Oscillation and asymptotic behavior of third-order nonlinear retarded dynamic equations

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\textbf{A B S T R A C T}

This paper is concerned with oscillation and asymptotic behavior of a class of third-order nonlinear delay dynamic equations on an arbitrary time scale. A new theorem is presented that improves a number of results reported in the literature. Examples are included to illustrate new results.

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\section{1. Introduction}

Following Hilger’s landmark contribution [21], the theory of time scales has recently received a lot of attention. Many authors have expounded on various aspects of this new theory; see the survey paper by Agarwal et al. [2] and references cited therein. A book on the subject of time scales, by Bohner and Peterson [8], summarizes and organizes much of the time scale calculus; we also refer to the book by Bohner and Peterson [9] for advances in dynamic equations on time scales.

Over the past few years, there has been much research activity concerning oscillation and nonoscillation of solutions of various classes of differential equations and dynamic equations on time scales, we refer the reader to the articles [1,3–7,10–20,22–32]. In particular, there are many results on the third-order dynamic equations, see, e.g., [10,12–18,20,22–26,29–32]. In what follows, we present some background details that motivate the contents of this paper. Baculíková et al. [7] studied a third-order nonlinear differential equation

\begin{equation}
(a((rx)'')'')(t) + p(t)x(t) = 0,
\end{equation}

where $1/a, 1/r,$ and $p$ are positive real-valued continuous functions defined on $[t_0, \infty)$. $\gamma$ is the quotient of odd positive integers. They established the following result.

\textbf{Theorem 1.1} (See [7, Theorem 3.1]). Let $t_1 \in [t_0, \infty)$ be large enough. Assume that

$$
\int_{t_0}^{\infty} \frac{dt}{a(t)} = \infty, \quad \int_{t_0}^{\infty} \frac{dt}{p(t)} = \infty,
$$

and

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If there exists a positive differentiable function \( \eta \) such that
\[
\limsup_{t \to \infty} \int_{t_1}^{t} \left( \frac{1}{a(s)} \int_s^t p(u) \, du \right) \, ds = \infty,
\]
then the solution \( x \) of (1.1) is oscillatory or satisfies \( \lim_{t \to \infty} x(t) = 0 \).

Li et al. [27] investigated a third-order delay differential equation
\[
(a(rx^\gamma))'(t) + p(t)x(t) = 0,
\]
where \( 1/a, 1/r, p, \) and \( \tau \) are positive real-valued continuous functions defined on \([t_0, \infty), \tau(t) \leq t\) with \( \lim_{t \to \infty} \tau(t) = \infty \). They obtained the following criterion.

**Theorem 1.2** (See [27, Theorem 2.1]). Let \( t_1 \in [t_0, \infty) \) be large enough. Assume that
\[
\int_{t_0}^{\infty} \frac{dt}{a(t)} = \infty, \quad \int_{t_0}^{\infty} \frac{dt}{r(t)} = \infty,
\]
and
\[
\int_{t_0}^{\infty} \frac{1}{r(t)} \int_{t}^{\infty} \frac{1}{a(s)} \int_{s}^{t} p(u) \, du \, ds \, dt = \infty.
\]
If there exists a positive differentiable function \( \eta \) such that for \( t_1 > t_2 > t_1 \),
\[
\limsup_{t \to \infty} \int_{t_1}^{t} \left( \frac{1}{a(s)} \int_s^t (1/a(u)) \, du \right) \left( \frac{1}{r(u)} \int_u^{t_1} p(v) \, dv \right) \, ds = \infty,
\]
then the solution \( x \) of (1.2) is oscillatory or satisfies \( \lim_{t \to \infty} x(t) = 0 \).

Erbe et al. [14] considered a third-order ordinary differential equation
\[
\left( a(rx^\gamma) \right)^{\Delta} (t) + p(t)x(t) = 0,
\]
where \( f \in C(\mathbb{R}, \mathbb{R}) \) is assumed to satisfy \( uf(u) > 0 \) and \( f(u)/u \geq K > 0 \) for \( u \neq 0, 1/a, 1/r, \) and \( p \) are positive real-valued rd-continuous functions defined on \( T \) which satisfy
\[
\int_{t_0}^{\infty} \frac{\Delta t}{a(t)} = \infty \quad \text{and} \quad \int_{t_0}^{\infty} \frac{\Delta t}{r(t)} = \infty.
\]
They proved several oscillation criteria for (1.3), one of which we present below for the convenience of the reader.

**Theorem 1.3** (See [14, Theorem 1 and Remark 1]). Let (1.4) hold and \( t_1 \in [t_0, \infty) \) be large enough. Assume that there exists a positive differentiable function \( \eta \) such that
\[
\limsup_{t \to \infty} \int_{t_1}^{t} \left( K \eta(s)p(s) - \frac{r(s)(\eta(s)^\Lambda)^2}{4(\eta(s) \frac{\Delta u}{u})} \right) \, ds = \infty.
\]
Then the solution \( x \) of (1.3) is oscillatory or \( \lim_{t \to \infty} x(t) \) (finite).

Later, Erbe et al. [15,16] and Wang and Xu [31] studied third-order dynamic equations
\[
x^\Delta(t) + p(t)x(t) = 0,
\]
and
\[
\left( a(\eta^\gamma) \right)^{\Delta} (t) + f(t,x(t)) = 0,
\]
respectively, where \( \gamma \geq 1 \) is the ratio of odd positive integers, \( 1/a \) and \( 1/r \) are positive rd-continuous functions defined on \( T, f \in C(T \times \mathbb{R}, \mathbb{R}), \) and there exists a positive rd-continuous function \( p(t) \) defined on \( T \) such that \( f(t,u)/u \geq p(t) \) for \( u \neq 0 \). Erbe et al. [16] presented several oscillation criteria for (1.5), one of which we present below for the convenience of the reader.
Theorem 1.4 (See [16, Theorem 2]). Let $t_1 \in [t_0, \infty)_T$ be large enough. Assume that
\begin{equation}
\int_{t_0}^{\infty} \frac{\Delta t}{a(t)} = \infty, \quad \int_{t_0}^{\infty} \frac{\Delta t}{p(t)} = \infty, \tag{1.6}
\end{equation}
and
\begin{equation}
\int_{t_0}^{\infty} \frac{1}{r(t)} \int_{t}^{\infty} \left[ \frac{1}{\alpha(s)} \int_{s}^{\infty} p(u) \Delta u \right] \Delta \Delta t = \infty. \tag{1.7}
\end{equation}
If there exists a positive differentiable function $\eta$ such that
\begin{equation}
\limsup_{t \to \infty} \int_{t_1}^{t} \left( \eta(s) p(s) - \frac{r''(s)(\eta^4(s))^{\gamma+1}}{(\gamma+1)(\gamma+2)(\eta^4(s))^{\frac{\gamma+2}{\alpha}}(r''(s) + \frac{\Delta u}{\alpha^2(s)})} \right) \Delta s = \infty,
\end{equation}
then the solution $x$ of (1.5) is oscillatory or satisfies $\lim_{t \to \infty} x(t) = 0$.

Yu and Wang [32] applied the ideas of [14] in a general third-order dynamic equation
\begin{equation}
\left( a((r(x^2)')^{\gamma})^{\lambda} \right)(t) + f(a(t)) = 0.
\end{equation}
Regarding oscillation of third-order dynamic equations with deviating arguments, Elabbasy and Hassan [10] investigated a third-order dynamic equation
\begin{equation}
(ax^2)'(t) + p(t)x(\tau(t)) = 0.
\end{equation}
in the case where $\int_{t_0}^{\infty} p(t) \tau'(t) \Delta t = \infty$. Han et al. [18] considered a third-order dynamic equation
\begin{equation}
\left( ax^2 \right)'(t) + p(t)x'(\tau(t)) = 0.
\end{equation}
Li et al. [25] investigated a third-order dynamic equation
\begin{equation}
\left( ax^2 \right)'(t) + p(t)x'(\tau(t)) = 0.
\end{equation}
in the case when
\begin{equation}
\int_{t_0}^{\infty} p(t) \tau'(t) \Delta t = \infty.
\end{equation}

Erbe et al. [12,13], Hassan [20], Kubiaczyk and Saker [23], Li et al. [24], and Saker [29,30] studied a third-order dynamic equation
\begin{equation}
\left( a((r(x^2))')^{\lambda} \right)(t) + f(t,x(t)) = 0, \tag{1.8}
\end{equation}
where $\gamma > 0$ is the ratio of odd positive integers, $1/a$ and $1/r$ are positive rd-continuous functions defined on $T$, $T \in C_{rd}(T, \mathbb{R}), \tau(t) \leq t, \lim_{t \to \infty} \tau(t) = \infty, f \in C(T \times \mathbb{R}, \mathbb{R})$ is assumed to satisfy $uf(t, u) > 0$ for $u \neq 0$, and there exists a positive rd-continuous function $p(t)$ defined on $T$ such that $f(t, u)/u^\gamma \geq p(t)$ for $u \neq 0$. In [13,20], the authors established some oscillation criteria for (1.8) in the case where
\begin{equation}
\tau(\sigma(t)) = \sigma(\tau(t)) \text{ and } \tau'(t) > 0. \tag{1.9}
\end{equation}
For the convenience of the reader, we introduce a result in [20].

Theorem 1.5 (See [20, Corollary 2.3]). Let $\gamma \geq 1$, (1.6), (1.7), and (1.9) hold. Assume that there exists a positive differentiable function $\eta$ such that
\begin{equation}
\limsup_{t \to \infty} \int_{t_2}^{t} \left( \eta(s) p(s) - \frac{r''(s)(\eta^4(s))^{\gamma+1}}{(\gamma+1)(\gamma+2)(\eta^4(s))^{\frac{\gamma+2}{\alpha}}(r''(s) + \frac{\Delta u}{\alpha^2(s)})} \right) \Delta s = \infty
\end{equation}
for some $t_2 > t_1 > t_0$. Then the solution $x$ of (1.5) is oscillatory or satisfies $\lim_{t \to \infty} x(t) = 0$.

Note that $\tau(\sigma(t)) = \sigma(\tau(t))$ depends on time scales and may be a restriction for applications. In order to remove this assumption, Li et al. [24] obtained some oscillation results for (1.8) provided that
\begin{equation}
r^4(t) \leq 0 \quad \text{and} \quad \int_{t_0}^{\infty} p(t) \tau'(t) \Delta t = \infty. \tag{1.10}
\end{equation}
Saker [29,30] studied asymptotic properties of nonoscillatory solutions to (1.8) under the case where
\[ a^4(t) \geq 0 \quad \text{and} \quad r(t) = 1, \tag{1.11} \]
and established some new criteria for (1.8), some of which we present below for the convenience of the reader. We use the following notation
\[ A(t) := p(t) \left( \frac{h_2(\tau(t), t_0)}{\sigma(t)} \right)^\gamma, \quad A_* := \liminf_{t \to \infty} \frac{t}{\sigma(t)} \int_{t}^{\infty} A(s) \Delta s, \]
\[ B_* := \liminf_{t \to \infty} \frac{1}{t} \int_{t}^{T} s^{\gamma+1} A(s) \Delta s, \quad l := \liminf_{t \to \infty} \frac{t}{\sigma(t)}, \]
\[ q(t) := p(t) \left( \frac{\alpha'(t) P(\tau(t), T)}{\alpha'(t) P(T, T) + \mu(t)} \right)^\gamma \theta'(\tau(t)), \quad \theta(\tau(t)) := \frac{h_2(\tau(t), T)}{\tau(t) - T}, \]
and
\[ P(t, T) := \int_{t}^{T} \left( \frac{1}{\alpha(s)} \right)^\frac{1}{2} \Delta s \]
for \( T \geq t_0. \)

**Theorem 1.6** (See [29, Theorem 3.4]). Let (1.6), (1.7), and (1.11) hold. Assume that
\[ A_* > \frac{\gamma}{F^2(\gamma + 1)^{\gamma+1}}, \]
or
\[ A_* + B_* > \frac{1}{F^{\gamma+1}}. \]
Then the solution \( x \) of (1.8) is oscillatory or satisfies \( \lim_{t \to \infty} x(t) = 0. \)

**Theorem 1.7** (See [30, Theorem 2.8]). Let (1.6), (1.7), and (1.11) hold, and \( t_1 \in [t_0, \infty)_T \) be large enough. If there exists a positive differentiable function \( \eta \) such that
\[ \limsup_{t \to \infty} \int_{t_1}^{t} \left( \eta(s) q(s) - \frac{a(s)(\eta^4(s))^{\gamma+1}}{(\gamma + 1)^{\gamma+1} \eta'(s)} \right) \Delta s = \infty, \]
then the solution \( x \) of (1.8) is oscillatory or satisfies \( \lim_{t \to \infty} x(t) = 0. \)

So the natural question now is: If one can find new oscillation conditions for (1.8) which dispel the assumptions \( \gamma \geq 1, (1.9), (1.10), \) and (1.11), and improve results obtained by Elabbasy and Hassan [10], Erbe et al. [12–16], Han et al. [18], Hassan [20], Li et al. [24,25], Saker [29,30], Wang and Xu [31], and Yu and Wang [32].

The objective of this paper is to give an affirmative answer to this question. Throughout the paper, we assume that \( \gamma > 0 \) is the quotient of odd positive integers, \( 1/a \) and \( 1/r \) are positive, real-valued rd-continuous functions defined on \( T \), the so-called delay function \( \tau(\tau(t), T) \) satisfies \( \tau(t) \leq t \) and \( \lim_{t \to \infty} \tau(t) = \infty, f \in C([t_0, \infty), T \times \mathbb{R}, \mathbb{R}) \) is assumed to satisfy \( uf(t, u) > 0 \) for \( u \neq 0 \) and there exists a positive rd-continuous function \( p(t) \) defined on \( T \) such that \( f(t, u)/u \geq p(t) \) for \( u \neq 0. \)

Since we are interested in oscillatory behavior, we assume throughout this paper that the given time scale \( T \) is unbounded above. Also we assume \( t_0 \in T \) and it is convenient to assume \( t_0 > 0 \), and define the time scale interval of the form \([t_0, \infty)_T \) by \([t_0, \infty)_T := [t_0, \infty) \cap T \). By a solution of (1.8) we mean a real-valued function \( x \in C^1_{rd}([T_0, \infty), [T_0, \infty) \times \mathbb{R}, \mathbb{R}) \) which has the properties of \( \alpha x^4 \in C^1_{rd}([T_0, \infty), [T_0, \infty)] \), \( \alpha''(\alpha x^4)^{\gamma} \in C^1_{rd}([T_0, \infty), [T_0, \infty)], \) and satisfies (1.8) on \([T_0, \infty)_T \). We consider only proper solutions \( x \) to (1.8) with the property \( \sup\{|x(t)| : t \in [t, \infty) \} > 0 \) for all sufficiently large \( t \in [T_0, \infty)_T \), tacitly assuming that such solutions exist. A solution \( x \) of (1.8) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory.

This note is organized as follows: In the next section, we present some basic definitions concerning the calculus on time scales. In Section 3, we will use the Riccati transformation technique to derive a new sufficient condition which ensures that nonoscillatory solutions of (1.8) tend to zero eventually. In Section 4, we shall give two examples to illustrate the main results.

**Remark 1.8.** All functional inequalities considered in the next sections are assumed to hold eventually, that is, they are satisfied for all \( t \) large enough.
2. Some preliminaries

A time scale $T$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above and is a time scale interval of the form $[t_0, \infty)_T$. On any time scale we define the forward and backward jump operators by

$$
\sigma(t) := \inf \{ s \in T | s > t \} \quad \text{and} \quad \rho(t) := \sup \{ s \in T | s < t \},
$$

where $\inf \emptyset := \sup T$ and $\sup \emptyset := \inf T$. $\emptyset$ denotes the empty set.

A point $t \in T$ is said to be left-dense if $\rho(t) = t$ and $t > \inf T$, right-dense if $\sigma(t) = t$ and $t < \sup T$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. The graininess $\mu$ of the time scale is defined by $\mu(t) = \sigma(t) - t$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left limit in all left-dense points. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_d(\mathbb{T}, \mathbb{R})$.

Fix $t \in \mathbb{T}$ and let $f : \mathbb{T} \to \mathbb{R}$. Define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $U$ of $t$ (i.e., $U = (t - \delta, t + \delta) \cap T$ for some $\delta > 0$) such that $|f(t) - f(s)| < \varepsilon |f(t) - f(s)|$ for all $s \in U$.

In this case, $f^\Delta(t)$ is defined as the (delta) derivative of $f$ at $t$. If $f$ is said to be differentiable if its derivative exists. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous function is denoted by $C_d(\mathbb{T}, \mathbb{R})$. If $f$ is differentiable at $t$, then $f$ is continuous at $t$. If $f$ is continuous at $t$ and $t$ is right-scattered, then $f$ is differentiable at $t$ with

$$
f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.\]

If $t$ is right-dense, then $f$ is differentiable at $t$ if the limit

$$
f^\Delta(t) = \lim_{t \to t} \frac{f(t) - f(s)}{t - s}
$$

exists as a finite number. In this case

$$
f^\Delta(t) = \lim_{t \to t} \frac{f(t) - f(s)}{t - s}.
$$

If $f$ is differentiable at $t$, then

$$
f^\Delta(t) = f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t).
$$

Let $f$ be a real-valued function defined on an interval $[a, b]$. We say that $f$ is increasing, decreasing, nondecreasing, and nonincreasing on $[a, b]$, if $t_1, t_2 \in [a, b]$ and $t_2 > t_1$ imply $f(t_2) > f(t_1)$. $f(t_2) < f(t_1)$. $f(t_2) \geq f(t_1)$, and $f(t_2) \leq f(t_1)$, respectively. Let $f$ be a differential function on $[a, b]$. Then $f$ is increasing, decreasing, nondecreasing, and nonincreasing on $[a, b]$, if $f^\Delta(t) > 0, f^\Delta(t) < 0, f^\Delta(t) \geq 0, f^\Delta(t) \leq 0$ for all $t \in [a, b]$, respectively.

We will make use of the following product and quotient rules for the derivative of the product $fg$ and the quotient $f/g$ (where $g(t)g(\sigma(t)) \neq 0$) of two differentiable functions $f$ and $g$

$$
(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)).
$$

$$
\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}.
$$

For $a, b \in \mathbb{T}$ and a differentiable function $f$, the Cauchy integral of $f^\Delta$ is defined by

$$
\int_a^b f^\Delta(t)\Delta t = f(b) - f(a).
$$

The integration by parts formula reads

$$
\int_a^b f^\Delta(t)g(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^\Delta(t)g^\Delta(t)\Delta t,
$$

and infinite integrals are defined as

$$
\int_a^\infty f(t)\Delta s = \lim_{t \to \infty} \int_a^t f(s)\Delta s.
$$

3. Main results

In this section we will establish a new oscillation criterion for (1.8). Below, we use the notation
The assumption regarding the existence of auxiliary function $m$ is valid, e.g., $m(t) = \delta(t, t_0) = \int_{t_0}^{t} a^{-\gamma}(s) \, ds$.

**Theorem 3.5.** Assume that (1.6) and (1.7) hold. Assume further that there exists a positive function $m \in C^1_{ad}(T, \mathbb{R})$ such that (3.3) holds. If there exists a positive function $x \in C^1_{ad}(t_0, \infty, \mathbb{R})$ such that for all sufficiently large $T_1 \in [t_0, \infty)$ and for some $T \in [T_1, \infty)$,

\[
(x^{\Delta})(t) := \max\{0, x(t)\}, \quad \varphi(t) := \begin{cases} \phi(t), & \text{if } 0 < \gamma \leq 1, \\ \psi(t), & \text{if } \gamma > 1, \end{cases}
\]

and

\[
\phi(t) := \frac{m(t)}{m(t) + \delta(t, u)} , \quad \delta(t, u) := \int_{u}^{t} \frac{\Delta s}{a^{\gamma}(s)},
\]

where $m$ is an auxiliary function that will be specified later.

To prove our main results, we use the formula

\[
(x^{\Delta \Delta})(t) = \gamma x^{\Delta}(t) \int_{0}^{t} \left[ h x^{\Delta}(t) + (1-h)x(t) \right]^{\gamma-1} \, dh.
\]  

(3.1)

where $x$ is delta differentiable, which is a simple consequence of Keller’s chain rule; see Bohner and Peterson [8, Theorem 1.90]. Moreover, we need the following lemmas which will be used in the proofs of the main results.

**Lemma 3.1** (See [20, Lemma 2.1]). Assume that (1.6) and (1.7) hold. Suppose also that (1.8) has a positive solution $x(t)$ on $[t_0, \infty)$, Then there exists a $T \in [t_0, \infty)$, sufficiently large such that

\[
\left( a((x^{\Delta})^{\gamma}) \right)(t) < 0, \quad (x^{\Delta})^{\Delta}(t) > 0 \quad \text{for} \quad t \in [T, \infty),
\]

and either $x^\Delta(t) > 0$ on $[T, \infty)$, or $\lim_{t \to \infty} x(t) = 0$.

**Lemma 3.2** (See [20, Lemma 2.2]). Assume that $x$ is a positive solution of (1.8) such that

\[
(x^{\Delta \Delta})(t) > 0 \quad \text{and} \quad x^\Delta(t) > 0
\]

on $[t_0, \infty)$, where $t_0 \in [t_0, \infty)$. Then

\[
x^{\Delta}(t) \geq \frac{\delta(t, t_0)}{r(t)} \Delta r(t) (x^{\Delta \Delta})(t)
\]

on $[t_0, \infty)$.

**Lemma 3.3.** Assume that $x$ is a positive solution of (1.8) such that (3.2) holds. If there exist a positive function $m \in C^1_{ad}(T, \mathbb{R})$ and a $T_1 \in [t_0, \infty)$ such that

\[
\frac{m(t)}{\delta(t, t_0) a(t)} - m^\theta(t) \leq 0 \quad \text{for} \quad t \in [T_1, \infty),
\]  

(3.3)

then $rx^\Delta/m$ is nonincreasing on $[T_1, \infty)$ and

\[
x(t) \geq \left( \frac{r(t)}{m(t)} \int_{T_1}^{t} \frac{m(s)}{r(s)} \Delta s \right) x^\Delta(t).
\]

**Proof.** Since

\[
\left( \frac{rx^\Delta}{m} \right)^\Delta (t) = \frac{(rx^\Delta)^\Delta(t)m(t) - r(t)x^\Delta(t)m^\Delta(t)}{m(t)m^\theta(t)} \leq \frac{r(t)x^\Delta(t)}{m(t)m^\theta(t)} \left[ \frac{m(t)}{\delta(t, t_0) a(t)} - m^\theta(t) \right] \leq 0,
\]

we see that $rx^\Delta/m$ is nonincreasing. Thus, we have

\[
x(t) = x(T_1) + \int_{T_1}^{t} r(s)x^\Delta(s) \frac{m(s)}{m(t)} \Delta s \geq \left( \frac{r(t)}{m(t)} \int_{T_1}^{t} \frac{m(s)}{r(s)} \Delta s \right) x^\Delta(t).
\]

The proof is complete. \( \Box \)

**Remark 3.4.** The assumption regarding the existence of auxiliary function $m$ is valid, e.g., $m(t) = \delta(t, t_0) = \int_{t_0}^{t} a^{-\gamma}(s) \, ds$. 

**Theorem 3.5.** Assume that (1.6) and (1.7) hold. Assume further that there exists a positive function $m \in C^1_{ad}(T, \mathbb{R})$ such that (3.3) holds. If there exists a positive function $x \in C^1_{ad}(t_0, \infty, \mathbb{R})$ such that for all sufficiently large $T_1 \in [t_0, \infty)$ and for some $T \in [T_1, \infty)$,
Proof. Assume (1.8) has a nonoscillatory solution $x$ on $[t_0, \infty)_T$. Without loss of generality, we may assume that $x(t) > 0$ and $x(\tau(t)) > 0$ for all $t \in [t_1, \infty)_T$, where $t_1 \in [t_0, \infty)_T$. We only consider this case, since the proof when $x$ is eventually negative is similar. Therefore, we get by Lemma 3.1 that

$$\left( a(t)\left(\frac{\alpha(t)}{(r(t)x(t))'}\right)^\Delta(t) \right)^\Delta(t) < 0, \quad (r(t)x(t))' > 0 \quad \text{for} \quad t \in [t_2, \infty)_T \subset [t_1, \infty)_T,$$

and either $x(t) > 0$ for $t \in [t_2, \infty)_T \subset [t_1, \infty)_T$, or $\lim_{t \to \infty} x(t) = 0$. Consider now $x(t) > 0$ on $[t_2, \infty)_T$. Define the Riccati substitution

$$\omega(t) := \alpha(t) \frac{a(t)\left(\frac{\alpha(t)}{(r(t)x(t))'}\right)^\Delta(t)}{(r(t)x(t))'(\sigma(t))}.$$  

Using the product rule and then the quotient rule

$$\omega(t) = \frac{x(t)}{(r(t)x(t))'(\sigma(t))} \frac{(x(t)\omega(t))'}{(r(t)x(t))'(\sigma(t))} + \frac{x(t)}{a(t)} \frac{(a(t)\left(\frac{\alpha(t)}{(r(t)x(t))'}\right)^\Delta(t) - a(t)\left(\frac{(r(t)x(t))'}{(r(t)x(t))'(\sigma(t))}\right)^\Delta(t)}{(r(t)x(t))'(\sigma(t))}.$$  

From (1.8) and (3.5), we have

$$\omega(t) \leq \frac{x(t)}{a(t)\left(\frac{(r(t)x(t))'}{(r(t)x(t))'(\sigma(t))}\right)}.$$  

Using Lemma 3.3, we find

$$\frac{x(t)}{(r(t)x(t))'(\sigma(t))} = \left(\frac{x(t)}{(r(t)x(t))'(\sigma(t))}\right)^\Delta(t) \frac{(x(t)\omega(t))'}{(r(t)x(t))'(\sigma(t))} \leq \frac{1}{r(t)(x(t))'} \frac{m(s)}{r(s)} \frac{1}{\Delta(s)} \left(\frac{\alpha(t)}{(r(t)x(t))'(\sigma(t))}\right)^\Delta(t) \leq \frac{1}{\Delta(t)} \left(\frac{\alpha(t)}{(r(t)x(t))'(\sigma(t))}\right)^\Delta(t).$$  

If $0 < \gamma \leq 1$, then we obtain by (3.1) that

$$\left( (r(t)x(t))' \right)^\gamma(t) \geq \gamma (r(t)x(t))^{-1} \left( (r(t)x(t))' \right)^\gamma(t).$$

Hence by (3.6) and Lemma 3.3, we get

$$\omega(t) \leq -\frac{x(t)}{a(t)\left(\frac{(r(t)x(t))'}{(r(t)x(t))'(\sigma(t))}\right)} \omega(t) - \frac{x(t)}{a(t)} \frac{\gamma x(t)\omega(t) - a(t)\left(\frac{(r(t)x(t))'}{(r(t)x(t))'(\sigma(t))}\right)^\Delta(t)}{(r(t)x(t))'(\sigma(t))} \frac{x(t)}{(r(t)x(t))'(\sigma(t))}.$$  

If $\gamma > 1$, then we have by (3.1) that

$$\left( (r(t)x(t))' \right)^\gamma(t) \geq \gamma (r(t)x(t))^{-1} \left( (r(t)x(t))' \right)^\gamma(t).$$

Then, by (3.6) and Lemma 3.3, we obtain

$$\omega(t) \leq -\frac{x(t)}{a(t)\left(\frac{(r(t)x(t))'}{(r(t)x(t))'(\sigma(t))}\right)} \omega(t) - \frac{x(t)}{a(t)} \frac{\gamma x(t)\omega(t) - a(t)\left(\frac{(r(t)x(t))'}{(r(t)x(t))'(\sigma(t))}\right)^\Delta(t)}{(r(t)x(t))'(\sigma(t))} \frac{x(t)}{(r(t)x(t))'(\sigma(t))}.$$
Using (3.7), (3.8), and the definitions of $\phi$ and $\varphi$, we have, for $\gamma > 0$,
\[
\omega^3(t) \leq -\varphi'(t)p(t)M(t, T_1) + \frac{\varphi(t)}{\varphi'(t)}\omega(t) - \gamma \varphi'(t)\varphi(t) \frac{1}{\varphi'^{-1}(t)}\frac{w'(t)}{\varphi'(t)},
\]
where $\lambda := (\gamma + 1)/\gamma$. Define $A \geq 0$ and $B \geq 0$ by
\[
A^i := \gamma \varphi'(t)\varphi(t) \frac{1}{\varphi'^{-1}(t)}\frac{w'(t)}{\varphi'(t)} \quad \text{and} \quad B^{i-1} := \frac{(\varphi'(t))_+}{\lambda(\varphi(t)\varphi'(t))^3}.
\]
Using the inequality $\lambda^{AB} - A^i \leq (\lambda - 1)B^i$, $A \geq 0$, $B \geq 0$, we have
\[
\frac{(\varphi'(t))_+}{\varphi(t)}\omega(t) - \gamma \varphi'(t)\varphi(t) \frac{1}{\varphi'^{-1}(t)}\frac{w'(t)}{\varphi'(t)} \leq \frac{\alpha(t)((\varphi'(t))_+)^{\gamma+1}}{(\gamma + 1)^{\gamma+1}(\varphi(t)\varphi'(t))^\gamma}.
\]
From the latter inequality and (3.9), we have
\[
\omega^3(t) \leq -\varphi'(t)p(t)M(t, T_1) + \frac{\alpha(t)((\varphi'(t))_+)^{\gamma+1}}{(\gamma + 1)^{\gamma+1}(\varphi(t)\varphi'(t))^\gamma}.
\]
Integrating the above inequality from $t_2$ to $t$ gives
\[
\int_{t_2}^t \left( \varphi(s)p(s)M(s, T_1) - \frac{\alpha(s)((\varphi'(s))_+)^{\gamma+1}}{(\gamma + 1)^{\gamma+1}(\varphi(s)\varphi'(s))^\gamma} \right) \Delta s \leq \omega(t_2) - \omega(t) \leq \omega(t_2),
\]
which contradicts condition (3.4). This completes the proof. $\square$

**Remark 3.6.** Based on Theorem 3.5, we can obtain different conditions for oscillation of (1.8) with different choices of $\varphi$.

**Remark 3.7.** The conclusion of Theorem 3.5 remains intact if assumption (3.4) is replaced by the following two conditions
\[
\limsup_{t \to \infty} \int_t^\infty \varphi(s)p(s)M(s, T_1)\Delta s = \infty,
\]
\[
\limsup_{t \to \infty} \int_t^\infty \frac{\alpha(s)((\varphi'(s))_+)^{\gamma+1}}{(\gamma + 1)^{\gamma+1}(\varphi(s)\varphi'(s))^\gamma} \Delta s < \infty.
\]

4. **Examples and discussions**

In this section, we give the following examples to compare the obtained results with those obtained in the literature. To obtain conditions for oscillation, we will use the formula
\[
\int_{t_0}^{t_\omega} \Delta t = \infty, \text{ if } 0 < \lambda < 1.
\]
For more details we refer the reader to [9, Theorem 5.68].

**Example 4.1.** Consider a third-order delay dynamic equation
\[
(\alpha(t))^{\lambda} + \frac{\sigma^i(t)}{t^{\gamma+1}}X'(t) = 0 \quad \text{for } t \in [t_0, \infty),
\]
where $\gamma > 0$ is the ratio of odd positive integers. Let $m(t) = t^2$ and $x(t) = 1$. It is not difficult to verify that all assumptions of Theorem 3.5 are satisfied when using [9, Theorem 5.68]. Hence every solution of (4.1) is either oscillatory or tends to zero as $t \to \infty$.

**Example 4.2.** For $\gamma > 0$, consider a third-order ordinary dynamic equation on $T = \mathbb{R}$
\[
(\alpha(t))^{\lambda} + \frac{\beta}{t^{\gamma+1}}X'(t) = 0.
\]
where $\beta > 0$ is a constant. Let $m(t) = t - t$ and $x(t) = \left( t + c_0 \right)^\gamma$, where $c_0 > 0$ is a constant and sufficiently large. Using Theorem 3.5, one easily obtains that every solution of (4.2) is either oscillatory or tends to zero as $t \to \infty$, if $\beta > 2^\gamma \gamma+1/(\gamma + 1)^{\gamma+1}$.

Remark 4.3. In this paper, we suggest a new oscillation criterion for a third-order delay dynamic Eq. (1.8) by using the Riccati transformation technique and inequalities technique. Our results can be applied to (1.8) on an arbitrary time scale since these results remove restrictive condition (1.9). Letting $a_t = \int_0 t a^2(s)ds$, Theorem 3.5 involves Theorem 1.2.

Remark 4.4. Regarding oscillation of (1.8) in the case where $a(t) = t$, our results are new. This can be shown by comparing oscillation result of Eq. (4.2) with those reported in the literature; see the following details.

1. Let $\eta(t) = t^2$. Using Theorem 1.3, we see that the solution $x$ of Eq. (4.2) with $\gamma = 1$ is either oscillatory or $\lim_{t \to \infty} x(t)$ exists (finite), if $\beta > k_0 > 1$;
2. Let $\eta(t) = t^{2^\gamma}$. Using Theorem 1.4 and Theorem 1.5, we find that the solution $x$ of Eq. (4.2) with $\gamma > 1$ is either oscillatory or satisfies $\lim_{t \to \infty} x(t) = 0$, if $\beta > k_0(2\gamma)^{\gamma+1}/(\gamma + 1)^{\gamma+1}$ for some constant $k_0 > 1$;
3. Conclusion of Example 4.2 for $\gamma = 1$ is the same as in [15, Example 3]; Using Theorem 1.6, we can obtain the same conclusion as in Example 4.2;
4. Let $\eta(t) = t + c_0 \gamma$, where $c_0 > 0$ is a constant and sufficiently large. Using Theorem 1.7, we can obtain the same conclusion as in Example 4.2;
5. Wang and Xu [31] obtained that the solution $x$ of Eq. (4.2) with $\gamma > 1$ is either oscillatory or satisfies $\lim_{t \to \infty} x(t) = 0$, if $\beta > 2^\gamma(\gamma+1)^{\gamma+1}$; see [31, Example 3.1].

From the above statements and details in Section 1, one can easily see that the main results obtained in this paper improve those by [10,12–16,18,20,24,25,29–32].

Remark 4.5. For $\gamma > 1$, one can obtain other results by defining a generalized Riccati substitution

$$\omega(t) := \frac{a(t)\left( x^\gamma \right)^\gamma(t) + a(t)\theta(t)}{\left( x^\gamma \right)^\gamma(t)}$$

where $\theta$ is a nonnegative function such that $(a\theta)^\gamma \in C_{\text{ad}}([t_0, \infty), \mathbb{R})$. The specific details are left to the reader; see the related inequalities technique in [20].

Remark 4.6. The question regarding the study of sufficient conditions which guarantee that all solutions of (1.8) oscillate remains open at the moment. As is well known (see Erbe [11]), the third-order Euler differential equation

$$x'''(t) + \frac{\beta}{t^2}x(t) = 0$$

is oscillatory when $\beta > 2/(3\sqrt{3})$. How to extend this sharp criterion to third-order dynamic equations on time scales also remains open.

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