{X}(a)$ is nonsingular, we have $\tilde{D}(a) = \tilde{X}(a)\tilde{X}^{-1}(a)\mathcal{B}(a) = \mathcal{B}(a) = \mathcal{B}(a) \geq 0$, a contradiction with our assumptions. Hence $\tilde{X}(a)$ cannot be nonsingular. If $\mu(a) > 0$, then using (6) for $t = a$ we have

$$\begin{aligned} X(X^\sigma)^{-1}\mathcal{B} &= X[(I + \mu\mathcal{D}^T)X^\sigma - \mu\mathcal{B}^T U^\sigma](X^\sigma)^{-1}\mathcal{B} \\ &= (I + \mu\mathcal{D}^T)\mathcal{B} - \mu\mathcal{B}^T Q^\sigma \mathcal{B}. \end{aligned}$$

Consequently,

$$\begin{aligned} \tilde{D}(a) - D(a) &= \tilde{X}(a)(\tilde{X}^\sigma(a))^{-1}\mathcal{B}(a) - X(a)(X^\sigma(a))^{-1}\mathcal{B}(a) \\ &= \mu(a)\mathcal{B}^T(a)[Q^\sigma(a) - \tilde{Q}^\sigma(a)]\mathcal{B}(a) \geq 0, \end{aligned}$$

a contradiction, i.e., $\tilde{X}(a)$ cannot be nonsingular also in this case.

Denote

$$\mathcal{G}(t;X) := \int_a^t (X^\sigma(s))^{-1}\mathcal{B}(s)\left(X^T(s)\right)^{-1} \Delta s$$

and let

$$\bar{X}(t) = X(t)[I + \mathcal{G}(t;X)], \quad \bar{U}(t) = U(t)[I + \mathcal{G}(t;X)] + (X^T(t))^{-1}.$$

Then $\binom{\bar{X}}{\bar{U}}$ is a conjoined basis of (1) for which $\bar{X}^T U - \bar{U}^T X = -I$. Since $\mathcal{G}(t;X)$ is nonnegative definite, $\bar{X}$ is nonsingular for $t \geq a$. Any conjoined basis of (1) is of the form

$$\begin{pmatrix} \bar{X}[M + \mathcal{G}(t;\bar{X})N] \\ \bar{U}[M + \mathcal{G}(t;\bar{X})N] + (\bar{X}^T)^{-1}N \end{pmatrix}, \tag{13}$$

where M, N are constant $n \times n$ matrices such that $M^T N$ is symmetric and

$$\mathcal{G}(t;\bar{X}) = \int_a^t (\bar{X}^\sigma(s))^{-1}\mathcal{B}(s)(\bar{X}^T(s))^{-1} \Delta s.$$

In particular, the solution $\binom{X}{U}$ is also of this form and substituting $t = a$ into (13) we get $M = I, N = -I$, hence

$$X = \bar{X}[I - \mathcal{G}(t;\bar{X})], \quad U = \bar{U}[I - \mathcal{G}(t;\bar{X})] - (\bar{X}^T)^{-1} \tag{14}$$

The first equalities in (13) and (14) imply that

$$I = [I - \mathcal{G}(t;\bar{X})][I + \mathcal{G}(t;X)].$$

Since the second factor in the last equality is a nondecreasing matrix-valued function, the first factor is nonincreasing and $0 \leq \mathcal{G}(t;\bar{X}) < I$, hence there exists a nonnegative definite matrix limit $\mathcal{G}_\infty = \lim_{t\to\infty} \mathcal{G}(t;\bar{X})$. Moreover,

$\mathcal{G}_\infty > 0$ because otherwise there would exist $0 \neq c \in \mathbb{R}^n$ such that $\mathcal{G}(t)c = 0$ for $t \geq a$ (use that fact that $\mathcal{G}(t, \bar{X})$ is nonincreasing) and then

$$\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} \bar{X}(t)\mathcal{G}(t;\bar{X})c \\ [\bar{U}(t)\mathcal{G}(t;\bar{X}) + (\bar{X}^T(t))^{-1}]c \end{pmatrix}$$

is a solution of (1) with $x(t) \equiv 0$ and $u(t) \neq 0$ on $[a, \infty)$ which contradicts eventual dense normality of this system. Consequently, $\hat{X}(a) = \bar{X}(a)\mathcal{G}_\infty$ is nonsingular.

Denote

$$\hat{X} = \bar{X}[\mathcal{G}_\infty - \mathcal{G}(t;\bar{X})], \quad \hat{U} = \bar{U}[\mathcal{G}_\infty - \mathcal{G}(t;\bar{X})] - (\bar{X}^T)^{-1}.$$

Then $\hat{X}^T\bar{U} - \hat{U}^T\bar{X} = I$ and

$$\lim_{t\to\infty} \bar{X}^{-1}(t)\hat{X}(t) = \lim_{t\to\infty}[\mathcal{G}_\infty - \mathcal{G}(t;\bar{X})] = 0.$$

Hence $\binom{\hat{X}}{\hat{U}}$ is a principal solution of (1) for which $\hat{X}(a)$ is nonsingular. By Theorem 3 there exists a nonsingular $n \times n$ matrix M such that $\binom{\tilde{X}}{\tilde{U}} = \binom{\hat{X}}{\hat{U}}M$, in particular, $\tilde{X}(a) = \hat{X}(a)M$ is nonsingular. However, this contradicts singularity of this matrix which we obtained at the beginning of the proof as a consequence of the assumption that $X(t)$ is nonsingular and $D(t) \geq 0$ for $t \geq a$. This completes the proof.

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