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MULTIPLE INTEGRATION ON TIME SCALES

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ABSTRACT. In this paper an introduction to integration theory for multivariable functions on time scales is given. Such an integral calculus can be used to develop a theory of partial dynamic equations on time scales in order to unify and extend the usual partial differential equations and partial difference equations.

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1. INTRODUCTION

A time scale is an arbitrary nonempty closed subset of the real numbers. For a general introduction to the calculus of time scales we refer the reader to the textbooks [6, 7]. In [5] a differential calculus for multivariable functions on time scales was presented by the authors in order to provide an instrument for introducing and investigating partial dynamic equations on time scales. The present paper continues [5] and discusses multiple integration on time scales.

In the original papers of B. Aulbach and S. Hilger [3, 10] on single variable time scales calculus the concept of integral was defined by means of an antiderivative (or pre-antiderivative) of a function and called the Cauchy integral. Next by S. Sailer [12] the Darboux definition of the integral was used for integral calculus on time scales. Further Riemann and Lebesgue definitions of the integral on time scales were introduced in [4, 7, 8, 9] and a complete, to a considered extent, theory of integration for single variable time scales was developed.

In [1], C. Ahlbrandt and C. Morian introduced double integrals over rectangles on time scales as iterated integrals defined by using antiderivatives of single variable functions, under the assumption that the order of integration in the iterated integral can be reversed. In the present paper we introduce Darboux and Riemann definitions of multiple integrals on time scales over arbitrary regions. For simplicity we confine ourselves to functions of two variables. Also we consider only delta integrals. Nabla

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integrals and mixed integrals involving delta integration with respect to a part of the variables and nabla integration with respect to the other part of the variables can be investigated in a similar manner.

The paper is organized as follows. In Section 2 we introduce double Darboux and Riemann Δ-integrals over rectangles. We show that the two definitions are equivalent and give several Cauchy criteria for Δ-integrability. Some basic examples are provided. Next, in Section 3 we present many properties of double Δ-integrals over rectangles, among them integrability of the product and of the composite function, additivity and linearity of the integral, and the mean value theorem. We also show that every continuous function is Δ-integrable and establish a reduction formula for a double integral to an iterated integral. Finally, in Section 4 we extend Riemann Δ-integrability over rectangles to more general sets, so-called Jordan Δ-measurable sets. To this end, the concept of Δ-boundary of a set is introduced. We give two definitions of the double integral over general sets and then prove their equivalence for Jordan Δ-measurable sets. The main properties of the double integral over Jordan Δ-measurable sets are presented. Lebesgue's definition of multiple integrals, line integrals, and Green's formula for time scales will be presented in a forthcoming paper of the authors.

2. **DOUBLE RIEMANN INTEGRALS OVER RECTANGLES**

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales. For $i = 1, 2$ let σ_i, ρ_i, and Δ_i denote the forward jump operator, the backward jump operator, and the delta differentiation operator, respectively, on $\mathbb{T}_i$. Suppose $a < b$ are points in $\mathbb{T}_1$, $c < d$ are points in $\mathbb{T}_2$, $[a, b)$ is the half-closed bounded interval in $\mathbb{T}_1$, and $[c, d)$ is the half-closed bounded interval in $\mathbb{T}_2$. Let us introduce a "rectangle" in $\mathbb{T}_1 \times \mathbb{T}_2$ by

$$R = [a, b) \times [c, d) = \{(t, s) :\ t \in [a, b)\ s \in [c, d)\}.$$

Let

$$\{t_0, t_1, \ldots, t_n\} \subset [a, b], \quad \text{where} \quad a = t_0 < t_1 < \ldots < t_n = b$$

and

$$\{s_0, s_1, \ldots, s_k\} \subset [c, d], \quad \text{where} \quad c = s_0 < s_1 < \ldots < s_k = d.$$

The numbers n and k may be arbitrary positive integers. We call the collection of intervals

$$P_1 = \{[t_{i-1}, t_i) :\ 1 \le i \le n\}$$

a Δ-*partition* of $[a, b)$ and denote the set of all Δ-partitions of $[a, b)$ by $\mathcal{P}([a, b))$. Similarly, the collection of intervals

$$P_2 = \{[s_{j-1}, s_j) :\ 1 \le j \le k\}$$

is called a Δ-partition of $[c, d)$ and the set of all Δ-partitions of $[c, d)$ is denoted by $\mathcal{P}([c, d))$. Let us set

$$R_{ij} = [t_{i-1}, t_i) \times [s_{j-1}, s_j), \quad \text{where} \quad 1 \le i \le n,\ 1 \le j \le k. \tag{2.1}$$

We call the collection

$$P = \{R_{ij} :\ 1 \le i \le n,\ 1 \le j \le k\} \tag{2.2}$$

a Δ-partition of R, generated by the Δ-partitions P_1 and P_2 of $[a, b)$ and $[c, d)$, respectively, and write $P = P_1 \times P_2$. The rectangles R_{ij}, $1 \le i \le n$, $1 \le j \le k$, are called the subrectangles of the partition P. The set of all Δ-partitions of R is denoted by $\mathcal{P}(R)$.

Let $f : R \to \mathbb{R}$ be a bounded function. We set

$$M = \sup\{f(t, s) :\ (t, s) \in R\} \quad \text{and} \quad m = \inf\{f(t, s) :\ (t, s) \in R\}$$

and for $1 \le i \le n$, $1 \le j \le k$,

$$M_{ij} = \sup\{f(t, s) :\ (t, s) \in R_{ij}\} \quad \text{and} \quad m_{ij} = \inf\{f(t, s) :\ (t, s) \in R_{ij}\}.$$

The *upper Darboux Δ-sum* $U(f, P)$ and the *lower Darboux Δ-sum* $L(f, P)$ of f with respect to P are defined by

$$U(f, P) = \sum_{i=1}^{n} \sum_{j=1}^{k} M_{ij}(t_i - t_{i-1})(s_j - s_{j-1})$$

and

$$L(f, P) = \sum_{i=1}^{n} \sum_{j=1}^{k} m_{ij}(t_i - t_{i-1})(s_j - s_{j-1}).$$

Note that

$$U(f, P) \le \sum_{i=1}^{n} \sum_{j=1}^{k} M(t_i - t_{i-1})(s_j - s_{j-1}) = M(b - a)(d - c)$$

and likewise $L(f, P) \ge m(b - a)(d - c)$ so that

$$m(b - a)(d - c) \le L(f, P) \le U(f, P) \le M(b - a)(d - c). \tag{2.3}$$

The *upper Darboux Δ-integral* $U(f)$ of f over R and the *lower Darboux Δ-integral* $L(f)$ of f over R are defined by

$$U(f) = \inf\{U(f, P) :\ P \in \mathcal{P}(R)\} \quad \text{and} \quad L(f) = \sup\{L(f, P) :\ P \in \mathcal{P}(R)\}.$$

In view of (2.3), $U(f)$ and $L(f)$ are finite real numbers. We will see in Theorem 2.5 that $L(f) \le U(f)$.

Definition 2.1. We say that f is *Δ-integrable* (or *delta integrable*) over R provided $L(f) = U(f)$. In this case, we write $\int\int_R f(t, s)\Delta_1 t \Delta_2 s$ for this common value. We call this integral the *Darboux Δ-integral*.

Riemann's definition of the integral is a little different (see Definition 2.13 below), but we will show in Theorem 2.14 that the two definitions are equivalent. For this reason, we will also call the integral defined in Definition 2.1 the *Riemann Δ-integral*.

Let $P, Q \in \mathcal{P}(R)$ and $P = P_1 \times P_2$, $Q = Q_1 \times Q_2$, where

$$P_1, Q_1 \in \mathcal{P}([a,b)) \quad \text{and} \quad P_2, Q_2 \in \mathcal{P}([c,d)).$$

We say that Q is a *refinement* of P if Q_1 is a refinement of P_1 and Q_2 is a refinement of P_2.

Lemma 2.2. *Let f be a bounded function on R. If P and Q are Δ-partitions of R and Q is a refinement of P, then*

$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P),$$

i.e., refining of a partition increases the lower sum and decreases the upper sum.

Proof. The middle inequality is obvious. The proofs of the first and third inequalities are similar, so we only prove $L(f,P) \le L(f,Q)$. An induction argument shows that we may assume that Q has only one more element than P. If P is given by

$$P = \{R_1, R_2, \dots, R_N\}$$

(every partition (2.2) can be labeled in this form, and the order in which those subrectangles are labeled makes no difference), then there exists some $k \in \{1, \dots, N\}$ such that Q is given by

$$Q = \{R_1, \dots, R_{k-1}, R_k', R_k'', R_{k+1}, \dots, R_N\},$$

where $R_k' \cup R_k'' = R_k$. Now setting $m_k = \inf_{(t,s)\in R_k} f(t,s)$, $m_k^{(1)} = \inf_{(t,s)\in R_k'} f(t,s)$, and $m_k^{(2)} = \inf_{(t,s)\in R_k''} f(t,s)$, we have $m_k^{(1)} \ge m_k$, $m_k^{(2)} \ge m_k$ so that

$$\begin{aligned} L(f,Q) - L(f,P) &= m_k^{(1)} m(R_k') + m_k^{(2)} m(R_k'') - m_k m(R_k) \\ &\ge m_k m(R_k') + m_k m(R_k'') - m_k m(R_k) = 0, \end{aligned}$$

where for a given rectangle $V = [\alpha, \beta) \times [\gamma, \delta) \subset \mathbb{T}_1 \times \mathbb{T}_2$ the "area" of V, i.e., $(\beta - \alpha)(\delta - \gamma)$, is denoted by $m(V)$. Therefore $L(f,P) \le L(f,Q)$. $\square$

Definition 2.3. Suppose $P = P_1 \times P_2$ and $Q = Q_1 \times Q_2$, where $P_1, Q_1 \in \mathcal{P}([a,b))$ and $P_2, Q_2 \in \mathcal{P}([c,d))$, are two Δ-partitions of $R = [a,b) \times [c,d)$. If P_1 is generated by a set

$$\{t_0, t_1, \dots, t_n\} \subset [a,b], \quad \text{where} \quad a = t_0 < t_1 < \dots < t_n = b$$

and Q_1 is generated by a set

$$\{\tau_0, \tau_1, \dots, \tau_p\} \subset [a,b], \quad \text{where} \quad a = \tau_0 < \tau_1 < \dots < \tau_p = b,$$

then by $P_1 + Q_1$ we denote the Δ-partition of $[a,b)$ generated by the set

$$\{t_0, t_1, \dots, t_n\} \cup \{\tau_0, \tau_1, \dots, \tau_p\}.$$

Similarly we define $P_2 + Q_2$, a Δ-partition of $[c, d)$. Then we denote the Δ-partition $(P_1 + Q_1) \times (P_2 + Q_2)$ of R by $P + Q$.

Obviously $P + Q$ is a refinement of both P and Q.

Lemma 2.4. *If f is a bounded function on R and if P and Q are any two Δ-partitions of R, then $L(f, P) \leq U(f, Q)$, i.e., every lower sum is less than or equal to every upper sum.*

Proof. Since $P + Q$ is a Δ-partition of R which is a refinement of both P and Q, we can apply Lemma 2.2 to obtain

$$L(f, P) \leq L(f, P + Q) \leq U(f, P + Q) \leq U(f, Q),$$

i.e., $L(f, P) \leq U(f, Q)$. □

Theorem 2.5. *If f is a bounded function on R, then $L(f) \leq U(f)$.*

Proof. Fix $P \in \mathcal{P}(R)$. By Lemma 2.4, $L(f, P)$ is a lower bound for the set

$$\{U(f, Q) : \ Q \in \mathcal{P}(R)\}.$$

Therefore $L(f, P)$ must be less than or equal to the greatest lower bound (infimum) of this set. That is,

$$L(f, P) \leq U(f). \tag{2.4}$$

Now (2.4) shows that $U(f)$ is an upper bound for the set

$$\{L(f, P) : \ P \in \mathcal{P}(R)\}$$

so that $U(f) \geq L(f)$. □

It follows that

$$L(f, P) \leq L(f) \leq U(f) \leq U(f, Q) \quad \text{for all} \quad P, Q \in \mathcal{P}(R).$$

In particular

$$L(f, P) \leq L(f) \leq U(f) \leq U(f, P) \quad \text{for all} \quad P \in \mathcal{P}(R). \tag{2.5}$$

From (2.5) we get the following result.

Theorem 2.6. *If $L(f, P) = U(f, P)$ for some $P \in \mathcal{P}(R)$, then the function f is Δ-integrable over R and*

$$\int\int_R f(t, s)\Delta_1 t \Delta_2 s = L(f, P) = U(f, P).$$

The next theorem gives a “Cauchy criterion” for integrability.

Theorem 2.7. *A bounded function f on R is Δ-integrable if and only if for each $\varepsilon > 0$ there exists $P \in \mathcal{P}(R)$ such that*

$$U(f,P) - L(f,P) < \varepsilon. \tag{2.6}$$

Proof. Suppose that f is Δ-integrable and consider $\varepsilon > 0$. By the definitions of supremum and infimum, there exist $H, Q \in \mathcal{P}(R)$ satisfying

$$L(f,H) > L(f) - \frac{\varepsilon}{2} \quad \text{and} \quad U(f,Q) < U(f) + \frac{\varepsilon}{2}.$$

Let now $P = H + Q$ (for the definition of $P + Q$ see Definition 2.3) which is a refinement of both H and Q. Therefore we can apply Lemma 2.2 to obtain

$$U(f,P) - L(f,P) \le U(f,Q) - L(f,H) < U(f) + \frac{\varepsilon}{2} - \Big(L(f) - \frac{\varepsilon}{2}\Big) = U(f) - L(f) + \varepsilon.$$

Since f is Δ-integrable, $U(f) = L(f)$ so that (2.6) holds.

Conversely, suppose that for each $\varepsilon > 0$ the inequality (2.6) holds for some $P \in \mathcal{P}(R)$. Then we have

$$U(f) \le U(f,P) = U(f,P) - L(f,P) + L(f,P) < \varepsilon + L(f,P) \le \varepsilon + L(f).$$

Since $\varepsilon > 0$ is arbitrary, it follows that $U(f) \le L(f)$, and in view of Theorem 2.5 we conclude that $U(f) = L(f)$, i.e., f is Δ-integrable. $\square$

We need the following auxiliary result. The proof can be found in [7, 9].

Lemma 2.8. *For every $\delta > 0$ there exists at least one partition $P_1 \in \mathcal{P}([a,b))$ generated by a set*

$$\{t_0, t_1, \ldots, t_n\} \subset [a,b], \quad \textit{where} \quad a = t_0 < t_1 < \ldots < t_n = b$$

such that for each $i \in \{1, 2, \ldots, n\}$ either

$$t_i - t_{i-1} \le \delta$$

or

$$t_i - t_{i-1} > \delta \quad \textit{and} \quad \rho_1(t_i) = t_{i-1}.$$

Definition 2.9. We denote by $\mathcal{P}_\delta([a,b))$ the set of all $P_1 \in \mathcal{P}([a,b))$ that possess the property indicated in Lemma 2.8. Similarly we define $\mathcal{P}_\delta([c,d))$. Further, by $\mathcal{P}_\delta(R)$ we denote the set of all $P \in \mathcal{P}(R)$ such that

$$P = P_1 \times P_2, \quad \text{where} \quad P_1 \in \mathcal{P}_\delta([a,b)) \quad \text{and} \quad P_2 \in \mathcal{P}_\delta([c,d)).$$

Lemma 2.10. *Let $P^0 \in \mathcal{P}(R)$ be given by $P^0 = P_1^0 \times P_2^0$ in which $P_1^0 \in \mathcal{P}([a,b))$ is generated by a set*

$$A_1^0 = \left\{t_0^0, t_1^0, \ldots, t_n^0\right\} \subset [a,b], \quad \textit{where} \quad a = t_0^0 < t_1^0 < \ldots < t_n^0 = b$$

and $P_2^0 \in \mathcal{P}([c,d))$ *is generated by a set*

$$A_2^0 = \{s_0^0, s_1^0, \ldots, s_l^0\} \subset [c,d], \quad \textit{where} \quad c = s_0^0 < s_1^0 < \ldots < s_l^0 = d.$$

Then for each $P \in \mathcal{P}_\delta(R)$ *we have*

$$L(f, P^0 + P) - L(f, P) \le (M - m)D(n + l - 2)\delta$$

and

$$U(f, P) - U(f, P^0 + P) \le (M - m)D(n + l - 2)\delta,$$

where the sum $P^0 + P$ *of the two partitions* $P^0, P \in \mathcal{P}(R)$ *is defined according to Definiton 2.3,* m *and* M *are the infimum and supremum of* f *on* R*, respectively, and* $D = \max\{b - a, d - c\}$*.*

Proof. Suppose the partition P is given by $P = P_1 \times P_2$ in which $P_1 \in \mathcal{P}([a,b))$ is generated by a set

$$A_1 = \{t_0, t_1, \ldots, t_p\} \subset [a,b], \quad \text{where} \quad a = t_0 < t_1 < \ldots < t_p = b$$

and $P_2 \in \mathcal{P}([c,d))$ is generated by a set

$$A_2 = \{s_0, s_1, \ldots, s_q\} \subset [c,d], \quad \text{where} \quad c = s_0 < s_1 < \ldots < s_q = d.$$

Let $Q = P^0 + P = Q_1 \times Q_2$, where $Q_1 \in \mathcal{P}([a,b))$ and $Q_2 \in \mathcal{P}([c,d))$ are generated by the sets

$$B_1 = A_1^0 \cup A_1 \quad \text{and} \quad B_2 = A_2^0 \cup A_2,$$

respectively. First we consider two particular cases.

(i) If B_1 has one more point, say t', than A_1 and $B_2 = A_2$, then $t' \in (t_{k-1}, t_k)$ for some $k \in \{1, 2, \ldots, p\}$, where $t_k - t_{k-1} \le \delta$. Indeed, if $t_k - t_{k-1} > \delta$, then by the condition $P \in \mathcal{P}_\delta(R)$ we have $\rho_1(t_k) = t_{k-1}$ and therefore $(t_{k-1}, t_k) = \emptyset$. Now denoting by m_{kj}, $m_{kj}^{(1)}$, and $m_{kj}^{(2)}$ the infima of f on

$$R_{kj} = [t_{k-1}, t_k) \times [s_{j-1}, s_j), \quad R_{kj}^{(1)} = [t_{k-1}, t') \times [s_{j-1}, s_j), \quad R_{kj}^{(2)} = [t', t_k) \times [s_{j-1}, s_j),$$

respectively, we have

$$m_{kj}^{(1)} \ge m_{kj}, \quad m_{kj}^{(2)} \ge m_{kj}, \quad m_{kj}^{(1)} - m_{kj} \le M - m, \quad m_{kj}^{(2)} - m_{kj} \le M - m,$$

and $m(R_{kj}) = m(R_{kj}^{(1)}) + m(R_{kj}^{(2)})$, so that

$$\begin{aligned} L(f,Q) - L(f,P) &= \sum_{j=1}^{q} \left\{ m_{kj}^{(1)} m(R_{kj}^{(1)}) + m_{kj}^{(2)} m(R_{kj}^{(2)}) - m_{kj} m(R_{kj}) \right\} \\ &= \sum_{j=1}^{q} \left\{ (m_{kj}^{(1)} - m_{kj}) m(R_{kj}^{(1)}) + (m_{kj}^{(2)} - m_{kj}) m(R_{kj}^{(2)}) \right\} \\ &\le (M - m) \sum_{j=1}^{q} \left\{ m(R_{kj}^{(1)}) + m(R_{kj}^{(2)}) \right\} \end{aligned}$$

$$\begin{aligned} &= (M-m)\sum_{j=1}^{q} m(R_{kj}) \;=\; (M-m)\sum_{j=1}^{q}(t_k-t_{k-1})(s_j-s_{j-1}) \\ &= (M-m)(t_k-t_{k-1})(d-c) \;\le\; (M-m)D\delta. \end{aligned}$$

(ii) If $B_1 = A_1$ and B_2 has one more point than A_2, then in a similar way as in the case (i) we again get

$$L(f,Q) - L(f,P) \le (M-m)D\delta.$$

Since B_1 has at most $n-1$ points that are not in A_1 and B_2 has at most $l-1$ points that are not in A_2, an induction argument based on the cases (i) and (ii) shows that

$$L(f,Q) - L(f,P) \le (M-m)D(n+l-2)\delta.$$

The proof for the other inequality is similar. □

The following is another Cauchy criterion for integrability.

Theorem 2.11. *A bounded function f on R is Δ-integrable if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$P \in \mathcal{P}_\delta(R) \quad \textit{implies} \quad U(f,P) - L(f,P) < \varepsilon. \tag{2.7}$$

Proof. Theorem 2.7 shows that the ε-δ condition (2.7) implies Δ-integrability. Conversely, suppose that f is Δ-integrable over R. Let $\varepsilon > 0$ and select $P^0 \in \mathcal{P}(R)$ such that

$$U(f,P^0) - L(f,P^0) < \frac{\varepsilon}{2}.$$

Suppose P^0 is given by $P^0 = P_1^0 \times P_2^0$ in which $P_1^0 \in \mathcal{P}([a,b))$ is generated by a set

$$\left\{t_0^0, t_1^0, \ldots, t_n^0\right\} \subset [a,b], \quad \text{where} \quad a = t_0^0 < t_1^0 < \ldots < t_n^0 = b$$

and $P_2^0 \in \mathcal{P}([c,d))$ is generated by a set

$$\left\{s_0^0, s_1^0, \ldots, s_l^0\right\} \subset [c,d], \quad \text{where} \quad c = s_0^0 < s_1^0 < \ldots < s_l^0 = d.$$

Let (without loss of generality f is not identically constant)

$$\delta = \frac{\varepsilon}{4(n+l)(M-m)D}, \quad \text{where} \quad D = \max\{b-a, d-c\}$$

and m and M are the infimum and supremum of f on R, respectively. Then for any $P \in \mathcal{P}_\delta(R)$ we have, by Lemma 2.10,

$$\begin{aligned} L(f,P^0+P) - L(f,P) &\le (M-m)D(n+l-2)\delta \\ &= (M-m)D(n+l-2)\frac{\varepsilon}{4(n+l)(M-m)D} \\ &= \frac{(n+l-2)\varepsilon}{4(n+l)} < \frac{\varepsilon}{4}. \end{aligned}$$

By Lemma 2.2 we have $L(f, P^0) \leq L(f, P^0 + P)$, and so

$$L(f, P^0) - L(f, P) < \frac{\varepsilon}{4} \quad \text{and similarly} \quad U(f, P) - U(f, P^0) < \frac{\varepsilon}{4}.$$

Hence

$$U(f, P) - L(f, P) < U(f, P^0) - L(f, P^0) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus we have verified (2.7). □

Theorem 2.12. *For every bounded function f on R the Darboux Δ-sums $L(f, P)$ and $U(f, P)$ evaluated for $P \in \mathcal{P}_\delta(R)$ have limits as $\delta \to 0$, uniformly with respect to P, and*

$$\lim_{\delta \to 0} L(f, P) = L(f) \quad \textit{and} \quad \lim_{\delta \to 0} U(f, P) = U(f).$$

Proof. Let us prove the statement for lower Darboux Δ-sums (the proof for upper Darboux Δ-sums is analogous). Fix an $\varepsilon > 0$ and choose a partition $P^0 \in \mathcal{P}(R)$ such that

$$L(f, P^0) > L(f) - \varepsilon, \quad \text{that is,} \quad L(f) - L(f, P^0) < \varepsilon.$$

Let P^0 be described as in Lemma 2.10. Then for any $P \in \mathcal{P}_\delta(R)$ we have, by Lemma 2.10,

$$L(f, P^0 + P) - L(f, P) \leq (M - m)D(n + l - 2)\delta.$$

Since $P^0 + P$ is a refinement of P^0, we have $L(f, P^0) \leq L(f, P^0 + P)$ by Lemma 2.2. Thus

$$L(f) - \varepsilon < L(f, P^0) \leq L(f, P^0 + P) \leq L(f) \quad \text{and hence} \quad L(f, P^0 + P) - L(f, P^0) < \varepsilon.$$

Therefore

$$\begin{aligned} |L(f) - L(f, P)| &\leq \left|L(f) - L(f, P^0)\right| + \left|L(f, P^0) - L(f, P^0 + P)\right| \\ &\quad + \left|L(f, P^0 + P) - L(f, P)\right| \\ &< \varepsilon + \varepsilon + (M - m)D(n + l - 2)\delta. \end{aligned}$$

Taking $\delta = \varepsilon/[(M - m)D(n + l - 2)]$ (since the case when f is constant is obvious, we may assume that $M - m \neq 0$), we get $|L(f) - L(f, P)| < 3\varepsilon$. This completes the proof. □

We now give Riemann's definition of integrability.

Definition 2.13. Let f be a bounded function on R and $P \in \mathcal{P}(R)$ be given by (2.1), (2.2). In each "rectangle" R_{ij} with $1 \leq i \leq n$, $1 \leq j \leq k$, choose an arbitrary point (ξ_{ij}, η_{ij}) and form the sum

$$S = \sum_{i=1}^{n} \sum_{j=1}^{k} f(\xi_{ij}, \eta_{ij})(t_i - t_{i-1})(s_j - s_{j-1}). \tag{2.8}$$

We call S a *Riemann Δ-sum* of f corresponding to $P \in \mathcal{P}(R)$. We say that f is *Riemann Δ-integrable* over R if there exists a number I with the following property: For each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|S - I| < \varepsilon$$

for every Riemann Δ-sum S of f corresponding to any $P \in \mathcal{P}_\delta(R)$ independent of the way in which we choose $(\xi_{ij}, \eta_{ij}) \in R_{ij}$ for $1 \le i \le n$, $1 \le j \le k$. The number I is the *Riemann Δ-integral* of f over R, and we write $I = \lim_{\delta \to 0} S$.

It is easy to see that the number I from Definition 2.13 is unique if it exists. Hence the Riemann Δ-integral is well defined. Note also that in the Riemann definition of the integral we need not assume the boundedness of f in advance. However, it easily follows that the Riemann integrability of a function f over R implies its boundedness on R.

Theorem 2.14. *A bounded function f on R is Riemann Δ-integrable if and only if it is Darboux Δ-integrable, in which case the values of the integrals are equal.*

Proof. Suppose first that f is Darboux Δ-integrable over R in the sense of Definition 2.1. Let $\varepsilon > 0$ and $\delta > 0$ be chosen so that (2.7) of Theorem 2.11 holds. We show that

$$\left| S - \int\int_R f(t,s)\Delta_1 t \Delta_2 s \right| < \varepsilon \tag{2.9}$$

for every Riemann Δ-sum (2.8) associated with some $P \in \mathcal{P}_\delta(R)$. Clearly we have $L(f,P) \le S \le U(f,P)$ and so (2.9) follows from the inequalities

$$U(f,P) < L(f,P) + \varepsilon \le L(f) + \varepsilon = \int\int_R f(t,s)\Delta_1 t \Delta_2 s + \varepsilon$$

and

$$L(f,P) > U(f,P) - \varepsilon \ge U(f) - \varepsilon = \int\int_R f(t,s)\Delta_1 t \Delta_2 s - \varepsilon.$$

This proves (2.9) and hence f is Riemann Δ-integrable and $I = \int\int_R f(t,s)\Delta_1 t \Delta_2 s$.

Now suppose that f is Riemann Δ-integrable in the sense of Definition 2.13. Select any $P \in \mathcal{P}_\delta(R)$ of the type (2.1), (2.2) and for each $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, k\}$ choose $(\xi_{ij}, \eta_{ij}) \in R_{ij}$ so that $f(\xi_{ij}, \eta_{ij}) < m_{ij} + \varepsilon$. The Riemann Δ-sum S for this choice of points (ξ_{ij}, η_{ij}) satisfies

$$S < L(f,P) + \varepsilon(b-a)(d-c) \quad \text{as well as} \quad |S - I| < \varepsilon.$$

It follows that

$$L(f) \ge L(f,P) > S - \varepsilon(b-a)(d-c) > I - \varepsilon - \varepsilon(b-a)(d-c).$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $L(f) \geq I$. A similar argument shows that $U(f) \leq I$. Since $L(f) \leq U(f)$, we obtain

$$L(f) = U(f) = I.$$

This shows that f is Darboux Δ-integrable and $\int\int_R f(t,s)\Delta_1 t\Delta_2 s = I$. ☐

In our definition of $\int\int_R f(t,s)\Delta_1 t\Delta_2 s$ with $R = [a,b) \times [c,d)$ we assumed that $a < b$ and $c < d$. We extend the definition to the case $a \leq b$ and $c \leq d$ by setting

$$\int\int_R f(t,s)\Delta_1 t\Delta_2 s = 0 \quad \text{if} \quad a = b \quad \text{or} \quad c = d. \tag{2.10}$$

Theorem 2.15. *Assume $a, b \in \mathbb{T}_1$ with $a \leq b$ and $c, d \in \mathbb{T}_2$ with $c \leq d$. Every constant function*

$$f(t,s) \equiv A \quad \textit{for} \quad (t,s) \in R = [a,b) \times [c,d)$$

is Δ-integrable over R and

$$\int\int_R f(t,s)\Delta_1 t\Delta_2 s = A(b-a)(d-c). \tag{2.11}$$

Proof. Let $a < b$ and $c < d$. Consider a partition P of $R = [a,b) \times [c,d)$ of the type (2.1), (2.2). Since

$$M_{ij} = m_{ij} = A \quad \text{for all} \quad 1 \leq i \leq n,\ 1 \leq j \leq k,$$

we have

$$U(f,P) = L(f,P) = A(b-a)(d-c),$$

and Theorem 2.6 shows that f is Δ-integrable and that (2.11) holds. For $a = b$ or $c = d$, (2.11) follows by (2.10). Note that every Riemann Δ-sum of f associated with P is also equal to $A(b-a)(d-c)$. ☐

Theorem 2.16. *Let $t^0 \in \mathbb{T}_1$ and $s^0 \in \mathbb{T}_2$. Every function $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is Δ-integrable over $R(t^0,s^0) = [t^0, \sigma_1(t^0)) \times [s^0, \sigma_2(s^0))$, and*

$$\int\int_{R(t^0,s^0)} f(t,s)\Delta_1 t\Delta_2 s = \mu_1(t^0)\mu_2(s^0)f(t^0,s^0). \tag{2.12}$$

Proof. If $\mu_1(t^0) = 0$ or $\mu_2(s^0) = 0$, then (2.12) is obvious as both sides of (2.12) are equal to zero in this case. If $\mu_1(t^0) > 0$ and $\mu_2(s^0) > 0$, then a single partition of $R(t^0,s^0)$ is $P = \{[t^0, \sigma_1(t^0)) \times [s^0, \sigma_2(s^0))\}$, and since

$$[t^0, \sigma_1(t^0)) \times [s^0, \sigma_2(s^0)) = \left\{(t^0, s^0)\right\},$$

we have

$$U(f,P) = \left(\sigma_1(t^0) - t^0\right)\left(\sigma_2(s^0) - s^0\right) f(t^0,s^0) = \mu_1(t^0)\mu_2(s^0)f(t^0,s^0) = L(f,P).$$

Therefore, Theorem 2.6 shows that f is Δ-integrable over $R(t^0, s^0)$ and (2.12) holds. Note that the Riemann Δ-sum associated with the above partition is also equal to $\mu_1(t^0)\mu_2(s^0)f(t^0, s^0)$. ☐

Theorem 2.17. *Let $a, b \in \mathbb{T}_1$ with $a \leq b$ and $c, d \in \mathbb{T}_2$ with $c \leq d$. Then we have the following.*

(i) *If $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$, then a bounded function f on $R = [a, b) \times [c, d)$ is Δ-integrable if and only if f is Riemann integrable on R in the classical sense, and in this case*

$$\int\int_R f(t, s)\Delta_1 t\Delta_2 s = \int\int_R f(t, s)\mathrm{d}t\mathrm{d}s,$$

where the integral on the right is the ordinary Riemann integral.

(ii) *If $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$, then every function f defined on $R = [a, b)\times[c, d)$ is Δ-integrable over R, and*

$$\int\int_R f(t, s)\Delta_1 t\Delta_2 s = \begin{cases} \sum\limits_{k=a}^{b-1}\sum\limits_{l=c}^{d-1} f(k, l) & \text{if} \quad a < b \quad \text{and} \quad c < d \\ 0 & \text{if} \quad a = b \quad \text{or} \quad c = d. \end{cases} \tag{2.13}$$

Proof. Clearly, the above given Definition 2.1 and Definition 2.13 of the Δ-integral coincide in case $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ with the usual Darboux and Riemann definitions of the integral, respectively (see e.g., [2, 11]). Notice that the classical definitions of Darboux's and Riemann's integral do not depend on whether the subrectangles of the partition are taken closed, half-closed, or open. Moreover, if $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$, then $\mathcal{P}_\delta(R)$ consists of all partitions of R with norm (mesh) less than or equal to $\delta\sqrt{2}$. So part (i) is valid.

To prove part (ii), let $a < b$ and $c < d$. Then $b = a + p$ and $d = c + q$ for some $p, q \in \mathbb{N}$. Consider the partition P^* of $R = [a, b) \times [c, d)$ given by (2.1), (2.2) with $n = p$, $k = q$, and

$$t_0 = a,\ t_1 = a + 1,\ \ldots,\ t_p = a + p \quad \text{and} \quad s_0 = c,\ s_1 = c + 1,\ \ldots,\ s_q = c + q.$$

Then R_{ij} contains the single point (t_{i-1}, s_{j-1}):

$$R_{ij} = [t_{i-1}, t_i) \times [s_{j-1}, s_j) = \{(t_{i-1}, s_{j-1})\} \quad \text{for all} \quad 1 \leq i \leq p,\ 1 \leq j \leq q.$$

Therefore

$$U(f, P^*) = \sum_{i=1}^{p}\sum_{j=1}^{q} M_{ij}(t_i - t_{i-1})(s_j - s_{j-1}) = \sum_{i=1}^{p}\sum_{j=1}^{q} f(t_{i-1}, s_{j-1})$$

and

$$L(f, P^*) = \sum_{i=1}^{p}\sum_{j=1}^{q} m_{ij}(t_i - t_{i-1})(s_j - s_{j-1}) = \sum_{i=1}^{p}\sum_{j=1}^{q} f(t_{i-1}, s_{j-1})$$

so that

$$U(f,P^*)=L(f,P^*)=\sum_{i=1}^{p}\sum_{j=1}^{q}f(t_{i-1},s_{j-1})=\sum_{k=a}^{b-1}\sum_{l=c}^{d-1}f(k,l).$$

Hence Theorem 2.6 shows that f is Δ-integrable over $R=[a,b)\times[c,d)$ and (2.13) holds for $a<b$ and $c<d$. If $a=b$ or $c=d$, then relation (2.10) shows the validity of (2.13). □

Remark 2.18. In the two variable time scales case four types of integrals can be defined:

(i) $\Delta\Delta$-integral over $[a,b)\times[c,d)$, which is introduced by using partitions consisting of subrectangles of the form $[\alpha,\beta)\times[\gamma,\delta)$;
(ii) $\nabla\nabla$-integral over $(a,b]\times(c,d]$, which is defined by using subrectangles of the form $(\alpha,\beta]\times(\gamma,\delta]$;
(iii) $\Delta\nabla$-integral over $[a,b)\times(c,d]$, which is defined by using subrectangles of the form $[\alpha,\beta)\times(\gamma,\delta]$;
(iv) $\nabla\Delta$-integral over $(a,b]\times[c,d)$, which is defined by using subrectangles of the form $(\alpha,\beta]\times[\gamma,\delta)$.

For brevity the first integral is called simply as Δ-integral, and in this paper we are dealing solely with such Δ-integrals. However, the presented theory can be easily adapted to study any of the four types of integrals described above.

3. PROPERTIES OF DOUBLE INTEGRALS OVER RECTANGLES

In this section we use the same notations as those in the preceding section. For given time scales $\mathbb{T}_1$ and $\mathbb{T}_2$, the set

$$\mathbb{T}_1\times\mathbb{T}_2=\{(t,s):\ t\in\mathbb{T}_1,\ s\in\mathbb{T}_2\}$$

is a complete metric space with the metric d defined by

$$d(x,y)=\sqrt{(t-t')^2+(s-s')^2}\quad\text{for}\quad x=(t,s),y=(t',s')\in\mathbb{T}_1\times\mathbb{T}_2,$$

and also with the equivalent metric

$$d(x,y)=\max\{|t-t'|,|s-s'|\}.$$

A function $f:\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{R}$ is said to be *continuous* at $x\in\mathbb{T}_1\times\mathbb{T}_2$ if for every $\varepsilon>0$ there exists $\delta>0$ such that

$$|f(x)-f(y)|<\varepsilon$$

for all points $y\in\mathbb{T}_1\times\mathbb{T}_2$ satisfying $d(x,y)<\delta$.

If x is an isolated point of $\mathbb{T}_1\times\mathbb{T}_2$, then our definition implies that every function $f:\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{R}$ is continuous at x. For, no matter which $\varepsilon>0$ we choose, we can pick $\delta>0$ so that the only point $y\in\mathbb{T}_1\times\mathbb{T}_2$ for which $d(x,y)<\delta$ is $y=x$; then

$|f(x) - f(y)| = 0 < \varepsilon$. In particular, every function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ is continuous at each point of $\mathbb{Z} \times \mathbb{Z}$.

Theorem 3.1. *Every continuous function on* $K = [a, b] \times [c, d]$ *is* Δ*-integrable over* $R = [a, b) \times [c, d)$.

Proof. In order to apply Theorem 2.7, let $\varepsilon > 0$. Since f is continuous, it is uniformly continuous on the compact subset K of $\mathbb{T}_1 \times \mathbb{T}_2$. Therefore there exists $\delta > 0$ such that

$$\begin{cases} (t, s), (t', s') \in R \quad \text{and} \quad \max\{|t - t'|, |s - s'|\} \le \delta \\ \quad \text{imply} \quad |f(t, s) - f(t', s')| < \frac{\varepsilon}{3(b-a+1)(d-c+1)}. \end{cases} \tag{3.1}$$

Consider any $P \in \mathcal{P}_\delta(R)$ given by (2.1), (2.2) and let $\tilde{R}_{ij} = [t_{i-1}, \rho_1(t_i)] \times [s_{j-1}, \rho_2(s_j)]$ and

$$\tilde{M}_{ij} = \sup\left\{f(t, s) : \ (t, s) \in \tilde{R}_{ij}\right\} \quad \text{and} \quad \tilde{m}_{ij} = \inf\left\{f(t, s) : \ (t, s) \in \tilde{R}_{ij}\right\}. \tag{3.2}$$

Then, since $R_{ij} \subset \tilde{R}_{ij}$, we have

$$\tilde{m}_{ij} \le m_{ij} \le M_{ij} \le \tilde{M}_{ij} \quad \text{for all} \quad 1 \le i \le n, \ 1 \le j \le k.$$

Therefore, taking into account that f assumes its maximum and minimum on each compact rectangle $\tilde{R}_{ij}$, it follows from (3.1) that

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^{n} \sum_{j=1}^{k} (M_{ij} - m_{ij})(t_i - t_{i-1})(s_j - s_{j-1}) \\ &\le \sum_{i=1}^{n} \sum_{j=1}^{k} (\tilde{M}_{ij} - \tilde{m}_{ij})(t_i - t_{i-1})(s_j - s_{j-1}) \\ &= \sum_{t_i - t_{i-1} \le \delta} \ \sum_{s_j - s_{j-1} \le \delta} (\tilde{M}_{ij} - \tilde{m}_{ij})(t_i - t_{i-1})(s_j - s_{j-1}) \\ &\quad + \sum_{t_i - t_{i-1} > \delta} \ \sum_{s_j - s_{j-1} \le \delta} (\tilde{M}_{ij} - \tilde{m}_{ij})(t_i - t_{i-1})(s_j - s_{j-1}) \\ &\quad + \sum_{t_i - t_{i-1} \le \delta} \ \sum_{s_j - s_{j-1} > \delta} (\tilde{M}_{ij} - \tilde{m}_{ij})(t_i - t_{i-1})(s_j - s_{j-1}) \\ &\quad + \sum_{t_i - t_{i-1} > \delta} \ \sum_{s_j - s_{j-1} > \delta} (\tilde{M}_{ij} - \tilde{m}_{ij})(t_i - t_{i-1})(s_j - s_{j-1}) \\ &< \frac{3\varepsilon}{3(b - a + 1)(d - c + 1)} \sum_{i=1}^{n} \sum_{j=1}^{k} (t_i - t_{i-1})(s_j - s_{j-1}) \\ &= \frac{\varepsilon(b - a)(d - c)}{(b - a + 1)(d - c + 1)} \ < \ \varepsilon, \end{aligned}$$

where we used the fact that if $t_i - t_{i-1} > \delta$, then $\rho_1(t_i) = t_{i-1}$ and if $s_j - s_{j-1} > \delta$, then $\rho_2(s_j) = s_{j-1}$, and hence

$$\tilde{M}_{ij} - \tilde{m}_{ij} < \frac{\varepsilon}{3(b-a+1)(d-c+1)}$$

in the first three sums, and $\tilde{M}_{ij} - \tilde{m}_{ij} = 0$ in the fourth sum. Thus $U(f,P) - L(f,P) < \varepsilon$ so that Theorem 2.7 yields that f is Δ-integrable. ☐

In the following theorem we say as usual that a function $\varphi : [\alpha, \beta] \subset \mathbb{R} \to \mathbb{R}$ satisfies a *Lipschitz condition* if there exists a constant $B > 0$ (the *Lipschitz constant*) such that

$$|\varphi(u) - \varphi(v)| \le B\,|u - v| \quad \text{for all} \quad u, v \in [\alpha, \beta].$$

Theorem 3.2. *Let f be bounded and Δ-integrable over $R = [a,b) \times [c,d)$ and let M and m be its supremum and infimum over R, respectively. Let, further, $\varphi : [m, M] \to \mathbb{R}$ be a function satisfying a Lipschitz condition. Then the composite function $h = \varphi \circ f$ is Δ-integrable over R.*

Proof. Let $\varepsilon > 0$. By Theorem 2.7 there exists $P \in \mathcal{P}(R)$ given by (2.1), (2.2) such that

$$U(f,P) - L(f,P) < \frac{\varepsilon}{B},$$

where B is a Lipschitz constant for φ. Let M_{ij} and m_{ij} be the supremum and infimum of f on R_{ij}, respectively, and let M_{ij}^* and m_{ij}^* be the corresponding numbers for h. Since φ satisfies a Lipschitz condition with Lipschitz constant B, we find that

$$\begin{aligned} h(t,s) - h(t',s') &\le |h(t,s) - h(t',s')| = |\varphi(f(t,s)) - \varphi(f(t',s'))| \\ &\le B\,|f(t,s) - f(t',s')| \le B\,(M_{ij} - m_{ij}) \end{aligned}$$

holds for all $(t,s), (t',s') \in R_{ij}$. Hence $M_{ij}^* - m_{ij}^* \le B(M_{ij} - m_{ij})$ because there exist two sequences $\{(t_p, s_p)\}$ and $\{(t_p', s_p')\}$ of points in R_{ij} such that

$$h(t_p, s_p) \to M_{ij}^* \quad \text{and} \quad h(t_p', s_p') \to m_{ij}^* \quad \text{as} \quad p \to \infty.$$

Consequently,

$$\begin{aligned} U(h,P) - L(h,P) &= \sum_{i=1}^{n} \sum_{j=1}^{k} (M_{ij}^* - m_{ij}^*)(t_i - t_{i-1})(s_j - s_{j-1}) \\ \le B \sum_{i=1}^{n} \sum_{j=1}^{k} &(M_{ij} - m_{ij})(t_i - t_{i-1})(s_j - s_{j-1}) = B\,[U(f,P) - L(f,P)] < \varepsilon. \end{aligned}$$

Therefore h is Δ-integrable by Theorem 2.7. ☐

Theorem 3.3. *Let f be a bounded function that is Δ-integrable over $R = [a,b) \times [c,d)$. Further, let $a', b' \in [a,b]$ with $a' < b'$ and $c', d' \in [c,d]$ with $c' < d'$. Then f is Δ-integrable over $R' = [a', b') \times [c', d')$.*

Proof. Let $\varepsilon > 0$ and $P \in \mathcal{P}(R)$ be such that $U(f,P) - L(f,P) < \varepsilon$. Let $P = P_1 \times P_2$, where $P_1 \in \mathcal{P}([a,b))$ and $P_2 \in \mathcal{P}([c,d))$. Suppose P_1 is generated by the set

$$\{t_0, t_1, \ldots, t_n\} \subset [a,b], \quad \text{where} \quad a = t_0 < t_1 < \ldots < t_n = b$$

and P_2 is generated by the set

$$\{s_0, s_1, \ldots, s_k\} \subset [c,d], \quad \text{where} \quad c = s_0 < s_1 < \ldots < s_k = d.$$

Let P_1' be the Δ-partition of $[a,b)$ generated by the set

$$\{t_0, t_1, \ldots, t_n\} \cup \{a', b'\}$$

and P_2' be the Δ-partition of $[c,d)$ generated by the set

$$\{s_0, s_1, \ldots, s_k\} \cup \{c', d'\}.$$

Let $P' = P_1' \times P_2'$. then P' is a refinement of P and by Lemma 2.2 we also have $U(f,P') - L(f,P') < \varepsilon$. Now consider $P'' \in \mathcal{P}(R')$ consisting of all subrectangles of P' belonging to R'. If $\tilde{U}$ and $\tilde{L}$ are upper and lower Δ-sums of f on R' associated with the partition P'', then

$$\tilde{U} - \tilde{L} \le U(f,P') - L(f,P') < \varepsilon,$$

and hence f is Δ-integrable over R' by Theorem 2.7. $\square$

The majority of the properties of Riemann one-fold Δ-integrals over a half-closed interval $[a,b)$ as given in [7, 8] can be carried accordingly over to the Riemann double Δ-integral over a rectangle $R = [a,b) \times [c,d)$. Let us present here without proof only the following six theorems.

Theorem 3.4 (Linearity). *Let f and g be bounded Δ-integrable functions on $R = [a,b) \times [c,d)$, and let $\alpha, \beta \in \mathbb{R}$. Then $\alpha f + \beta g$ is also Δ-integrable on R and*

$$\int\int_R [\alpha f(t,s) + \beta g(t,s)]\,\Delta_1 t \Delta_2 s = \alpha \int\int_R f(t,s)\Delta_1 t\Delta_2 s + \beta \int\int_R g(t,s)\Delta_1 t \Delta_2 s.$$

Theorem 3.5. *If f and g are bounded Δ-integrable functions on R, then so is their product fg.*

Theorem 3.6 (Additivity). *Let the rectangle $R = [a,b) \times [c,d)$ be the union of two disjoint rectangles of the forms $R_1 = [a_1,b_1) \times [c_1,d_1)$ and $R_2 = [a_2,b_2) \times [c_2,d_2)$. If f is a bounded Δ-integrable function on each of R_1 and R_2, then f is Δ-integrable on R and*

$$\int\int_R f(t,s)\Delta_1 t\Delta_2 s = \int\int_{R_1} f(t,s)\Delta_1 t\Delta_2 s + \int\int_{R_2} f(t,s)\Delta_1 t\Delta_2 s.$$

Theorem 3.7. *If f and g are bounded Δ-integrable functions on R satisfying the inequality $f(t,s) \leq g(t,s)$ for all $(t,s) \in R$, then*

$$\int\int_R f(t,s)\Delta_1 t\Delta_2 s \leq \int\int_R g(t,s)\Delta_1 t\Delta_2 s.$$

Theorem 3.8. *If f is a bounded Δ-integrable function on R, then so is $|f|$ and*

$$\left|\int\int_R f(t,s)\Delta_1 t\Delta_2 s\right| \leq \int\int_R |f(t,s)|\,\Delta_1 t\Delta_2 s.$$

Theorem 3.9 (Mean Value Theorem)**.** *Let f and g be bounded Δ-integrable functions on R, and let g be nonnegative (or nonpositive) on R. Let us set*

$$m = \inf\{f(t,s):\ (t,s) \in R\} \quad \text{and} \quad M = \sup\{f(t,s):\ (t,s) \in R\}.$$

Then there exists a real number $\Lambda \in [m,M]$ such that

$$\int\int_R f(t,s)g(t,s)\Delta_1 t\Delta_2 s = \Lambda \int\int_R g(t,s)\Delta_1 t\Delta_2 s.$$

An effective way for evaluating multiple integrals is to reduce them to iterated (successive) integrations with respect to each of the variables.

Theorem 3.10. *Let f be bounded and Δ-integrable over $R = [a,b) \times [c,d)$ and suppose that the single integral*

$$I(t) = \int_c^d f(t,s)\Delta_2 s \tag{3.3}$$

exists for each $t \in [a,b)$. Then the iterated integral

$$\int_a^b I(t)\Delta_1 t = \int_a^b \Delta_1 t \int_c^d f(t,s)\Delta_2 s$$

exists and the equality

$$\int\int_R f(t,s)\Delta_1 t\Delta_2 s = \int_a^b \Delta_1 t \int_c^d f(t,s)\Delta_2 s \tag{3.4}$$

holds.

Proof. Let $P \in \mathcal{P}(R)$ be given by (2.1), (2.2). Obviously,

$$m_{ij} \leq f(t,s) \leq M_{ij} \quad \text{on} \quad R_{ij}, \tag{3.5}$$

where m_{ij} and M_{ij} are the infimum and supremum of f on R_{ij}, respectively. Choose any point $\xi_i \in [t_{i-1}, t_i)$ and set $t = \xi_i$ in (3.5), then integrate (3.5) with respect to s from s_{j-1} to s_j. We obtain

$$m_{ij}(s_j - s_{j-1}) \leq \int_{s_{j-1}}^{s_j} f(\xi_i, s)\Delta_2 s \leq M_{ij}(s_j - s_{j-1}). \tag{3.6}$$

Note that the integral in (3.6) exists because the existence of the integral in (3.6) is assumed over the entire interval $[c, d)$. Multiplying (3.6) by $t_i - t_{i-1}$ and summing then with respect to i and j, where $1 \le i \le n$ and $1 \le j \le k$, we obtain

$$L(f, P) \le \sum_{i=1}^{n} I(\xi_i)(t_i - t_{i-1}) \le U(f, P). \tag{3.7}$$

By the hypothesis the function f is Δ-integrable over R. Therefore taking into account Theorem 2.11 and the inequalities

$$L(f, P) \le \int\int_R f(t, s)\Delta_1 t\Delta_2 s \le U(f, P),$$

for arbitrary $\varepsilon > 0$ we can find $\delta > 0$ such that $P \in \mathcal{P}_\delta(R)$ implies

$$\left|L(f, P) - \int\int_R f(t, s)\Delta_1 t\Delta_2 s\right| < \frac{\varepsilon}{2} \quad \text{and} \quad \left|U(f, P) - \int\int_R f(t, s)\Delta_1 t\Delta_2 s\right| < \frac{\varepsilon}{2}.$$

For such partitions P we get from (3.7)

$$\left|\sum_{i=1}^{n} I(\xi_i)(t_i - t_{i-1}) - \int\int_R f(t, s)\Delta_1 t\Delta_2 s\right| < \varepsilon.$$

This means, by the Riemann definition of the single integral, that the function $I(t)$ is Δ-integrable from a to b and

$$\int_a^b I(t)\Delta_1 t = \int\int_R f(t, s)\Delta_1 t\Delta_2 s.$$

Thus we have established the existence of the iterated integral and the equality (3.4). □

Remark 3.11. It is evident from the proof of Theorem 3.10 that we can interchange the rôles of t and s, that is, we may assume the existence of the double integral and the existence of the single integral

$$K(s) = \int_a^b f(t, s)\Delta_1 t \tag{3.8}$$

for each $s \in [c, d)$. Then the theorem will state the existence of the iterated integral

$$\int_c^d K(s)\Delta_2 s = \int_c^d \Delta_2 s \int_a^b f(t, s)\Delta_1 t$$

and the equality

$$\int\int_R f(t, s)\Delta_1 t\Delta_2 s = \int_c^d \Delta_2 s \int_a^b f(t, s)\Delta_1 t. \tag{3.9}$$

Remark 3.12. If together with the double integral $\int\int_R f(t, s)\Delta_1 t\Delta_2 s$ there exist both single integrals (3.3) and (3.8), then the formulas (3.4) and (3.9) will hold simultaneously, i.e.,

$$\int_a^b \Delta_1 t \int_c^d f(t, s)\Delta_2 s = \int_c^d \Delta_2 s \int_a^b f(t, s)\Delta_1 t = \int\int_R f(t, s)\Delta_1 t\Delta_2 s.$$

Remark 3.13. If the function f is continuous on $[a, b] \times [c, d]$, then the existence of all the above mentioned integrals is guaranteed. In this case any of the formulas (3.4) and (3.9) may be used to calculate the double integral.

4. DOUBLE INTEGRATION OVER MORE GENERAL SETS

So far the double Riemann Δ-integral $\int\int_R f(t, s)\Delta_1 t\Delta_2 s$ has been defined only for rectangles of the form $R = [a, b) \times [c, d) \subset \mathbb{T}_1 \times \mathbb{T}_2$. In this section we extend the definition to more general sets in $\mathbb{T}_1 \times \mathbb{T}_2$, called *Jordan Δ-measurable sets.* The definition makes use of the Δ-*boundary* of a set $E \subset \mathbb{T}_1 \times \mathbb{T}_2$.

Definition 4.1. Let $E \subset \mathbb{T}_1 \times \mathbb{T}_2$. A point $x = (t, s) \in \mathbb{T}_1 \times \mathbb{T}_2$ is called a *boundary point* of E if every open (two-dimensional) ball $B(x; r) = \{y \in \mathbb{T}_1 \times \mathbb{T}_2 : d(x, y) < r\}$ of radius r and center x contains at least one point of E and at least one point of $(\mathbb{T}_1 \times \mathbb{T}_2) \setminus E$. The set of all boundary points of E is called the *boundary* of E and is denoted by ∂E.

Definition 4.2. Let $E \subset \mathbb{T}_1 \times \mathbb{T}_2$. A point $x = (t, s) \in \mathbb{T}_1 \times \mathbb{T}_2$ is called a Δ-*boundary point* of E if every rectangle of the form $V = [t, t') \times [s, s') \subset \mathbb{T}_1 \times \mathbb{T}_2$ with $t' \in \mathbb{T}_1$, $t' > t$ and $s' \in \mathbb{T}_2$, $s' > s$, contains at least one point of E and at least one point of $(\mathbb{T}_1 \times \mathbb{T}_2) \setminus E$. The set of all Δ-boundary points of E is called the Δ-*boundary* of E and is denoted by $\partial_\Delta E$.

For $i = 1, 2$ let us introduce the set $\mathbb{T}_i^0$ as follows: If $\mathbb{T}_i$ has a finite maximum t^*, then $\mathbb{T}_i^0 = \mathbb{T}_i \setminus \{t^*\}$, otherwise $\mathbb{T}_i^0 = \mathbb{T}_i$. Briefly we will write $\mathbb{T}_i^0 = \mathbb{T}_i \setminus \{\max \mathbb{T}_i\}$. Evidently, for every point $t \in \mathbb{T}_i^0$ there exists an interval of the form $[\alpha, \beta) \subset \mathbb{T}_i$ (with $\alpha, \beta \in \mathbb{T}_i$ and $\alpha < \beta$) that contains the point t.

Definition 4.3. A point $(t^0, s^0) \in \mathbb{T}_1^0 \times \mathbb{T}_2^0$ is called Δ-*dense* if every rectangle of the form $V = [t^0, t) \times [s^0, s) \subset \mathbb{T}_1 \times \mathbb{T}_2$ with $t \in \mathbb{T}_1$, $t > t^0$ and $s \in \mathbb{T}_2$, $s > s^0$, contains at least one point of $\mathbb{T}_1 \times \mathbb{T}_2$ distinct from (t^0, s^0). Otherwise the point (t^0, s^0) is called Δ-*scattered.*

Note that in the single variable case Δ-dense points are precisely the right-dense points, and Δ-scattered points are precisely the right-scattered points. Also, a point $(t^0, s^0) \in \mathbb{T}_1^0 \times \mathbb{T}_2^0$ is Δ-dense if and only if at least one of t^0 and s^0 is right-dense in $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively.

Obviously, each Δ-boundary point of E is a boundary point of E, but the converse is not necessarily true. Also, each Δ-boundary point of E must belong to $\mathbb{T}_1^0 \times \mathbb{T}_2^0$ and must be a Δ-dense point in $\mathbb{T}_1 \times \mathbb{T}_2$.

Example 4.4. (i) For arbitrary time scales $\mathbb{T}_1$ and $\mathbb{T}_2$, the rectangle of the form $E = [a, b) \times [c, d) \subset \mathbb{T}_1 \times \mathbb{T}_2$, where $a, b \in \mathbb{T}_1$, $a < b$ and $c, d \in \mathbb{T}_2$, $c < d$, has no Δ-boundary point, i.e., $\partial_\Delta E = \emptyset$.

(ii) If $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$, then any set $E \subset \mathbb{Z} \times \mathbb{Z}$ has no boundary as well as no Δ-boundary points.

(iii) Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ and $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$. Let us set

$$E_1 = [a, b) \times [c, d), \quad E_2 = (a, b] \times (c, d], \quad \text{and} \quad E_3 = [a, b] \times [c, d].$$

Then all three rectangles E_1, E_2, and E_3 have the boundary consisting of the union of all four sides of the rectangle. Moreover, $\partial_\Delta E_1$ is empty, $\partial_\Delta E_2$ consists of the union of all four sides of the rectangle E_2, and $\partial_\Delta E_3$ consists of the union of the right and upper sides of E_3.

(iv) Let $\mathbb{T}_1 = \mathbb{T}_2 = [0, 1] \cup \{2\}$, where $[0, 1]$ is the real number interval, and let $E = [0, 1) \times [0, 1)$. Then the boundary ∂E of E consists of the union of the right and upper sides of the rectangle E whereas $\partial_\Delta E = \emptyset$.

(v) Let $\mathbb{T}_1 = \mathbb{T}_2 = [0, 1] \cup \left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$, where $[0, 1]$ is the real number interval, and let $E = [0, 1] \times [0, 1]$. Then the boundary ∂E as well as the Δ-boundary $\partial_\Delta E$ of E conincide with the union of the right and upper sides of E.

Definition 4.5. Let $E \subset \mathbb{T}_1^0 \times \mathbb{T}_2^0$ be a bounded set and let $\partial_\Delta E$ be its boundary. Let $R = [a, b) \times [c, d)$ be a rectangle in $\mathbb{T}_1 \times \mathbb{T}_2$ such that $E \cup \partial_\Delta E \subset R$. Further, let $\mathcal{P}(R)$ denote the set of all Δ-partitions of R of type (2.1), (2.2). For every $P \in \mathcal{P}(R)$ define $J_*(E, P)$ to be the sum of the areas of those subrectangles of P which are entirely contained in E, and let $J^*(E, P)$ be the sum of the areas of those subrectangles of P each of which contains at least one point of $E \cup \partial_\Delta E$. The numbers

$$J_*(E) = \sup\{J_*(E, P) : \ P \in \mathcal{P}(R)\} \quad \text{and} \quad J^*(E) = \inf\{J^*(E, P) : \ P \in \mathcal{P}(R)\}$$

are called the (two-dimensional) *inner* and *outer Jordan Δ-measure* of E, respectively. The set E is said to be *Jordan Δ-measurable* if $J_*(E) = J^*(E)$, in which case this common value is called the *Jordan Δ-measure* of E, denoted by $J(E)$.

It is easy to verify that $J_*(E)$ and $J^*(E)$ depend only on E and not on the rectangle R which contains $E \cup \partial_\Delta E$. Also, $0 \le J_*(E) \le J^*(E)$. If E has Jordan Δ-measure zero, then $J_*(E) = J^*(E) = 0$. Hence we have the following statement.

Lemma 4.6. *A bounded set $E \subset \mathbb{T}_1^0 \times \mathbb{T}_2^0$ has Jordan Δ-measure zero if and only if for every $\varepsilon > 0$, the set E can be covered by a finite collection of rectangles of type $V_j = [\alpha_j, \beta_j) \times [\gamma_j, \delta_j) \subset \mathbb{T}_1 \times \mathbb{T}_2$, $j = 1, \ldots, n$, the sum of whose areas is less than ε:*

$$E \subset \bigcup_{j=1}^{n} V_j \quad \text{and} \quad \sum_{j=1}^{n} m(V_j) < \varepsilon.$$

It follows that if E is a set of Jordan Δ-measure zero, then so is any set $\tilde{E} \subset E$.

Lemma 4.7. *The union of a finite number of bounded subsets $E_1, \ldots, E_m \subset \mathbb{T}_1^0 \times \mathbb{T}_2^0$ each of which has Jordan Δ-measure zero is in turn a set of Jordan Δ-measure zero.*

Proof. Given $\varepsilon > 0$, we can construct for each $k \in \{1, \ldots, m\}$ a finite covering $\{V_j^{(k)}\}_{j=1}^{n_k}$ of E_k by rectangles of the needed type, the sum of whose areas is less than $\varepsilon/2^k$:

$$E_k \subset \bigcup_{j=1}^{n_k} V_j^{(k)} \quad \text{and} \quad \sum_{j=1}^{n_k} m\left(V_j^{(k)}\right) < \frac{\varepsilon}{2^k} \quad \text{for all} \quad k \in \{1, \ldots, m\}.$$

The union of all these coverings is itself a finite covering of $E = \cup_{k=1}^m E_k$ by rectangles, and the sum of the areas of all rectangles is less than $\sum_{k=1}^{\infty} \varepsilon/2^k = \varepsilon$. Since $\varepsilon > 0$ was arbitrary, the set E is of Jordan Δ-measure zero. ☐

The empty set is regarded as a Jordan Δ-measurable set and its Jordan Δ-measure is understood as being zero.

Lemma 4.8. *For each point $x^0 = (t^0, s^0) \in \mathbb{T}_1^0 \times \mathbb{T}_2^0$, the single point set $\{x^0\}$ is Jordan Δ-measurable, and its Jordan Δ-measure is given by*

$$J\left(\{x^0\}\right) = \left(\sigma_1(t^0) - t^0\right)\left(\sigma_2(s^0) - s^0\right) = \mu_1(t^0)\mu_2(s^0).$$

Proof. If $t^0 < \sigma_1(t^0)$ and $s^0 < \sigma_2(s^0)$, then $\{x^0\} = [t^0, \sigma_1(t^0)) \times [s^0, \sigma_2(s^0))$. Therefore $\{x^0\}$ is Jordan Δ-measurable with

$$J\left(\{x^0\}\right) = m\left([t^0, \sigma_1(t^0)) \times [s^0, \sigma_2(s^0))\right) = \left(\sigma_1(t^0) - t^0\right)\left(\sigma_2(s^0) - s^0\right),$$

which is the desired result. Further consider the cases when at least one of t^0 and s^0 is right-dense. To illustrate the proof, suppose $t^0 = \sigma_1(t^0)$ and $s^0 < \sigma_2(s^0)$. In this case there exists a point $t \in \mathbb{T}_1$ sufficiently close to t^0 and such that $t > t^0$. Therefore the rectangle $[t^0, t) \times [s^0, \sigma_2(s^0))$ covers the point x^0 and has a sufficiently small area. This means that the single point set $\{x^0\}$ has Jordan Δ-measure zero in the considered case. On the other hand, in this case we also have $(\sigma_1(t^0) - t^0)(\sigma_2(s^0) - s^0) = 0$ as $\sigma_1(t^0) = t^0$. ☐

The following lemma is an immediate consequence of Lemma 4.8.

Lemma 4.9. *Every Δ-dense point of $\mathbb{T}_1^0 \times \mathbb{T}_2^0$ has Jordan Δ-measure zero.*

Theorem 4.10. *Let $E \subset \mathbb{T}_1^0 \times \mathbb{T}_2^0$ be a bounded set and $\partial_\Delta E$ denote its Δ-boundary. Then we have*

$$J^*(\partial_\Delta E) = J^*(E) - J_*(E).$$

Hence E is Jordan Δ-measurable iff its Δ-boundary $\partial_\Delta E$ has Jordan Δ-measure zero.

Proof. Let $R = [a, b) \times [c, d)$ be a rectangle in $\mathbb{T}_1 \times \mathbb{T}_2$ containing $E \cup \partial_\Delta E$. Then it is not difficult to see that for every $P \in \mathcal{P}(R)$ we have

$$J^*(\partial_\Delta E, P) = J^*(E, P) - J_*(E, P).$$

Therefore $J^*(\partial_\Delta E, P) \geq J^*(E) - J_*(E)$ and hence $J^*(\partial_\Delta E) \geq J^*(E) - J_*(E)$. To obtain the reverse inequality, let $\varepsilon > 0$ be given and choose $H, Q \in \mathcal{P}(R)$ so that

$$J_*(E, H) > J_*(E) - \frac{\varepsilon}{2} \quad \text{and} \quad J^*(E, Q) < J^*(E) + \frac{\varepsilon}{2}.$$

Let $P = H + Q$ so that P is a refinement of both H and Q (for the definition of $H+Q$ see Definition 2.3). Since refinement increases the inner sums J_* and decreases the outer sums J^*, we find

$$\begin{aligned} J^*(\partial_\Delta E) &\leq J^*(\partial_\Delta E, P) = J^*(E, P) - J_*(E, P) \\ &\leq J^*(E, Q) - J_*(E, H) < J^*(E) - J_*(E) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $J^*(\partial_\Delta E) \leq J^*(E) - J_*(E)$. Therefore $J^*(\partial_\Delta E) = J^*(E) - J_*(E)$ and the theorem is proved. □

Note that every rectangle $R = [a, b) \times [c, d) \subset \mathbb{T}_1 \times \mathbb{T}_2$, where $a, b \in \mathbb{T}_1$, $a < b$ and $c, d \in \mathbb{T}_2$, $c < d$, is Jordan Δ-measurable with Jordan Δ-measure $J(R) = (b-a)(d-c)$. Indeed, it is easily seen that the Δ-boundary of R is empty (see Example 4.4 (i)), and therefore it has Jordan Δ-measure zero.

Also note that, for an arbitrary set $E \subset \mathbb{T}_1^0 \times \mathbb{T}_2^0$, a boundary point of E may have nonzero Jordan Δ-measure whereas Δ-boundary points of E (being Δ-dense points) have always Jordan Δ-measure zero. In fact, in Example 4.4 (iv), the point $(1, 1)$ is a boundary point of E, and the Jordan Δ-measure of that point is equal to 1.

The following lemma can be checked directly by using Definition 4.2.

Lemma 4.11. *For arbitrary sets $E_1, E_2 \subset \mathbb{T}_1 \times \mathbb{T}_2$, we have the following relations:*

(i) $\partial_\Delta(E_1 \cup E_2) \subset \partial_\Delta E_1 \cup \partial_\Delta E_2$;
(ii) $\partial_\Delta(E_1 \cap E_2) \subset \partial_\Delta E_1 \cup \partial_\Delta E_2$;
(iii) $\partial_\Delta(E_1 \setminus E_2) \subset \partial_\Delta E_1 \cup \partial_\Delta E_2$.

Hence, in view of Theorem 4.10 and Lemma 4.7, we get the following result.

Lemma 4.12. *The union and intersection of a finite number of Jordan Δ-measurable sets is Jordan Δ-measurable. Also, the difference of two Jordan Δ-measurable sets is Jordan Δ-measurable.*

Now we want to define and compute double Δ-integrals over Jordan Δ-measurable sets.

Definition 4.13. Let f be defined and bounded on a bounded Jordan Δ-measurable set $E \subset \mathbb{T}_1^0 \times \mathbb{T}_2^0$. Let $R = [a, b) \times [c, d) \subset \mathbb{T}_1 \times \mathbb{T}_2$ be a rectangle containing E and put $K = [a, b] \times [c, d]$. Define F on K as follows:

$$F(t, s) = \begin{cases} f(t, s) & \text{if} \quad (t, s) \in E \\ 0 & \text{if} \quad (t, s) \in K \setminus E. \end{cases} \tag{4.1}$$

Then f is said to be *Riemann Δ-integrable* over E if F is Riemann Δ-integrable over R in the sense of Section 2, and we write

$$\int\int_E f(t,s)\Delta_1 t\Delta_2 s = \int\int_R F(t,s)\Delta_1 t\Delta_2 s.$$

Remark 4.14. Considering Riemann Δ-sums which approximate $\int\int_R F(t,s)\Delta_1 t\Delta_2 s$, it is easy to see that the integral $\int\int_E f(t,s)\Delta_1 t\Delta_2 s$ does not depend on the choice of the rectangle R used to enclose E.

Let us also give another definition of the Riemann double Δ-integral over arbitrary bounded Jordan Δ-measurable sets. Let the function f be defined and bounded on a bounded Jordan Δ-measurable set $E \subset \mathbb{T}_1^0 \times \mathbb{T}_2^0$. Let $R = [a,b) \times [c,d) \subset \mathbb{T}_1 \times \mathbb{T}_2$ be a rectangle such that $E \subset R$. To define the double Δ-integral of f over E, we begin with a Δ-partition $P \in \mathcal{P}(R)$ of type (2.1), (2.2). Some of the subrectangles of P will lie entirely within E, some will be outside of E, and some will lie partly within and partly outside E. We consider the collection $P' = \{R_1, R_2, \ldots, R_k\}$ of all those subrectangles in P that lie *completely within* the set E. This collection P' is called the *inner Δ-partition* of the set E, determined by the partition P of the rectangle R. Using the inner Δ-partition P' of the set E, we can proceed in much the same way as in Section 2. By choosing an arbitrary point (ξ_i, η_i) in the ith subrectangle R_i of P' for $i \in \{1, \ldots, k\}$, we obtain a *selection* for the inner Δ-partition P'. Let us denote by $m(R_i)$ the area of R_i. Then this selection gives the sum

$$S = \sum_{i=1}^{k} f(\xi_i, \eta_i) m(R_i).$$

We call S a *Riemann Δ-sum* of f corresponding to the partition $P \in \mathcal{P}(R)$.

Definition 4.15. We say that f is *Riemann Δ-integrable* over $E \subset \mathbb{T}_1^0 \times \mathbb{T}_2^0$ if there exists a number I with the property that for each $\varepsilon > 0$ there exists a number $\delta > 0$ such that $|S - I| < \varepsilon$ for every Riemann Δ-sum S of f corresponding to any inner Δ-partition $P' = \{R_1, R_2, \ldots, R_k\}$ of E, determined by a partition $P \in \mathcal{P}(R)$ independent of the way in which we choose $(\xi_i, \eta_i) \in R_i$ for $1 \leq i \leq k$. The number I is called the *Riemann double Δ-integral* of f over E, and we write $I = \lim_{\delta \to 0} S$.

Remark 4.16. If E is a rectangle of the form $[a,b) \times [c,d) \subset \mathbb{T}_1 \times \mathbb{T}_2$ and we choose $R = E$ (so that an inner Δ-partition of E is simply a Δ-partition of R), then the preceding definition reduces to our earlier definition (Definition 2.13) of a double Δ-integral over a rectangle.

Now we want to prove the equivalence of Definition 4.13 and Definition 4.15. To this end, we first prove two auxiliary results.

Lemma 4.17. *Let $E \subset \mathbb{T}_1^0 \times \mathbb{T}_2^0$ be a bounded set and let $\partial_\Delta E$ denote its Δ-boundary. Let $R = [a, b) \times [c, d) \subset \mathbb{T}_1 \times \mathbb{T}_2$ be a rectangle that contains $E \cup \partial_\Delta E$. Next, for every $P \in \mathcal{P}_\delta(R)$, let $J_*(E, P)$ and $J^*(E, P)$ be defined as in Definition 4.5. Then*

$$\lim_{\delta \to 0} J_*(E, P) = J_*(E) \quad \textit{and} \quad \lim_{\delta \to 0} J^*(E, P) = J^*(E).$$

Proof. Define the functions $g_1 : R \to \mathbb{R}$ and $g_2 : R \to \mathbb{R}$ by

$$g_1(t, s) = \begin{cases} 1 & \text{if } (t, s) \in E \\ 0 & \text{if } (t, s) \in R \setminus E \end{cases} \quad \text{and} \quad g_2(t, s) = \begin{cases} 1 & \text{if } (t, s) \in E \cup \partial_\Delta E \\ 0 & \text{if } (t, s) \in R \setminus (E \cup \partial_\Delta E). \end{cases}$$

Then it is easily seen that

$$J_*(E, P) = L(g_1, P), \quad J_*(E) = L(g_1), \quad J^*(E, P) = U(g_2, P), \quad J^*(E) = U(g_2).$$

On the other hand, by Theorem 2.12 we have

$$\lim_{\delta \to 0} L(g_1, P) = L(g_1) \quad \text{and} \quad \lim_{\delta \to 0} U(g_2, P) = U(g_2).$$

This completes the proof. ☐

Lemma 4.18. *Let $\Gamma \subset \mathbb{T}_1^0 \times \mathbb{T}_2^0$ be a set of Jordan Δ-measure zero. Moreover, let $R = [a, b) \times [c, d) \subset \mathbb{T}_1 \times \mathbb{T}_2$ be a rectangle in $\mathbb{T}_1 \times \mathbb{T}_2$ that contains Γ. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that for every partition $P \in \mathcal{P}_\delta(R)$ the sum of areas of subrectangles of P which have a common point with Γ is less than ε.*

Proof. It is sufficient to apply Lemma 4.17 to the set $E = \Gamma$ and take into account that the assumption implies $J^*(\Gamma) = 0$. ☐

Theorem 4.19. *Let $E \subset \mathbb{T}_1^0 \times \mathbb{T}_2^0$ be a bounded and Jordan Δ-measurable set and let f be a bounded function on E. Then Definition 4.13 and Definition 4.15 of the Riemann Δ-integrability of f over E are equivalent to each other.*

Proof. Suppose $R = [a, b) \times [c, d) \subset \mathbb{T}_1 \times \mathbb{T}_2$ contains E and let $K = [a, b] \times [c, d]$. Define F on K by the formula (4.1). Let P be a Δ-partition of R into subrectangles R_{ij} $(1 \le i \le n,\ 1 \le j \le k)$ defined by (2.1), (2.2). For every selection $(\xi_{ij}, \eta_{ij}) \in R_{ij}$ we have

$$\sum_{i=1}^{n} \sum_{j=1}^{k} F(\xi_{ij}, \eta_{ij}) m(R_{ij}) = \sum_{(i,j) \in A} f(\xi_{ij}, \eta_{ij}) m(R_{ij}) + \sum_{(i,j) \in B} F(\xi_{ij}, \eta_{ij}) m(R_{ij}), \tag{4.2}$$

where

$$A = \{(i, j) : \ R_{ij} \subset E\} \quad \text{and} \quad B = \{(i, j) : \ R_{ij} \not\subset E \text{ and } R_{ij} \cap \partial_\Delta E \neq \emptyset\}. \tag{4.3}$$

Now the statement of the theorem follows from (4.2) because, by Lemma 4.18, the second sum on the right-hand side can be made sufficiently small for $P \in \mathcal{P}_\delta(R)$ as $\delta \to 0$, since $\partial_\Delta E$ has Jordan Δ-measure zero. ☐

Theorem 4.20. *Let $E \subset \mathbb{T}_1^0 \times \mathbb{T}_2^0$ be a bounded and Jordan Δ-measurable set. Then the integral $\int\int_E 1\Delta_1 t\Delta_2 s$ exists and we have*

$$J(E) = \int\int_E 1\Delta_1 t\Delta_2 s.$$

Proof. Suppose $R = [a, b) \times [c, d) \subset \mathbb{T}_1 \times \mathbb{T}_2$ contains E and let $K = [a, b] \times [c, d]$. Set

$$F(t, s) = \begin{cases} 1 & \text{if} \quad (t, s) \in E \\ 0 & \text{if} \quad (t, s) \in K \setminus E. \end{cases}$$

Further, let P be a Δ-partition of R into subrectangles defined by (2.1), (2.2), and let A and B be defined as in (4.3). If $(i, j) \in A$, then we have $F(\xi_{ij}, \eta_{ij}) = 1$, and so (4.2) with $f = 1$ becomes

$$\sum_{i=1}^{n}\sum_{j=1}^{k} F(\xi_{ij}, \eta_{ij})m(R_{ij}) = J_*(E, P) + \sum_{(i,j)\in B} F(\xi_{ij}, \eta_{ij})m(R_{ij}). \tag{4.4}$$

Now if $P \in \mathcal{P}_\delta(R)$ and $\delta \to 0$, then by Lemma 4.17 and the Jordan Δ-measurability of E, the first term on the right-hand side of (4.4) tends to $J(E)$ while the second term tends to zero by Lemma 4.18 since $\partial_\Delta E$ has Jordan Δ-measure zero. Therefore it follows from (4.4) that 1 is integrable over E and $\int\int_E 1\Delta_1 t\Delta_2 s = J(E)$. □

Example 4.21. Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$ and consider any bounded set $E \subset \mathbb{T}_1 \times \mathbb{T}_2 = \mathbb{Z}\times\mathbb{Z}$. Then $\partial_\Delta E = \emptyset$ and therefore E is Jordan Δ-measurable. For any function $f : E \to \mathbb{R}$ we have (see Definition 4.13 and Theorem 2.17 (ii))

$$\int\int_E f(t, s)\Delta_1 t\Delta_2 s = \sum_{(t,s)\in E} f(t, s).$$

The Jordan Δ-measure of E coincides with the number of points of E.

Theorem 4.22 (Additivity)**.** *Let $E_1, E_2 \subset \mathbb{T}_1^0 \times \mathbb{T}_2^0$ be bounded Jordan Δ-measurable sets such that $J(E_1 \cap E_2) = 0$, and let $E = E_1 \cup E_2$. Assume $f : E \to \mathbb{R}$ is a bounded function which is Δ-integrable over each of E_1 and E_2. Then f is Δ-integrable over E, and we have*

$$\int\int_E f(t, s)\Delta_1 t\Delta_2 s = \int\int_{E_1} f(t, s)\Delta_1 t\Delta_2 s + \int\int_{E_2} f(t, s)\Delta_1 t\Delta_2 s. \tag{4.5}$$

Proof. Suppose $R = [a, b) \times [c, d) \subset \mathbb{T}_1 \times \mathbb{T}_2$ contains E and let $K = [a, b] \times [c, d]$. Define F as in (4.1). Let $P = \{R_1, R_2, \ldots, R_k\}$ be a Δ-partition of R and form a Riemann Δ-sum

$$S(F, P) = \sum_{i=1}^{k} F(\xi_i, \eta_i)m(R_i).$$

If S_1 denotes the part of the sum arising from those subrectangles containing only points of E_1, and if S_2 is similarly defined by E_2, then we can write

$$S(F, P) = S_1 + S_2 + S_3,$$

where S_3 contains those terms coming from subrectangles which contain points of $E_1 \cap E_2$. Then $|S_3|$ can be made arbitrarily small when P is sufficiently fine, S_1 approximates the integral $\int\int_{E_1} f(t,s)\Delta_1 t\Delta_2 s$, and S_2 approximates $\int\int_{E_2} f(t,s)\Delta_1 t\Delta_2 s$. The equation (4.5) is an easy consequence of these remarks. ☐

Remark 4.23. It can be shown that the converse of Theorem 4.22 is also true: Δ-integrability of f over E implies Δ-integrability of f over each of E_1 and E_2, and the equation (4.5) holds.

The following properties of the Riemann Δ-integral over a Jordan Δ-measurable set, given in Theorems 4.24 – 4.28, follow by using Definition 4.13 and Theorems 3.4, 3.7 – 3.9. We assume E is an arbitrary bounded Jordan Δ-measurable set in $\mathbb{T}_1^0 \times \mathbb{T}_2^0$, and the considered functions are assumed to be bounded.

Theorem 4.24 (Linearity). *Let f and g be Δ-integrable over E, and let $\alpha, \beta \in \mathbb{R}$. Then $\alpha f + \beta g$ is also Δ-integrable over E and*

$$\int\int_E [\alpha f(t,s) + \beta g(t,s)]\,\Delta_1 t\Delta_2 s = \alpha \int\int_E f(t,s)\Delta_1 t\Delta_2 s + \beta \int\int_E g(t,s)\Delta_1 t\Delta_2 s.$$

Theorem 4.25. *If f and g are Δ-integrable over E, then so is their product fg.*

Theorem 4.26. *If f and g are Δ-integrable over E satisfying $f(t,s) \le g(t,s)$ for all $(t,s) \in E$, then*

$$\int\int_E f(t,s)\Delta_1 t\Delta_2 s \le \int\int_E g(t,s)\Delta_1 t\Delta_2 s.$$

Theorem 4.27. *If f is Δ-integrable over E, then so is $|f|$ and*

$$\left|\int\int_E f(t,s)\Delta_1 t\Delta_2 s\right| \le \int\int_E |f(t,s)|\,\Delta_1 t\Delta_2 s.$$

Theorem 4.28 (Mean Value Theorem). *Let f and g be Δ-integrable over E, and let g be nonnegative (or nonpositive) on E. Let us set*

$$m = \inf\{f(t,s):\ (t,s) \in E\} \quad \text{and} \quad M = \sup\{f(t,s):\ (t,s) \in E\}.$$

Then there exists a real number $\Lambda \in [m,M]$ such that

$$\int\int_E f(t,s)g(t,s)\Delta_1 t\Delta_2 s = \Lambda \int\int_E g(t,s)\Delta_1 t\Delta_2 s.$$

For sets $E \subset \mathbb{T}_1 \times \mathbb{T}_2$ whose structure is relatively simple, Theorem 3.10 can be used to obtain formulas for evaluating double integrals by iterated integration. In order to present one of such formulas, we first give the following lemma.

Lemma 4.29. *Let $[a,b] \subset \mathbb{T}_1^0$ and $\varphi : [a,b] \to \mathbb{T}_2^0$ be a continuous function. Let Γ be the set (graph of φ) in $\mathbb{T}_1 \times \mathbb{T}_2$ given by*

$$\Gamma = \{(t,\varphi(t)):\ t \in [a,b)\}.$$

Then the subset Γ' of Γ consisting of all Δ-dense points of Γ has Jordan Δ-measure zero in $\mathbb{T}_1 \times \mathbb{T}_2$.

Proof. Since φ is continous on the compact interval $[a,b]$, it is uniformly continuous on $[a,b]$. Therefore, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$t, t' \in [a,b] \quad \text{and} \quad |t - t'| < \delta \quad \text{imply} \quad |\varphi(t) - \varphi(t')| < \frac{\varepsilon}{b-a}.$$

Take a partition $P \in \mathcal{P}_\delta([a,b))$ determined by $a = t_0 < t_1 < \ldots < t_k = b$. For each $i \in \{1, \ldots, k\}$, let us set

$$d_i = \min\{\varphi(t) : \ t \in [t_{i-1}, t_i]\} \quad \text{and} \quad D_i = \max\{\varphi(t) : \ t \in [t_{i-1}, t_i]\}.$$

Denote

$$I = \{i \in \{1, \ldots, k\} : \ t_i - t_{i-1} \leq \delta\} \quad \text{and} \quad I' = \{i \in \{1, \ldots, k\} : \ t_i - t_{i-1} > \delta\}.$$

Consider rectangles $R_i \subset \mathbb{T}_1 \times \mathbb{T}_2$ $(i = 1, \ldots, k)$ defined by $R_i = [t_{i-1}, t_i) \times [d_i, D_i)$. Obviously, all Δ-dense points of Γ may lie only in rectangles R_i for $i \in I$. On the other hand,

$$\sum_{i \in I} m(R_i) = \sum_{i \in I} (t_i - t_{i-1})(D_i - d_i) \leq \frac{\varepsilon}{b-a} \sum_{i \in I} (t_i - t_{i-1}) \leq \frac{\varepsilon}{b-a}(b-a) = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this completes the proof. □

Theorem 4.30. *Let $[a,b] \subset \mathbb{T}_1^0$ and let $\varphi : [a,b] \to \mathbb{T}_2^0$ and $\psi : [a,b] \to \mathbb{T}_2^0$ be two continuous functions such that $\varphi(t) < \psi(t)$ for all $t \in [a,b]$. Let E be the bounded set in $\mathbb{T}_1 \times \mathbb{T}_2$ given by*

$$E = \{(t,s) \in \mathbb{T}_1 \times \mathbb{T}_2 : \ a \leq t < b, \ \varphi(t) \leq s < \psi(t)\}.$$

Then E is Jordan Δ-measurable, and if $f : E \to \mathbb{R}$ is Δ-integrable over E and if the single integral

$$\int_{\varphi(t)}^{\psi(t)} f(t,s) \Delta_2 s$$

exists for each $t \in [a,b)$, then the iterated integral

$$\int_a^b \Delta_1 t \int_{\varphi(t)}^{\psi(t)} f(t,s) \Delta_2 s$$

exists and we have

$$\int\int_E f(t,s) \Delta_1 t \Delta_2 s = \int_a^b \Delta_1 t \int_{\varphi(t)}^{\psi(t)} f(t,s) \Delta_2 s.$$

Proof. It follows by using Lemma 4.29 that $J(\partial_\Delta E) = 0$ and hence E is Jordan Δ-measurable. Choose an interval $[c,d] \subset \mathbb{T}_2^0$ such that the rectangle $R = [a,b) \times [c,d)$

contains E. Define the function F as in (4.1). For the function F, all conditions of Theorem 3.10 are satisfied because

$$\begin{aligned}\int_c^d F(t,s)\Delta_2 s &= \int_c^{\varphi(t)} F(t,s)\Delta_2 s + \int_{\varphi(t)}^{\psi(t)} F(t,s)\Delta_2 s + \int_{\psi(t)}^{d} F(t,s)\Delta_2 s\\ &= \int_{\varphi(t)}^{\psi(t)} f(t,s)\Delta_2 s.\end{aligned}$$

Therefore we have

$$\int\int_R F(t,s)\Delta_1 t\Delta_2 s = \int_a^b \Delta_1 t \int_c^d F(t,s)\Delta_2 s = \int_a^b \Delta_1 t \int_{\varphi(t)}^{\psi(t)} f(t,s)\Delta_2 s.$$

On the other hand, by Definition 4.13,

$$\int\int_E f(t,s)\Delta_1 t\Delta_2 s = \int\int_R F(t,s)\Delta_1 t\Delta_2 s$$

so that the theorem is proved. ☐

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