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Line integrals and Green's formula on time scales

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Abstract

In this paper we study curves parametrized by a time scale parameter, introduce line delta and nabla integrals along time scale curves, and obtain an analog of Green's formula in the time scale setting.

Keywords: Time scales; Time scale curves; Line delta and nabla integrals; Green's formula

1. Introduction

Differentiation and integration theory for functions of one time scale variable was developed earlier in [1–3,7,8,10–12]. Recently we gave in [4] a differential calculus and in [5,6] an integral calculus for functions of several time scale variables. The present paper continues [4–6] and develops line integration along time scale curves. As an application of the introduced line integrals we derive a time scale version of Green's formula that establishes a connection between the double and line integrals.

The paper is organized as follows. In Section 2, we introduce the concept of a curve parametrized by a time scale parameter and give integral formulas for computation of its length. In Section 3, we define line delta and nabla integrals of the first and second types on time scales and give sufficient conditions for the existence of these integrals and also offer formulas for their evaluation. Next, we present some properties of line delta integrals. Finally, in Section 4, we

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establish a version of the classical Green formula suitable to time scales. To this end, the concepts of connectedness, domain, and fence of a set are introduced for sets in the time scale plane. Surface integrals in the time scale setting will be discussed in a forthcoming paper of the authors.

2. Length of time scale curves

Let $\mathbb{T}$ be a time scale with the forward jump operator σ and the delta differentiation operator Δ. Given the points $a, b \in \mathbb{T}$ with $a \leqslant b$, let $[a, b] = \{t \in \mathbb{T}\colon\ a \leqslant t \leqslant b\}$ be the closed interval in $\mathbb{T}$. Further, let $\varphi : [a, b] \to \mathbb{R}$ and $\psi : [a, b] \to \mathbb{R}$ be continuous (in the time scale topology) on $[a, b]$. Consider the xy-plane, i.e., the set of all ordered pairs (x, y) of real numbers x and y. Each such pair is called a point of the plane and the numbers x and y are called the coordinates of that point. The point (x, y) can also be denoted by a single letter A, and writing $A(x, y)$ will mean that the point A has the coordinates x and y.

Definition 2.1. The pair of functions

$$x = \varphi(t), \qquad y = \psi(t), \quad t \in [a, b] \subset \mathbb{T} \tag{2.1}$$

is said to define a (*time scale continuous*) *curve* Γ.

The points $A(x, y)$ with the coordinates x and y defined by (2.1) are called the *points* of the curve, and the set of all points of the curve, i.e., the range of the mapping (2.1), will often be referred to as simply the *curve* (when no ambiguity can arise). In particular, the points $A_0(\varphi(a), \psi(a))$ and $A_1(\varphi(b), \psi(b))$ are called the *initial* and *final points* of the curve, respectively, and A_0, A_1 are called the *end points* of the curve. The initial and final points of a curve may coincide, in which case the curve is said to be *closed*. The time scale variable t is called the *parameter* of the curve, and Eq. (2.1), mapping the values of the parameter onto the points of the curve, is called the (*parametric*) *equation* of the curve. We can also think of Γ as an *oriented curve*, in the sense that a point $(x', y') = (\varphi(t'), \psi(t')) \in \Gamma$ is regarded as *distinct* from a point $(x'', y'') = (\varphi(t''), \psi(t'')) \in \Gamma$ if $t' \neq t''$ and as *preceding* (x'', y'') if $t' < t''$. The oriented curve Γ is then said to be "traversed in the direction of increasing t." It will always be clear from the context whether Γ is a curve in the set-theoretic sense, i.e., the continuous image of a closed time scale interval, or an oriented curve as just described. Two curves Γ_1 and Γ_2 with equations

$$x = \varphi_1(t), \qquad y = \psi_1(t), \quad t \in [a, b] \subset \mathbb{T}_1,$$

and

$$x = \varphi_2(\tau), \qquad y = \psi_2(\tau), \quad \tau \in [\alpha, \beta] \subset \mathbb{T}_2,$$

are regarded as *identical* if the equation of one curve can be transformed into the equation of the other by means of a continuous (strictly) increasing change of parameter, i.e., if there is a continuous increasing function $\tau = \lambda(t)$, $t \in [a, b]$, with the range $[\alpha, \beta]$, such that

$$\varphi_2\big(\lambda(t)\big) = \varphi_1(t), \qquad \psi_2\big(\lambda(t)\big) = \psi_1(t), \quad t \in [a, b].$$

(Then, of course, $\varphi_1(\lambda^{-1}(\tau)) = \varphi_2(\tau)$ and $\psi_1(\lambda^{-1}(\tau)) = \psi_2(\tau)$ for $\tau \in [\alpha, \beta]$, where λ^{-1} is the continuous inverse of the function λ.) We then say that the two curves have the *same* direction. We say that the two curves have *opposite* directions if the function λ above is decreasing. In this case, the initial point of Γ_1 is the same as the final point of Γ_2, and vice versa. The curve differing from Γ only by the direction in which it is traversed will be denoted by $-\Gamma$. If the same point (x, y) corresponds to more than one parameter value in the half-open (time scale) interval $[a, b)$, then we say that (x, y) is a *multiple point* of the curve (2.1). A curve with no multiple points is called a *simple curve* (or *Jordan curve*). It is to allow for the possibility of closed simple curves, i.e., simple curves whose end points coincide, that we consider parameter values in the half-open interval $[a, b)$.

Example 2.2. Let $\mathbb{T} = \mathbb{R}$. The mappings

$$\begin{cases} x = t, \\ y = 0, \end{cases} \qquad \begin{cases} x = t^2, \\ y = 0, \end{cases} \qquad \begin{cases} x = 1 - t, \\ y = 0, \end{cases} \qquad 0 \leqslant t \leqslant 1, \tag{2.2}$$

all define the same continuous curve (except for direction), corresponding to the segment of the x-axis lying between the points $(0, 0)$ and $(1, 0)$. The first and second representations in (2.2) have the same direction (with initial point $(0, 0)$ and final point $(1, 0)$), but the third representation in (2.2) has the opposite direction (with initial point $(1, 0)$ and final point $(0, 0)$). Obviously, the curve (2.2) has no multiple points and hence is a simple curve. Since its initial and final points do not coincide, (2.2) is not a closed simple curve.

Example 2.3. Let $\mathbb{T} = \mathbb{R}$. The mappings

$$\begin{cases} x = t, \\ y = 0, \end{cases} \qquad \begin{cases} x = \sin(\pi t), \\ y = 0, \end{cases} \qquad 0 \leqslant t \leqslant 1, \tag{2.3}$$

define different continuous curves Γ_1 and Γ_2, since there is no continuous increasing change of parameter carrying one of these curves into the other. In fact, such a transformation would have to carry the increasing function $\varphi(t) = t$ into another increasing function, but the function $\sin(\pi t)$ is not monotonic in [0, 1]. We note that although the two curves (2.3) are different, they consist of the same set of points, i.e., the points in [0, 1] on the x-axis. Therefore, the fact that two continuous curves consist of the same points is not a sufficient condition (but is obviously a necessary condition) for the curves to be identical. We also note the following facts: (i) the curve Γ_1 is a simple curve, but not a closed curve; (ii) the curve Γ_2 is not a simple curve, since it has multiple points, $\sin(\pi t) = \sin(\pi(1 - t))$, $0 < t < 1/2$; (iii) the curve Γ_2 is a closed curve, since its initial and final points coincide.

Let Γ be a (time scale) continuous curve with Eq. (2.1). A partition of $[a, b]$ is any finite ordered set

$$P = \{t_0, t_1, \ldots, t_n\} \subset [a, b], \quad \text{where } a = t_0 < t_1 < \cdots < t_n = b. \tag{2.4}$$

For a given partition, we denote by $A_0, A_1, \ldots, A_n$ the corresponding points of the curve Γ, i.e., the points with the coordinates $A_i(\varphi(t_i), \psi(t_i))$, $i \in \{0, 1, \ldots, n\}$. Let us set

$$\ell(\Gamma, P) = \sum_{i=1}^{n} d(A_{i-1}, A_i) = \sum_{i=1}^{n} \sqrt{\left[\varphi(t_i) - \varphi(t_{i-1})\right]^2 + \left[\psi(t_i) - \psi(t_{i-1})\right]^2}, \tag{2.5}$$

where $d(A_{i-1}, A_i)$ is the distance from the point A_{i-1} to the point A_i.

Definition 2.4. The curve Γ is said to be *rectifiable* if

$$\sup\bigl\{\ell(\Gamma, P)\colon\ P \text{ is a partition of } [a, b]\bigr\} =: \ell(\Gamma) < \infty,$$

where the least upper bound is taken over all possible partitions (2.4). The nonnegative real number $\ell = \ell(\Gamma)$ is called the *length* of the curve Γ. If the supremum does not exist, the curve is said to be *nonrectifiable*. In this case, Γ is considered to have no length at all (or, if preferred, infinite length).

If P and Q are two partitions of $[a,b]$ such that every point of P belongs to Q, i.e., $P \subset Q$, then we say that Q is a refinement of P.

Lemma 2.5. *If P and Q are partitions of $[a,b]$ and Q is a refinement of P, then $\ell(\Gamma, P) \leqslant \ell(\Gamma, Q)$.*

Proof. An induction argument shows that we may assume that Q has only one more point, say s, than P. If P is of the form (2.4), then

$$a = t_0 < t_1 < \cdots < t_{k-1} < s < t_k < \cdots < t_n = b \quad \text{for some } k \in \{1, 2, \ldots, n\}.$$

Let us set

$$B = B\bigl(\varphi(s), \psi(s)\bigr) \quad \text{and} \quad A_i = A_i\bigl(\varphi(t_i), \psi(t_i)\bigr) \quad \text{for } i \in \{0, 1, \ldots, n\}.$$

Then

$$\ell(\Gamma, Q) - \ell(\Gamma, P) = d(A_{k-1}, B) + d(B, A_k) - d(A_{k-1}, A_k) \geqslant 0$$

in virtue of the triangle inequality for the distance between points. □

Let us present some properties of rectifiable curves.

Theorem 2.6. *If the curve Γ is rectifiable, then its length $\ell(\Gamma)$ does not depend on the parametrization of this curve.*

Proof. Assume we have two parametrizations of the curve Γ and let t and τ be parameters of these parametrizations defined on the intervals $[a,b]$ and $[\alpha,\beta]$, respectively. Since t and τ are strictly increasing and continuous functions of each other, to each partition P of $[a,b]$ there corresponds a definite partition P' of $[\alpha,\beta]$, and vice versa. Evidently, $\ell(\Gamma, P) = \ell(\Gamma, P')$, and therefore the lengths of Γ corresponding to the two parametrizations of Γ are equal. □

Theorem 2.7. *If a rectifiable curve Γ is split by means of a finite number of points $A_0, A_1, \ldots, A_n$ into a finite number of curves Γ_i and if the points A_i correspond to the values t_i of the parameter t such that $a = t_0 < t_1 < \cdots < t_n = b$, then each of the curves Γ_i also is rectifiable and the sum of the lengths $\ell(\Gamma_i)$ of all curves Γ_i equals the length $\ell(\Gamma)$ of the curve Γ.*

Proof. Obviously, it is sufficient to prove this property for the case when the curve Γ is split only into two curves Γ_1 and Γ_2 by means of a point A. Then the points of the curve Γ_1 correspond to the values of $t \in [a,c]$ and the points of Γ_2 to the values of $t \in [c,b]$. Let P_1 and P_2 be arbitrary partitions of $[a,c]$ and $[c,b]$, respectively. Then their union $P = P_1 \cup P_2$ is a partition of $[a,b]$ and, evidently, we have

$$\ell(\Gamma_1, P_1) + \ell(\Gamma_2, P_2) = \ell(\Gamma, P). \tag{2.6}$$

Hence by the rectifiability of the curve Γ, the numbers $\ell(\Gamma_1, P_1)$ and $\ell(\Gamma_2, P_2)$ corresponding to all possible partitions of the intervals $[a,c]$ and $[c,b]$ are bounded, i.e., the curves Γ_1 and Γ_2

are rectifiable. From (2.6) and the definition of the least upper bound, it follows that the lengths $\ell(\Gamma_1)$, $\ell(\Gamma_2)$, and $\ell(\Gamma)$ of the curves Γ_1, Γ_2, and Γ satisfy the inequality

$$\ell(\Gamma_1) + \ell(\Gamma_2) \leqslant \ell(\Gamma). \tag{2.7}$$

Let us show that in (2.7) in fact equality holds. Assume the contrary, i.e., $\ell(\Gamma_1) + \ell(\Gamma_2) < \ell(\Gamma)$. Then the number

$$\varepsilon := \ell(\Gamma) - \ell(\Gamma_1) - \ell(\Gamma_2) \tag{2.8}$$

is positive. From the definition of $\ell(\Gamma)$, it follows that corresponding to ε we can find a partition P_0 of $[a, b]$ such that $\ell(\Gamma) - \ell(\Gamma, P_0) < \varepsilon$. Adding the point c to the partition P_0 and denoting the obtained partition by P and taking into account Lemma 2.5, we also have $\ell(\Gamma) - \ell(\Gamma, P) < \varepsilon$. Since the partition P of $[a, b]$ is a union of some partitions P_1 and P_2 of $[a, c]$ and $[c, b]$, respectively, for these partitions we have the relation (2.6). Therefore

$$\ell(\Gamma) - \ell(\Gamma_1, P_1) - \ell(\Gamma_2, P_2) < \varepsilon.$$

Since $\ell(\Gamma_1, P_1) + \ell(\Gamma_2, P_2) \leqslant \ell(\Gamma_1) + \ell(\Gamma_2)$, we conclude

$$\ell(\Gamma) - \ell(\Gamma_1) - \ell(\Gamma_2) < \varepsilon.$$

But this contradicts the equality (2.8). This concludes the proof. □

Now we present a sufficient condition for rectifiability of curves and give a formula for evaluating their lengths.

Theorem 2.8. *Let the functions φ and ψ be continuous on $[a, b]$ and Δ-differentiable on $[a, b)$. If their Δ-derivatives φ^Δ and ψ^Δ are bounded and Δ-integrable over $[a, b)$, then the curve Γ defined by the parametric equation* (2.1) *is rectifiable and its length $\ell(\Gamma)$ can be evaluated by the formula*

$$\ell(\Gamma) = \int_a^b \sqrt{\left[\varphi^\Delta(t)\right]^2 + \left[\psi^\Delta(t)\right]^2}\,\Delta t. \tag{2.9}$$

Proof. First we show that the curve Γ is rectifiable. Let P be an arbitrary partition of $[a, b]$ of the form (2.4). Consider $\ell(\Gamma, P)$ defined by (2.5). Applying to each of the functions φ and ψ the mean value theorem (see [4, Theorem 4.2]) on $[t_{i-1}, t_i]$ for $i \in \{1, 2, \ldots, n\}$, we get that there exist points ξ_i, ξ_i' and η_i, η_i' in $[t_{i-1}, t_i)$ such that

$$\varphi^\Delta(\xi_i)(t_i - t_{i-1}) \leqslant \varphi(t_i) - \varphi(t_{i-1}) \leqslant \varphi^\Delta\left(\xi_i'\right)(t_i - t_{i-1}), \tag{2.10}$$

$$\psi^\Delta(\eta_i)(t_i - t_{i-1}) \leqslant \psi(t_i) - \psi(t_{i-1}) \leqslant \psi^\Delta\left(\eta_i'\right)(t_i - t_{i-1}). \tag{2.11}$$

From (2.10) and (2.11) it follows that

$$\left|\varphi(t_i) - \varphi(t_{i-1})\right| \leqslant A_i(t_i - t_{i-1}), \qquad \left|\psi(t_i) - \psi(t_{i-1})\right| \leqslant B_i(t_i - t_{i-1}),$$

where

$$A_i = \max\left\{\left|\varphi^\Delta(\xi_i)\right|, \left|\varphi^\Delta\left(\xi_i'\right)\right|\right\}, \qquad B_i = \max\left\{\left|\psi^\Delta(\eta_i)\right|, \left|\psi^\Delta\left(\eta_i'\right)\right|\right\}.$$

Further, by the condition of the theorem, the derivatives φ^Δ and ψ^Δ are bounded on $[a, b)$. Then there is a finite positive constant C such that

$$\left|\varphi^\Delta(t)\right| \leqslant C \quad \text{and} \quad \left|\psi^\Delta(t)\right| \leqslant C \quad \text{for all } t \in [a, b).$$

Consequently,

$$\left|\varphi(t_i)-\varphi(t_{i-1})\right| \leqslant C(t_i-t_{i-1}), \qquad \left|\psi(t_i)-\psi(t_{i-1})\right| \leqslant C(t_i-t_{i-1})$$

for all $i \in \{1,2,\ldots,n\}$ and we have from (2.5)

$$\ell(\Gamma,P) \leqslant \sum_{i=1}^{n}\sqrt{C^2+C^2}(t_i-t_{i-1}) = C\sqrt{2}\sum_{i=1}^{n}(t_i-t_{i-1}) = C\sqrt{2}(b-a).$$

This shows that the set $\{\ell(\Gamma,P)\colon\ P$ is a partition of $[a,b]\}$ is bounded, and hence by Definition 2.4, the curve Γ is rectifiable.

Now we prove that the length $\ell(\Gamma)$ of the curve Γ can be evaluated by the formula (2.9). Consider the Riemann Δ-sum

$$S = \sum_{i=1}^{n}\sqrt{\left[\varphi^\Delta(\xi_i)\right]^2+\left[\psi^\Delta(\xi_i)\right]^2}(t_i-t_{i-1})$$

of the Δ-integrable function $\sqrt{[\varphi^\Delta(t)]^2+[\psi^\Delta(t)]^2}$, corresponding to the partition P of $[a,b]$ and the choice of intermediate points ξ_i defined in (2.10). For every $\delta>0$ there exists (see [8, Lemma 5.7]) at least one partition P of $[a,b]$ of the form (2.4) such that for each $i\in\{1,2,\ldots,n\}$ either $t_i-t_{i-1}\leqslant\delta$ or $t_i-t_{i-1}>\delta$ and $\sigma(t_{i-1})=t_i$, where σ denotes the forward jump operator in $\mathbb{T}$. We denote by $\mathcal{P}_\delta([a,b])$ the set of all such partitions P of $[a,b]$. Let us show that for an arbitrary $\varepsilon>0$ there exists $\delta>0$ such that

$$\left|\ell(\Gamma,P)-I\right| < \frac{\varepsilon}{2} \quad \text{for all } P\in\mathcal{P}_\delta\big([a,b]\big), \tag{2.12}$$

where

$$I = \int_a^b \sqrt{\left[\varphi^\Delta(t)\right]^2+\left[\psi^\Delta(t)\right]^2}\,\Delta t. \tag{2.13}$$

From (2.10) and (2.11), we get

$$0 \leqslant \varphi(t_i)-\varphi(t_{i-1})-\varphi^\Delta(\xi_i)(t_i-t_{i-1}) \leqslant \left[\varphi^\Delta\big(\xi_i'\big)-\varphi^\Delta(\xi_i)\right](t_i-t_{i-1}),$$
$$0 \leqslant \psi(t_i)-\psi(t_{i-1})-\psi^\Delta(\eta_i)(t_i-t_{i-1}) \leqslant \left[\psi^\Delta\big(\eta_i'\big)-\psi^\Delta(\eta_i)\right](t_i-t_{i-1}),$$

and, consequently,

$$\varphi(t_i)-\varphi(t_{i-1}) = \left[\varphi^\Delta(\xi_i)+\alpha_i\right](t_i-t_{i-1}),$$
$$\psi(t_i)-\psi(t_{i-1}) = \left[\psi^\Delta(\eta_i)+\beta_i\right](t_i-t_{i-1}),$$

where

$$0 \leqslant \alpha_i \leqslant \varphi^\Delta\big(\xi_i'\big)-\varphi^\Delta(\xi_i) \leqslant M_i-m_i,$$
$$0 \leqslant \beta_i \leqslant \psi^\Delta\big(\eta_i'\big)-\psi^\Delta(\eta_i) \leqslant N_i-n_i,$$

in which M_i and m_i are the supremum and infimum of φ^Δ on $[t_{i-1},t_i)$, respectively, and N_i and n_i are the corresponding numbers for ψ^Δ. Further, using the inequality (for arbitrary real numbers x,y,x_1,y_1)

$$\left|\sqrt{x^2+y^2}-\sqrt{x_1^2+y_1^2}\right| \leqslant |x-x_1|+|y-y_1|,$$

we obtain

$$\begin{aligned}
&\left|\sqrt{\left[\varphi^{\Delta}(\xi)+\alpha_i\right]^2+\left[\psi^{\Delta}(\eta_i)+\beta_i\right]^2}-\sqrt{\left[\varphi^{\Delta}(\xi)\right]^2+\left[\psi^{\Delta}(\xi_i)\right]^2}\,\right| \\
&\quad \leqslant |\alpha_i|+\left|\psi^{\Delta}(\eta_i)+\beta_i-\psi^{\Delta}(\xi_i)\right| \leqslant |\alpha_i|+|\beta_i|+\left|\psi^{\Delta}(\eta_i)-\psi^{\Delta}(\xi_i)\right| \\
&\quad \leqslant M_i-m_i+2(N_i-n_i).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left|\ell(\Gamma,P)-S\right| \\
&\quad =\left|\sum_{i=1}^{n}\left(\sqrt{\left[\varphi^{\Delta}(\xi_i)+\alpha_i\right]^2+\left[\psi^{\Delta}(\eta_i)+\beta_i\right]^2}-\sqrt{\left[\varphi^{\Delta}(\xi_i)\right]^2+\left[\psi^{\Delta}(\xi_i)\right]^2}\right)(t_i-t_{i-1})\right| \\
&\quad \leqslant \sum_{i=1}^{n}\left[M_i-m_i+2(N_i-n_i)\right](t_i-t_{i-1}) \\
&\quad = U\left(\varphi^{\Delta},P\right)-L\left(\varphi^{\Delta},P\right)+2\left[U\left(\psi^{\Delta},P\right)-L\left(\psi^{\Delta},P\right)\right],
\end{aligned}$$

where U and L denote the upper and lower Darboux Δ-sums, respectively. Since the functions $\sqrt{[\varphi^{\Delta}]^2+[\psi^{\Delta}]^2}$, φ^{Δ}, and ψ^{Δ} are Δ-integrable over $[a,b)$, for arbitrary $\varepsilon>0$ there is $\delta>0$ such that

$$|S-I|<\frac{\varepsilon}{8},\qquad U\left(\varphi^{\Delta},P\right)-L\left(\varphi^{\Delta},P\right)<\frac{\varepsilon}{8},\qquad U\left(\psi^{\Delta},P\right)-L\left(\psi^{\Delta},P\right)<\frac{\varepsilon}{8}$$

for all $P\in\mathcal{P}_\delta([a,b])$ (see [8, Theorem 5.9] and the Riemann definition of Δ-integrability therein), where I is defined by (2.13). Therefore, taking into account the above estimate, we get

$$\left|\ell(\Gamma,P)-I\right| \leqslant \left|\ell(\Gamma,P)-S\right|+|S-I|<\frac{\varepsilon}{8}+2\frac{\varepsilon}{8}+\frac{\varepsilon}{8}=\frac{\varepsilon}{2},$$

and so the validity of (2.12) is proved. On the other hand, among the partitions P for which (2.12) is satisfied, there is a partition P such that

$$\left|\ell(\Gamma,P)-\ell(\Gamma)\right|<\frac{\varepsilon}{2}. \tag{2.14}$$

Indeed, from Definition 2.4 it follows that there is a partition P^* of $[a,b]$ such that

$$0\leqslant \ell(\Gamma)-\ell(\Gamma,P^*)<\frac{\varepsilon}{2}. \tag{2.15}$$

Next, we refine the partition P^* adding to it new partition points so that we get a partition P that belongs to $\mathcal{P}_\delta([a,b])$. Then, by Lemma 2.5, $\ell(\Gamma,P)\geqslant \ell(\Gamma,P^*)$, and (2.15) yields

$$0\leqslant \ell(\Gamma)-\ell(\Gamma,P)<\frac{\varepsilon}{2},$$

so that (2.14) is shown. By (2.12) and (2.14), we get that

$$\left|\ell(\Gamma)-I\right|<\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we have $\ell(\Gamma)=I$ and the proof is complete. □

Remark 2.9. Let the curve Γ be given as the graph of a function $y = f(x)$, $x \in [a,b] \subset \mathbb{T}$, where f is continuous on $[a,b]$ and Δ-differentiable on $[a,b)$. If f^Δ is bounded and Δ-integrable over $[a,b)$, then Theorem 2.8 implies (by taking $x = t$, $y = f(t)$) that the curve Γ is rectifiable, and its length $\ell(\Gamma)$ can be evaluated by the formula

$$\ell(\Gamma) = \int_a^b \sqrt{1 + \left[f^\Delta(x)\right]^2}\, \Delta x.$$

Example 2.10. Let $\mathbb{T} = (-\infty, 0] \cup \mathbb{N}$, where $(-\infty, 0]$ is the real line interval, and let $[a,b] = [-1,3] = [-1,0] \cup \{1,2,3\}$. Define the curve Γ by

$$x = \varphi(t) = t^3, \qquad y = \psi(t) = t^2, \quad t \in [-1,3] \subset \mathbb{T}.$$

Setting $h(t) = \sqrt{[\varphi^\Delta(t)]^2 + [\psi^\Delta(t)]^2}$, we have

$$h(t) = \begin{cases} \sqrt{9t^4 + 4t^2} & \text{if } t \in [-1,0), \\ \sqrt{(3t^2+3t+1)^2 + (2t+1)^2} & \text{if } t \in \{0,1,2,3\}. \end{cases}$$

Therefore

$$\begin{aligned} \ell(\Gamma) &= \int_{-1}^{3} h(t)\Delta t = \int_{-1}^{0} h(t)\Delta t + \int_{0}^{3} h(t)\Delta t \\ &= \int_{-1}^{0} \sqrt{9t^4 + 4t^2}\, \mathrm{d}t + h(0) + h(1) + h(2) \\ &= \frac{1}{27}\left(13\sqrt{13} - 8\right) + \sqrt{2} + \sqrt{58} + \sqrt{386}. \end{aligned}$$

The following theorem can be proved similarly to Theorem 2.8 by using the mean value theorem for nabla derivatives given in [10].

Theorem 2.11. *Let the functions φ and ψ be continuous on $[a,b]$ and ∇-differentiable on $(a,b]$. If their ∇-derivatives φ^∇ and ψ^∇ are bounded and ∇-integrable over $(a,b]$, then the curve Γ defined by the parametric equation (2.1) is rectifiable and its length $\ell(\Gamma)$ can be evaluated by the formula*

$$\ell(\Gamma) = \int_a^b \sqrt{\left[\varphi^\nabla(t)\right]^2 + \left[\psi^\nabla(t)\right]^2}\, \nabla t.$$

Remark 2.12. Let the curve Γ be given as the graph of a function $y = f(x)$, $x \in [a,b] \subset \mathbb{T}$, where f is continuous on $[a,b]$ and ∇-differentiable on $(a,b]$. If f^∇ is bounded and ∇-integrable over $(a,b]$, then Theorem 2.11 implies (by taking $x = t$, $y = f(t)$) that the curve Γ is rectifiable, and its length $\ell(\Gamma)$ can be evaluated by the formula

$$\ell(\Gamma) = \int_a^b \sqrt{1 + \left[f^\nabla(x)\right]^2}\, \nabla x.$$

3. Line delta and nabla integrals of the first and second types

Let $\mathbb{T}$ be a time scale with the forward jump operator σ and the delta differentiation operator Δ. Let a and b be points in $\mathbb{T}$ with $a \leqslant b$ and $[a, b]$ be a closed interval in $\mathbb{T}$. Let $\varphi : [a, b] \to \mathbb{R}$ and $\psi : [a, b] \to \mathbb{R}$ be two functions that are continuous (in the time scales topology) on $[a, b]$. Denote by Γ the curve defined by the parametric equation

$$x = \varphi(t), \qquad y = \psi(t), \quad t \in [a, b] \subset \mathbb{T}. \tag{3.1}$$

The points (x, y) with the coordinates x and y defined by (3.1) are the points of the curve Γ. Let A and B denote the initial and final points $(\varphi(a), \psi(a))$ and $(\varphi(b), \psi(b))$ of the curve Γ, respectively. The range of the mapping (3.1) will also be denoted by Γ (provided no ambiguity arises).

Let three functions $f(x, y)$, $M(x, y)$, and $N(x, y)$ be defined and continuous on the curve Γ (for example, for the function $f(x, y)$ this means that for each $A_0 \in \Gamma$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(A) - f(A_0)| < \varepsilon$ whenever $A \in \Gamma$ and $d(A, A_0) < \delta$, where $d(A, A_0)$ denotes the Euclidean distance between the points A and A_0).

Let $P = \{t_0, t_1, \dots, t_n\} \subset [a, b]$, where $a = t_0 < t_1 < \cdots < t_n = b$, be a partition of $[a, b]$ and let $A_0, A_1, \dots, A_n$ be points of the curve Γ with coordinates $(\varphi(t_i), \psi(t_i))$, $i \in \{0, 1, \dots, n\}$, respectively. Take any $\tau_k \in [t_{k-1}, t_k)$ for $k \in \{1, 2, \dots, n\}$. Denote by ℓ_k the length of the piece of the curve Γ between its points A_{k-1} and A_k. By Theorem 2.8, the formula

$$\ell_k = \int_{t_{k-1}}^{t_k} \sqrt{\left[\varphi^\Delta(t)\right]^2 + \left[\psi^\Delta(t)\right]^2}\,\Delta t \tag{3.2}$$

holds. Let us introduce the three integral sums

$$S_1 = \sum_{k=1}^{n} f\big(\varphi(\tau_k), \psi(\tau_k)\big)\ell_k, \tag{3.3}$$

$$S_2 = \sum_{k=1}^{n} M\big(\varphi(\tau_k), \psi(\tau_k)\big)\big[\varphi(t_k) - \varphi(t_{k-1})\big], \tag{3.4}$$

$$S_3 = \sum_{k=1}^{n} N\big(\varphi(\tau_k), \psi(\tau_k)\big)\big[\psi(t_k) - \psi(t_{k-1})\big]. \tag{3.5}$$

Definition 3.1. We say that a complex number I_1 is the *line delta integral of the first type* of the function f along the curve Γ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|S_1 - I_1| < \varepsilon$ for every integral sum S_1 of f corresponding to a partition $P \in \mathcal{P}_\delta([a, b])$ independent of the way in which we choose $\tau_k \in [t_{k-1}, t_k)$ for $k \in \{1, 2, \dots, n\}$. We denote the number I_1, symbolically, by

$$\int_\Gamma f(x, y)\Delta\ell \quad \text{or} \quad \int_{AB} f(x, y)\Delta\ell, \tag{3.6}$$

where A and B denote the initial and final points of the curve Γ, respectively.

Definition 3.2. We say that a complex number I_2 (respectively I_3) is the *line delta integral of the second type* of the function M (respectively N) along the curve Γ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|S_2 - I_2| < \varepsilon$ (respectively $|S_3 - I_3| < \varepsilon$) for every integral sum S_2 (respectively S_3) of M (respectively N) corresponding to a partition $P \in \mathcal{P}_\delta([a,b])$ independent of the way in which we choose $\tau_k \in [t_{k-1}, t_k)$ for $k \in \{1, 2, \ldots, n\}$. We denote the number I_2 (respectively I_3), symbolically, by

$$\int_\Gamma M(x,y)\Delta_1 x \quad \text{or} \quad \int_{AB} M(x,y)\Delta_1 x$$

$$\left(\text{respectively } \int_\Gamma N(x,y)\Delta_2 y \text{ or } \int_{AB} N(x,y)\Delta_2 y\right). \tag{3.7}$$

The sum

$$\int_\Gamma M(x,y)\Delta_1 x + \int_\Gamma N(x,y)\Delta_2 y$$

is called a *general line delta integral of the second type* and is denoted by

$$\int_\Gamma M(x,y)\Delta_1 x + N(x,y)\Delta_2 y. \tag{3.8}$$

The following theorem gives conditions sufficient for the existence of the line delta integrals.

Theorem 3.3. *Suppose that the curve Γ is given by the parametric equation* (3.1)*, where φ and ψ are continuous on $[a,b]$ and Δ-differentiable on $[a,b)$. If φ^Δ and ψ^Δ are bounded and Δ-integrable over $[a,b)$ and if the functions f, M, and N are continuous on Γ, then the line delta integrals* (3.6) *and* (3.7) *exist and can be computed by the formulas*

$$\int_\Gamma f(x,y)\Delta\ell = \int_a^b f\big(\varphi(t),\psi(t)\big)\sqrt{\big[\varphi^\Delta(t)\big]^2 + \big[\psi^\Delta(t)\big]^2}\,\Delta t, \tag{3.9}$$

$$\int_\Gamma M(x,y)\Delta_1 x = \int_a^b M\big(\varphi(t),\psi(t)\big)\varphi^\Delta(t)\Delta t, \tag{3.10}$$

$$\int_\Gamma N(x,y)\Delta_2 y = \int_a^b N\big(\varphi(t),\psi(t)\big)\psi^\Delta(t)\Delta t. \tag{3.11}$$

Proof. First of all, note that the Δ-integrals standing on the right-hand sides of formulas (3.9)–(3.11) exist. Note also that the derivations of the relations (3.10) and (3.11) are analogous to each other, and therefore we will derive only relations (3.9) and (3.10) and prove only the existence of the integrals (3.6) and (3.7).

Take as above an arbitrary partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[a,b]$ and introduce the integral sums (3.3) and (3.4). Taking into account (3.2) and the relation

$$\varphi(t_k) - \varphi(t_{k-1}) = \int_{t_{k-1}}^{t_k} \varphi^\Delta(t)\Delta t,$$

we represent the integral sums (3.3) and (3.4) in the forms

$$S_1=\sum_{k=1}^{n} f\big(\varphi(\tau_k),\psi(\tau_k)\big)\int_{t_{k-1}}^{t_k}\sqrt{\big[\varphi^{\Delta}(t)\big]^2+\big[\psi^{\Delta}(t)\big]^2}\,\Delta t,$$

$$S_2=\sum_{k=1}^{n} M\big(\varphi(\tau_k),\psi(\tau_k)\big)\int_{t_{k-1}}^{t_k}\varphi^{\Delta}(t)\Delta t.$$

Denote the Δ-integrals on the right-hand sides of (3.9) and (3.10) by I_1 and I_2, respectively, and represent them as sums of integrals over the subintervals $[t_{k-1},t_k)$, $k\in\{1,2,\dots,n\}$. Let us estimate the differences

$$S_1-I_1=\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\big[f\big(\varphi(\tau_k),\psi(\tau_k)\big)-f\big(\varphi(t),\psi(t)\big)\big]\sqrt{\big[\varphi^{\Delta}(t)\big]^2+\big[\psi^{\Delta}(t)\big]^2}\,\Delta t, \tag{3.12}$$

$$S_2-I_2=\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\big[M\big(\varphi(\tau_k),\psi(\tau_k)\big)-M\big(\varphi(t),\psi(t)\big)\big]\varphi^{\Delta}(t)\Delta t. \tag{3.13}$$

Under the conditions imposed on f, M, and (3.1), we have that $f(\varphi(t),\psi(t))$ and $M(\varphi(t),\psi(t))$ being composite functions of the argument t are continuous on $[a,b]$ and therefore are uniformly continuous on this interval. Consequently, for arbitrary $\varepsilon>0$ there exists $\delta>0$ such that $t,\tau\in[a,b]$ and $|t-\tau|\leqslant\delta$ imply

$$\big|f\big(\varphi(t),\psi(t)\big)-f\big(\varphi(\tau),\psi(\tau)\big)\big|<\frac{\varepsilon}{\ell(\Gamma)},$$

$$\big|M\big(\varphi(t),\psi(t)\big)-M\big(\varphi(\tau),\psi(\tau)\big)\big|<\frac{\varepsilon}{M(b-a)},$$

where $\ell(\Gamma)$ denotes the length of the curve Γ and $M=\sup\{|\varphi^{\Delta}(t)|:\ t\in[a,b)\}$. Now, assuming that $P\in\mathcal{P}_\delta([a,b])$, we have

$$\begin{aligned}|S_1-I_1|&\leqslant\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\big|f\big(\varphi(\tau_k),\psi(\tau_k)\big)-f\big(\varphi(t),\psi(t)\big)\big|\sqrt{\big[\varphi^{\Delta}(t)\big]^2+\big[\psi^{\Delta}(t)\big]^2}\Delta t\\&=\sum_{t_k-t_{k-1}\leqslant\delta}(\cdots)+\sum_{t_k-t_{k-1}>\delta}(\cdots)=\sum_{t_k-t_{k-1}\leqslant\delta}(\cdots)\\&\leqslant\sum_{t_k-t_{k-1}\leqslant\delta}\frac{\varepsilon}{\ell(\Gamma)}\int_{t_{k-1}}^{t_k}\sqrt{\big[\varphi^{\Delta}(t)\big]^2+\big[\psi^{\Delta}(t)\big]^2}\Delta t\\&\leqslant\frac{\varepsilon}{\ell(\Gamma)}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\sqrt{\big[\varphi^{\Delta}(t)\big]^2+\big[\psi^{\Delta}(t)\big]^2}\Delta t\end{aligned}$$

$$= \frac{\varepsilon}{\ell(\Gamma)} \int_a^b \sqrt{\left[\varphi^\Delta(t)\right]^2 + \left[\psi^\Delta(t)\right]^2}\,\Delta t$$

$$= \frac{\varepsilon}{\ell(\Gamma)} \cdot \ell(\Gamma) = \varepsilon,$$

$$\begin{aligned}
|S_2 - I_2| &\leqslant \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left|M\big(\varphi(\tau_k), \psi(\tau_k)\big) - M\big(\varphi(t), \psi(t)\big)\right| \left|\varphi^\Delta(t)\right| \Delta t \\
&= \sum_{t_k - t_{k-1} \leqslant \delta} (\cdots) + \sum_{t_k - t_{k-1} > \delta} (\cdots) = \sum_{t_k - t_{k-1} \leqslant \delta} (\cdots) \\
&\leqslant \sum_{t_k - t_{k-1} \leqslant \delta} \frac{\varepsilon}{M(b-a)} \int_{t_{k-1}}^{t_k} \left|\varphi^\Delta(t)\right| \Delta t \\
&\leqslant \frac{\varepsilon}{M(b-a)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left|\varphi^\Delta(t)\right| \Delta t \\
&= \frac{\varepsilon}{M(b-a)} \int_a^b \left|\varphi^\Delta(t)\right| \Delta t \\
&\leqslant \frac{\varepsilon}{M(b-a)} \cdot M(b-a) = \varepsilon.
\end{aligned}$$

Thus we have proved the existence of the line integrals (3.6) and (3.7) and the validity of the formulas (3.9) and (3.10), respectively. Note that we did not use the boundedness and Δ-integrability of ψ^Δ in deriving the formula (3.10). The proof is complete. □

Remark 3.4. We call the curve Γ given by (3.1) *piecewise* Δ*-smooth* if φ and ψ are continuous on $[a, b]$ and there is a partition $a = c_0 < c_1 < \cdots < c_m = b$ of $[a, b]$ such that φ and ψ have bounded and Δ-integrable Δ-derivatives on each of the intervals $[c_{i-1}, c_i)$, $i \in \{1, 2, \ldots, m\}$. In case of a piecewise Δ-smooth curve Γ, it is natural to define line Δ-integrals along this curve as sums of line Δ-integrals along all Δ-smooth parts constituting the curve Γ. Then the equalities (3.9)–(3.11) hold for piecewise Δ-smooth curves Γ as well. These equalities are valid also in case when the functions f, M, and N are only piecewise continuous along the curve Γ.

Remark 3.5. In case when it is necessary to emphasize that the contour Γ is closed, we will use for the line integral the symbol $\oint_\Gamma$ instead of $\int_\Gamma$.

Line Δ-integrals possess the same properties as those of ordinary Δ-integrals (proofs are similar to those given in [8, Chapter 5]). Note that under the more rigid conditions of Theorem 3.3 these properties follow at once from the formulas (3.9)–(3.11). Let us formulate these properties in connection with the first type line Δ-integrals.

Theorem 3.6 *(Linearity). Let the functions f and g be Δ-integrable along the curve Γ and let $\alpha, \beta \in \mathbb{R}$. Then $\alpha f + \beta g$ is also Δ-integrable along the curve Γ and*

$$\int_\Gamma \big[\alpha f(x,y) + \beta g(x,y)\big]\Delta\ell = \alpha \int_\Gamma f(x,y)\Delta\ell + \beta \int_\Gamma g(x,y)\Delta\ell.$$

Theorem 3.7 *(Additivity). If the curve AB consists of two parts AC and CB and if the function f is Δ-integrable along the curve AB, then it is Δ-integrable along each of the curves AC and CB and*

$$\int_{AB} f(x,y)\Delta\ell = \int_{AC} f(x,y)\Delta\ell + \int_{CB} f(x,y)\Delta\ell.$$

Theorem 3.8 *(Estimate of modulus of integral). If f is Δ-integrable along the curve Γ, then so is $|f|$ and*

$$\left| \int_\Gamma f(x,y)\Delta\ell \right| \leqslant \int_\Gamma \big|f(x,y)\big|\Delta\ell.$$

Theorem 3.9 *(Mean value formula). Let f be bounded and Δ-integrable along the curve Γ. Let us set*

$$m = \inf\big\{f(x,y)\colon (x,y) \in \Gamma\big\} \quad \text{and} \quad M = \sup\big\{f(x,y)\colon (x,y) \in \Gamma\big\}.$$

Then there exists a real number $\Lambda \in [m, M]$ such that

$$\int_{AB} f(x,y)\Delta\ell = \Lambda\ell(\Gamma),$$

where $\ell(\Gamma)$ is the length of the curve Γ.

Similarly to line delta integrals introduced above we can define *line nabla integrals*. Let Γ be a curve defined by the parametric equation (3.1) and let f, M, and N be functions defined on the curve Γ. Let $P = \{t_0, t_1, \ldots, t_n\} \subset [a, b]$, where $a = t_0 < t_1 < \cdots < t_n = b$, be a partition of $[a, b]$ and let $A_0, A_1, \ldots, A_n$ be points of the curve Γ with coordinates $(\varphi(t_i), \psi(t_i))$, $i \in \{0, 1, \ldots, n\}$, respectively. Denote by ℓ_k the length of the piece of Γ between its points A_{k-1} and A_k. By Theorem 2.11, the formula

$$\ell_k = \int_{t_{k-1}}^{t_k} \sqrt{\big[\varphi^\nabla(t)\big]^2 + \big[\psi^\nabla(t)\big]^2}\,\nabla t$$

holds. Take any $\tau_k \in (t_{k-1}, t_k]$ for $k \in \{1, 2, \ldots, n\}$ (note that in contrast to the delta integral we now choose the intermediate point τ_k in $(t_{k-1}, t_k]$ rather than in $[t_{k-1}, t_k)$) and introduce the three integral sums S_1', S_2', and S_3' as the right-hand sides of Eqs. (3.3), (3.4), and (3.5), respectively. Then we define the line nabla integrals

$$\int_\Gamma f(x,y)\nabla\ell, \qquad \int_\Gamma M(x,y)\nabla_1 x, \quad \text{and} \quad \int_\Gamma N(x,y)\nabla_2 y \tag{3.14}$$

to be the limits as $\delta \to 0$ of S_1', S_2', and S_3', respectively, assuming that S_1', S_2', and S_3' correspond to $P \in \mathcal{P}_\delta([a,b])$. The following theorem can then be proved similarly to the proof of Theorem 3.3.

Theorem 3.10. *Suppose that the curve Γ is given by the parametric equation* (3.1)*, where φ and ψ are continuous on $[a,b]$ and ∇-differentiable on $(a,b]$. If φ^∇ and ψ^∇ are bounded and ∇-integrable over $(a,b]$ and if the functions f, M, and N are continuous on Γ, then the line nabla integrals* (3.14) *exist and can be computed by the formulas*

$$\int_\Gamma f(x,y)\nabla\ell = \int_a^b f\big(\varphi(t),\psi(t)\big)\sqrt{\big[\varphi^\nabla(t)\big]^2+\big[\psi^\nabla(t)\big]^2}\,\nabla t,$$

$$\int_\Gamma M(x,y)\nabla_1 x = \int_a^b M\big(\varphi(t),\psi(t)\big)\varphi^\nabla(t)\nabla t,$$

$$\int_\Gamma N(x,y)\nabla_2 y = \int_a^b N\big(\varphi(t),\psi(t)\big)\psi^\nabla(t)\nabla t.$$

Remark 3.11. As was noted above in Section 2, the curve Γ can be thought of as an oriented curve "traversed" in the direction of increasing t from a to b. A second possible orientation of Γ arises by traversing it in the direction of decreasing t from b to a. Denote by $-\Gamma$ the curve obtained by reversing the orientation of Γ. It follows from the definitions of the line delta and nabla integrals that we have

$$\int_\Gamma f(x,y)\Delta\ell = \int_{-\Gamma} f(x,y)\nabla\ell,$$

$$\int_\Gamma M(x,y)\Delta_1 x = -\int_{-\Gamma} M(x,y)\nabla_1 x, \qquad \int_\Gamma N(x,y)\Delta_2 y = -\int_{-\Gamma} N(x,y)\nabla_2 y.$$

As an illustration of the last equalities we now consider the following example.

Example 3.12. Any ordered collection of points $\{A_0, A_1, \dots, A_n\}$ with the coordinates (x_k, y_k), respectively, determines an oriented (time scale) curve in $\mathbb{R}\times\mathbb{R}$, with the parametric equation

$$x = \varphi(t), \qquad y = \psi(t), \quad t \in [0,n] \subset \mathbb{Z},$$

where $\varphi(k) = x_k$ and $\psi(k) = y_k$ for $k \in [0,n] = \{0,1,\dots,n\}$. For any functions $f, M, N : \Gamma \to \mathbb{R}$ we have

$$\int_\Gamma f(x,y)\Delta\ell = \sum_{k=1}^n f(x_{k-1},y_{k-1})\sqrt{(x_k-x_{k-1})^2+(y_k-y_{k-1})^2},$$

$$\int_\Gamma M(x,y)\Delta_1 x = \sum_{k=1}^n M(x_{k-1},y_{k-1})(x_k-x_{k-1}),$$

$$\int_\Gamma N(x,y)\Delta_2 y = \sum_{k=1}^{n} N(x_{k-1}, y_{k-1})(y_k - y_{k-1}).$$

The curve $-\Gamma$ is given by the parametric equation

$$x = \varphi_1(t), \qquad y = \psi_1(t), \quad t \in [0,n] \subset \mathbb{Z},$$

where $\varphi_1(t) = \varphi(n-t)$ and $\psi_1(t) = \psi(n-t)$. Therefore

$$\begin{aligned}\int_{-\Gamma} f(x,y)\nabla\ell &= \sum_{k=1}^{n} f(x_{n-k}, y_{n-k})\sqrt{(x_{n-k} - x_{n-k+1})^2 + (y_{n-k} - y_{n-k+1})^2} \\ &= \sum_{j=1}^{n} f(x_{j-1}, y_{j-1})\sqrt{(x_{j-1} - x_j)^2 + (y_{j-1} - y_j)^2} \\ &= \int_\Gamma f(x,y)\Delta\ell, \\ \int_{-\Gamma} M(x,y)\nabla_1 x &= \sum_{k=1}^{n} M(x_{n-k}, y_{n-k})(x_{n-k} - x_{n-k+1}) \\ &= \sum_{j=1}^{n} M(x_{j-1}, y_{j-1})(x_{j-1} - x_j) \\ &= -\sum_{j=1}^{n} M(x_{j-1}, y_{j-1})(x_j - x_{j-1}) \\ &= -\int_\Gamma M(x,y)\Delta_1 x,\end{aligned}$$

and similarly for the remaining integral.

4. A time scale version of Green's formula

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales and put $\mathbb{T}_1 \times \mathbb{T}_2 = \{(x,y)\colon x \in \mathbb{T}_1,\ y \in \mathbb{T}_2\}$. Since $\mathbb{T}_1$ and $\mathbb{T}_2$ are closed subsets of $\mathbb{R}$, the set $\mathbb{T}_1 \times \mathbb{T}_2$ is a complete metric space with the metric (distance) d defined by

$$d\big((x,y),(x',y')\big) = \sqrt{(x-x')^2 + (y-y')^2} \quad \text{for } (x,y),(x',y') \in \mathbb{T}_1 \times \mathbb{T}_2.$$

Consequently, according to the well-known theory of general metric spaces, we have for $\mathbb{T}_1 \times \mathbb{T}_2$ the fundamental concepts such as open balls (disks), neighborhoods of points, open sets, closed sets, compact sets, boundary of a set, and so on. There is also the concept of a continuous curve for general metric spaces and associated with it the concept of connectedness (arcwise connectedness). Namely, if $\mathcal{M}$ is a metric space, then any continuous mapping $h : [a,b] \to \mathcal{M}$ of the real line interval $[a,b]$ into the metric space $\mathcal{M}$ is called a (continuous) curve in $\mathcal{M}$. Above, in Section 2, we generalized the concept of continuous curve taking as $[a,b]$ an interval of a time scale instead of the reals $\mathbb{R}$. Accordingly, we can generalize the concept of arcwise connectedness to $\mathbb{T}_1 \times \mathbb{T}_2$.

Definition 4.1. Let $[a,b]$ be an interval in $\mathbb{T}_1$ with $a,b\in\mathbb{T}_1$ and $y_0\in\mathbb{T}_2$. The set

$$\big\{(x,y_0):\ x\in[a,b]\big\}$$

is called a horizontal line segment in $\mathbb{T}_1\times\mathbb{T}_2$ and denoted by AB, where $A=(a,y_0)$ and $B=(b,y_0)$. Similarly, taking $x_0\in\mathbb{T}_1$ and $[c,d]\subset\mathbb{T}_2$, we define a vertical line segment in $\mathbb{T}_1\times\mathbb{T}_2$ as the set

$$\big\{(x_0,y):\ y\in[c,d]\big\}$$

and denote it by CD, where $C=(x_0,c)$ and $D=(x_0,d)$.

Definition 4.2. A finite sequence $P_1Q_1, P_2Q_2, \dots, P_nQ_n$, each of whose term P_kQ_k is a horizontal or vertical line segment in $\mathbb{T}_1\times\mathbb{T}_2$, is said to form a *polygonal path* (or *broken line*) in $\mathbb{T}_1\times\mathbb{T}_2$ with terminal points P_1 and Q_n if $Q_1=P_2, Q_2=P_3,\dots,Q_{n-1}=P_n$. A set of points of $\mathbb{T}_1\times\mathbb{T}_2$ is said to be *connected* if any two of its points are terminal points of a polygonal path of points contained in the set. A *component* of a set $\Omega\subset\mathbb{T}_1\times\mathbb{T}_2$ is a nonempty maximal connected subset of Ω. A nonempty open connected set of points of $\mathbb{T}_1\times\mathbb{T}_2$ is called a *domain*. A *closed domain* is a subset in $\mathbb{T}_1\times\mathbb{T}_2$ being the closure of a domain in $\mathbb{T}_1\times\mathbb{T}_2$.

Suppose $a<b$ are points in $\mathbb{T}_1$, $c<d$ are points in $\mathbb{T}_2$, $[a,b)$ is the half-closed bounded interval in $\mathbb{T}_1$, and $[c,d)$ is the half-closed bounded interval in $\mathbb{T}_2$. Let us introduce a "rectangle" in $\mathbb{T}_1\times\mathbb{T}_2$ by

$$R=[a,b)\times[c,d)=\big\{(x,y):\ x\in[a,b),\ y\in[c,d)\big\}. \tag{4.1}$$

Let us set

$$L_1=\big\{(x,c):\ x\in[a,b]\big\},\qquad L_2=\big\{(b,y):\ y\in[c,d]\big\},$$
$$L_3=\big\{(x,d):\ x\in[a,b]\big\},\qquad L_4=\big\{(a,y):\ y\in[c,d]\big\}.$$

Each of L_j for $j\in\{1,2,3,4\}$ is an oriented line segment; for example, the positive orientation of L_1 arises according to the increase of x from a to b and the positive orientation of L_2 arises according to the increase of y from c to d. The set (closed curve)

$$\Gamma:=L_1\cup L_2\cup(-L_3)\cup(-L_4) \tag{4.2}$$

is called the *positively oriented fence* of R. Positivity of orientation of Γ means that the rectangle R remains on the "left" side as we describe the fence curve Γ.

Definition 4.3. We say that $E\subset\mathbb{T}_1\times\mathbb{T}_2$ is a *set of the type* ω if it is a connected set in $\mathbb{T}_1\times\mathbb{T}_2$ being the union of a finite number of rectangles of the form (4.1) that are pairwise disjoint and adjoining to each other. Let $E\subset\mathbb{T}_1\times\mathbb{T}_2$ be a set of the type ω so that $E=\bigcup_{k=1}^m R_k$, where $R_k=[a_k,b_k)\times[c_k,d_k)\subset\mathbb{T}_1\times\mathbb{T}_2$ for each $k\in\{1,2,\dots,m\}$ and $R_1,R_2,\dots,R_m$ are pairwise disjoint and adjoining to each other. Let Γ_k be the positively oriented fence of the rectangle R_k. Let us set $X=\bigcup_{k=1}^m\Gamma_k$. Further, let X_0 consist of a finite number of line segments each of which serves as a common part of fences of two adjoining rectangles belonging to $\{R_1,R_2,\dots,R_m\}$. Then the set $\Gamma=X\setminus X_0$ forms a positively oriented closed "polygonal curve," which we call the *positively oriented fence of the set* E (the set E remains on the "left" as we describe the fence curve Γ).

Theorem 4.4 *(Green's formula). Let $E \subset \mathbb{T}_1 \times \mathbb{T}_2$ be a set of the type ω and let Γ be its positively oriented fence. If the functions M and N are continuous and have continuous partial delta derivatives $\partial M/\Delta_2 y$ and $\partial N/\Delta_1 x$ on $E \cup \Gamma$, then*

$$\iint_E \left(\frac{\partial N}{\Delta_1 x} - \frac{\partial M}{\Delta_2 y}\right)\Delta_1 x\,\Delta_2 y = \int_\Gamma M\,\mathrm{d}^* x + N\,\mathrm{d}^* y, \tag{4.3}$$

where the "star line integrals" on the right-hand side in (4.3) *denote the sum of line delta integrals taken over the line segment constituents of Γ directed to the right or upwards and line nabla integrals of f taken over the line segment constituents of Γ directed to the left or downwards.*

Proof. First we assume that E is a rectangle R of the form (4.1). Let L_1, L_2, L_3, and L_4 be the oriented line segments defined above and let Γ be the positively oriented fence of R defined by (4.2). Using the formula reducing double delta integrals to repeated delta integrals (see [5, Theorem 3.10]), we have

$$\iint_R \frac{\partial M}{\Delta_2 y}\Delta_1 x\,\Delta_2 y = \int_a^b \Delta_1 x \int_c^d \frac{\partial M}{\Delta_2 y}\Delta_2 y.$$

Further, by the fundamental theorem of time scales calculus (see [8, Theorem 5.34] or [10, Theorem 4.1]) we have

$$\int_c^d \frac{\partial M}{\Delta_2 y}\Delta_2 y = M(x,d) - M(x,c).$$

Therefore

$$\iint_R \frac{\partial M}{\Delta_2 y}\Delta_1 x\,\Delta_2 y = \int_a^b M(x,d)\Delta_1 x - \int_a^b M(x,c)\Delta_1 x.$$

Next, each of the last two integrals can be replaced by a line delta integral. Indeed, taking into account Theorem 3.3, we see that

$$\int_a^b M(x,d)\Delta_1 x = \int_{L_3} M(x,y)\Delta_1 x \quad \text{and} \quad \int_a^b M(x,c)\Delta_1 x = \int_{L_1} M(x,y)\Delta_1 x.$$

Hence

$$\begin{aligned}\iint_R \frac{\partial M}{\Delta_2 y}\Delta_1 x\,\Delta_2 y &= \int_{L_3} M(x,y)\Delta_1 x - \int_{L_1} M(x,y)\Delta_1 x \\ &= -\int_{-L_3} M(x,y)\nabla_1 x - \int_{L_1} M(x,y)\Delta_1 x,\end{aligned}$$

where in the last equality we have used Remark 3.11. Now considering the integral along the whole fence Γ of R, we add to the right-hand side of the obtained equality the two integrals

$$-\int_{L_2} M(x,y)\Delta_1 x \quad \text{and} \quad -\int_{-L_4} M(x,y)\nabla_1 x,$$

each of which is, obviously, equal to zero, since the variable x remains fixed over L_2 and $-L_4$. We obtain

$$\iint\limits_R \frac{\partial M}{\Delta_2 y} \Delta_1 x\, \Delta_2 y = -\int\limits_\Gamma M(x, y)\, \mathrm{d}^* x. \tag{4.4}$$

Similarly we can show

$$\iint\limits_R \frac{\partial N}{\Delta_1 x} \Delta_1 x\, \Delta_2 y = \int\limits_\Gamma N(x, y)\, \mathrm{d}^* y. \tag{4.5}$$

Subtracting (4.4) from (4.5) yields (4.3). Thus we have proved (4.3) if $E = R$ is a rectangle of the form (4.1).

Now we assume that E is a set of the type ω, i.e., we assume that $E = \bigcup_{k=1}^m R_k$, where $R_k = [a_k, b_k) \times [c_k, d_k) \subset \mathbb{T}_1 \times \mathbb{T}_2$ for each $k \in \{1, 2, \ldots, m\}$ and $R_1, R_2, \ldots, R_m$ are pairwise disjoint. Let Γ_k be the positively oriented fence of the rectangle R_k. Then, since Green's formula already applies to each of the rectangles R_k, we obtain

$$\iint\limits_E \left(\frac{\partial N}{\Delta_1 x} - \frac{\partial M}{\Delta_2 y}\right) \Delta_1 x\, \Delta_2 y = \sum_{k=1}^m \iint\limits_{R_k} \left(\frac{\partial N}{\Delta_1 x} - \frac{\partial M}{\Delta_2 y}\right) \Delta_1 x\, \Delta_2 y$$
$$= \sum_{k=1}^n \int\limits_{\Gamma_k} M\, \mathrm{d}^* x + N\, \mathrm{d}^* y = \int\limits_\Gamma M\, \mathrm{d}^* x + N\, \mathrm{d}^* y.$$

The last of these equalities is explained as follows. The fence Γ of E is obtained by removing from the union of $\Gamma_1, \ldots, \Gamma_m$ a finite number of line segments each of which serves as a common part of the fences of two adjoining rectangles belonging to $\{R_1, R_2, \ldots, R_m\}$. Every such line segment is traversed twice in the opposite directions; therefore the integrals corresponding to these paths of integration mutually cancel out and thus only the integral over Γ remains. □

For a version of Green's formula in the case $\mathbb{Z} \times \mathbb{Z}$, we refer the reader to [9].

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