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On analytical solutions of the Black–Scholes equation

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ABSTRACT

This work presents a theoretical analysis for the Black–Scholes equation. Given a terminal condition, the analytical solution of the Black–Scholes equation is obtained by using the Adomian approximate decomposition technique. The mathematical technique employed in this work also has significance in studying some other problems in finance theory.

1. Introduction

The pricing of options is a central problem in financial investment. It is of both theoretical and practical importance since the use of options thrives in the financial market. In option pricing theory, the Black–Scholes equation is one of the most effective models for pricing options. The equation assumes the existence of perfect capital markets and the security prices are log normally distributed or, equivalently, the log-returns are normally distributed. To these, one adds the assumptions that trading in all securities is continuous and that the distribution of the rates of return is stationary. The partial differential equation of Black–Scholes is written as [4,6,8,9]

$$u_t + ax^2 u_{xx} + bxu_x - ru = 0, \tag{1.1}$$

where

$$a = \frac{\sigma^2}{2} \quad \text{and} \quad b = r - \delta,$$

r is the risk-free rate, σ is the volatility, and δ is the dividend yield.

In this work, we investigate the Black–Scholes equation with a view to obtaining the analytical solution to the terminal value problem consisting of the partial differential equation (1.1) and the terminal condition

$$u(x, T) = g(x) \tag{1.2}$$

by using the Adomian decomposition technique [1–3,5,7]. We assume throughout that g has derivatives of all orders. An advantage of the Adomian decomposition method is that it can provide analytical approximations to a rather wide class of linear or nonlinear equations without perturbation, closure approximations, or discretization, methods which can result in massive numerical computation.

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2. Approximate analytical solution

Our main result is an application of [7] in which the Adomian decomposition of the solution for

$$\sum_{n=0}^{N} \alpha_n(x,t) G_n u(x,t) = \sum_{m=1}^{M} \beta_m(x,t) F_m u(x,t) + f(x,t),$$
$$G_n u(x,0) = g_n(x), \quad 0 \le n \le N-1,$$

where

$$G_n = \frac{\partial^n}{\partial t^n}, \quad 0 \le n \le N \quad \text{and} \quad F_m = \frac{\partial^m}{\partial x^m}, \quad 0 \le m \le M,$$

is used to represent the solution as

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t),$$

where

$$u_0(x,t) = G_N^{-1}\left(\frac{f(x,t)}{\alpha_N(x,t)}\right) + \sum_{l=0}^{N-1} \frac{t^l}{l!} g_l(x)$$

and for $k \in \mathbb{N}_0$,

$$u_{k+1}(x,t) = \left[\sum_{m=1}^{M} G_N^{-1}\left(\frac{\beta_m(x,t)}{\alpha_N(x,t)} F_m\right) - \sum_{n=0}^{N-1} G_N^{-1}\left(\frac{\alpha_n(x,t)}{\alpha_N(x,t)} G_n\right)\right] u_k(x,t).$$

Here we consider the Black–Scholes equation (1.1), i.e., we have

$$N = 1, \qquad M = 2, \qquad f = 0, \qquad \alpha_0 = r, \qquad \alpha_1 = -1, \qquad \beta_0 = 0, \qquad \beta_1 = bx, \qquad \beta_2 = ax^2.$$

Applying the above mentioned formulas to the terminal value problem (1.1) and (1.2), we therefore obtain

$$u_0(x,t) = g(x) \tag{2.1}$$

and for $k \in \mathbb{N}_0$,

$$\begin{aligned} u_{k+1}(x,t) &= \left[\sum_{m=1}^{2} G_1^{-1}\left(\frac{\beta_m(x,t)}{\alpha_1(x,t)} F_m\right) - \sum_{n=0}^{0} G_1^{-1}\left(\frac{\alpha_n(x,t)}{\alpha_1(x,t)} G_n\right)\right] u_k(x,t) \\ &= G_1^{-1}\left[-bxF_1 - ax^2 F_2 + rG_0\right] u_k(x,t) \\ &= \int_T^t \left[-bx \frac{\partial u_k(x,\tau)}{\partial x} - ax^2 \frac{\partial^2 u_k(x,\tau)}{\partial x^2} + ru_k(x,\tau)\right] \mathrm{d}\tau, \end{aligned}$$

i.e.,

$$u_{k+1}(x,t) = \int_t^T \left[ax^2 \frac{\partial^2 u_k(x,\tau)}{\partial x^2} + bx \frac{\partial u_k(x,\tau)}{\partial x} - ru_k(x,\tau)\right] \mathrm{d}\tau. \tag{2.2}$$

We now prove the following main result of this work.

Theorem 2.1. *The functions defined recursively by (2.1) and (2.2) can be represented explicitly as*

$$u_k(x,t) = \left[\sum_{m=0}^{2k} \left\{\sum_{\nu=0}^{m} \frac{(-1)^{m-\nu}}{\nu!(m-\nu)!} \rho_\nu^k\right\} x^m g^{(m)}(x)\right] \frac{(T-t)^k}{k!} \quad \text{for all } k \in \mathbb{N}_0, \tag{2.3}$$

where

$$\rho_m = (am+r)(m-1) - \delta m \quad \text{for all } m \in \mathbb{N}_0. \tag{2.4}$$

Proof. For convenience we introduce the notation

$$v_k(x) = \sum_{m=0}^{2k} \left\{ \sum_{\nu=0}^{m} \frac{(-1)^{m-\nu}}{\nu!(m-\nu)!} \rho_\nu^k \right\} x^m g^{(m)}(x). \tag{2.5}$$

Then u_k from (2.3) can be written as

$$u_k(x,t) = v_k(x) \frac{(T-t)^k}{k!} \quad \text{for all } k \in \mathbb{N}_0.$$

We show that this u_k satisfies equations (2.1) and (2.2). Clearly, (2.1) is satisfied as $v_0(x) = g(x)$. Now we calculate

$$\begin{aligned}
&\int_t^T \left[ax^2 \frac{\partial^2 u_k(x,\tau)}{\partial x^2} + bx \frac{\partial u_k(x,\tau)}{\partial x} - ru_k(x,\tau) \right] d\tau \\
&= \int_t^T \left[ax^2 v_k''(x) \frac{(T-\tau)^k}{k!} + bxv_k'(x) \frac{(T-\tau)^k}{k!} - rv_k(x) \frac{(T-\tau)^k}{k!} \right] d\tau \\
&= \left[ax^2 v_k''(x) + bxv_k'(x) - rv_k(x) \right] \frac{(T-t)^{k+1}}{(k+1)!} \\
&= v_{k+1}(x) \frac{(T-t)^{k+1}}{(k+1)!} = u_{k+1}(x,t),
\end{aligned}$$

where we used (3.1) from Lemma 3.1 in the next section. □

3. Auxiliary results

In this section we complete the proof of Theorem 2.1 by proving a series of identities that are collected in the following lemmas. Throughout, we use the convention that "empty" sums are equal to zero while "empty" products are equal to one.

Lemma 3.1. *The functions v_k as defined in (2.5) satisfy the recursion*

$$ax^2 v_k''(x) + bxv_k'(x) - rv_k(x) = v_{k+1}(x) \quad \textit{for all } k \in \mathbb{N}_0. \tag{3.1}$$

Proof. For convenience we introduce the notation

$$\gamma_m^{(k)} = \sum_{\nu=0}^{m} \frac{(-1)^{m-\nu}}{\nu!(m-\nu)!} \rho_\nu^k. \tag{3.2}$$

Then v_k from (2.5) can be written as

$$v_k(x) = \sum_{m=0}^{2k} \gamma_m^{(k)} x^m g^{(m)}(x) \quad \text{for all } k \in \mathbb{N}_0.$$

Now

$$v_k'(x) = \sum_{m=0}^{2k} \gamma_m^{(k)} \left[mx^{m-1} g^{(m)}(x) + x^m g^{(m+1)}(x) \right]$$

and

$$v_k''(x) = \sum_{m=0}^{2k} \gamma_m^{(k)} \left[m(m-1)x^{m-2} g^{(m)}(x) + 2mx^{m-1} g^{(m+1)}(x) + x^m g^{(m+2)}(x) \right]$$

so that

$$\begin{aligned}
&ax^2 v_k''(x) + bxv_k'(x) - rv_k(x) \\
&= \sum_{m=0}^{2k} \gamma_m^{(k)} \left\{ [am(m-1) + bm - r] x^m g^{(m)}(x) + [2am + b] x^{m+1} g^{(m+1)}(x) + ax^{m+2} g^{(m+2)}(x) \right\} \\
&= \sum_{m=0}^{2k} [am(m-1) + bm - r] \gamma_m^{(k)} x^m g^{(m)}(x) + \sum_{m=1}^{2k+1} [2a(m-1) + b] \gamma_{m-1}^{(k)} x^m g^{(m)}(x) + \sum_{m=2}^{2k+2} a\gamma_{m-2}^{(k)} x^m g^{(m)}(x)
\end{aligned}$$

$$= \sum_{m=0}^{2k+2} \gamma_m^{(k+1)} x^m g^{(m)}(x) = v_{k+1}(x),$$

where we used first (3.3) and then (3.4) from the subsequent Lemma 3.2. □

Lemma 3.2. *The functions $\gamma_m^{(k)}$ as defined in (3.2) satisfy*

$$\gamma_m^{(k)} = 0 \quad \text{if } m < 0 \quad \text{or} \quad m > 2k \tag{3.3}$$

and the recursion

$$[am(m-1) + bm - r]\,\gamma_m^{(k)} + [2a(m-1) + b]\,\gamma_{m-1}^{(k)} + a\gamma_{m-2}^{(k)} = \gamma_m^{(k+1)}. \tag{3.4}$$

Proof. We first show the recursion (3.4). Note that

$$\gamma_m^{(k+1)} = \sum_{\nu=0}^{m} \frac{(-1)^\nu}{\nu!(m-\nu)!} \rho_\nu^{k+1} = \sum_{\nu=0}^{m} \frac{(-1)^\nu}{\nu!(m-\nu)!} \rho_\nu^k \rho_\nu$$

and

$$\rho_m - \rho_\nu = (m-\nu)\,[a(m+\nu-1) + b] \quad \text{and so} \quad \rho_m - \rho_{m-1} = [2a(m-1) + b]\,.$$

Thus we may calculate

$$\begin{aligned}
&[am(m-1) + bm - r]\,\gamma_m^{(k)} + [2a(m-1) + b]\,\gamma_{m-1}^{(k)} + a\gamma_{m-2}^{(k)} - \gamma_m^{(k+1)} \\
&= \sum_{\nu=0}^{m} \frac{(-1)^\nu}{\nu!(m-\nu)!} \rho_\nu^k\,[\rho_m - \rho_\nu] - \sum_{\nu=0}^{m-1} \frac{(-1)^\nu}{\nu!(m-1-\nu)!} \rho_\nu^k\,[\rho_m - \rho_{m-1}] + \sum_{\nu=0}^{m-2} \frac{(-1)^\nu}{\nu!(m-2-\nu)!} \rho_\nu^k a \\
&= \sum_{\nu=0}^{m-2} \frac{(-1)^\nu}{\nu!(m-\nu)!} \rho_\nu^k\,[\rho_m - \rho_\nu - (m-\nu)(\rho_m - \rho_{m-1}) + (m-\nu)(m-\nu-1)a] \\
&= 0
\end{aligned}$$

so that the recursion (3.4) holds. Now we show (3.3). By the definition (3.2) it is clear that $\gamma_m^{(k)} = 0$ whenever $m < 0$. Note that

$$\gamma_m^{(0)} = \sum_{\nu=0}^{m} \frac{(-1)^{m-\nu}}{\nu!(m-\nu)!} \rho_\nu^0 = \frac{1}{m!} \sum_{\nu=0}^{m} \binom{m}{\nu} (-1)^{m-\nu} = (1-1)^m = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

Because of the recursion (3.4), $\gamma_m^{(1)} = 0$ for all $m \geq 3$, $\gamma_m^{(2)} = 0$ for all $m \geq 5$ and in general $\gamma_m^{(k)} = 0$ for all $m \geq 2k+1$. This shows (3.3) and completes the proof. □

4. Examples

We start with the following lemma.

Lemma 4.1. *Let $s \in \mathbb{R}$. The functions $\gamma_m^{(k)}$ as defined in (3.2) satisfy*

$$\sum_{m=0}^{2k} s(s-1)\cdot\cdots\cdot(s-m+1)\gamma_m^{(k)} = \rho_s^k \quad \text{for all } k \in \mathbb{N}_0, \tag{4.1}$$

where

$$\rho_s = (as + r)(s-1) - \delta s. \tag{4.2}$$

Proof. We show (4.1) by induction. First, (4.1) holds for $k = 0$ as $\gamma_0^{(0)} = 1 = \rho_n^0$ due to (3.5). Now, assuming that (4.1) holds for $k \in \mathbb{N}_0$, we use first (3.4), shift the indices in the last two sums, and then use (3.3) to obtain

$$\begin{aligned}
\sum_{m=0}^{2k+2} s(s-1)\cdot\cdots\cdot(s-m+1)\gamma_m^{(k+1)} &= \sum_{m=0}^{2k+2} s(s-1)\cdot\cdots\cdot(s-m+1)\,[am(m-1) + bm - r]\,\gamma_m^{(k)} \\
&\quad + \sum_{m=-1}^{2k+1} s(s-1)\cdot\cdots\cdot(s-m)\,[2am + b]\,\gamma_m^{(k)} + \sum_{m=-2}^{2k+1} s(s-1)\cdot\cdots\cdot(s-m-1)a\gamma_m^{(k)}
\end{aligned}$$

$$= \sum_{m=0}^{2k} s(s-1)\cdot\cdots\cdot(s-m+1)\left[\rho_m + (s-m)(2am+b) + (s-m)(s-m-1)a\right]\gamma_m^{(k)}$$
$$= \sum_{m=0}^{2k} s(s-1)\cdot\cdots\cdot(s-m+1)\rho_s\gamma_m^{(k)}$$
$$= \rho_s^k\rho_s = \rho_s^{k+1}$$

and hence (4.1) holds for $k+1$ as well. This completes the proof. □

We now consider the special case $g(x) = x^s$ for $s \in \mathbb{R}$. An application of Theorem 2.1 together with Lemma 4.1 immediately yields the following result.

Theorem 4.2. *Let $g(x) = x^s$ for $s \in \mathbb{R}$. The functions defined recursively by (2.1) and (2.2) can be represented explicitly as*

$$u_k(x,t) = \frac{(\rho_s(T-t))^k}{k!}x^s \quad \text{for all } k \in \mathbb{N}_0, \tag{4.3}$$

where ρ_s is given in (4.2).

Remark 4.3. Theorem 4.2 can also be obtained directly using (3.1). Moreover, the Adomian solution is given by

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t) = x^s e^{\rho_s(T-t)}, \tag{4.4}$$

and this solution can also be obtained by separation of variables. Let $u(x,t) = \varphi(x)\psi(t)$ be a solution of (1.1) and (1.2). Then neither of the two sides of the equation

$$-\frac{\psi'(t)}{\psi(t)} + r = ax^2\frac{\varphi''(x)}{\varphi(x)} + bx\frac{\varphi'(x)}{\varphi(x)}$$

can depend on t or on x and therefore can be put equal to a constant λ. We obtain ($c \neq 0$)

$$\psi(t) = ce^{(r-\lambda)t}, \qquad \varphi(x) = \frac{g(x)}{\psi(T)} = \frac{g(x)}{ce^{(r-\lambda)T}}, \quad \text{and} \quad ax^2\frac{g''(x)}{g(x)} + bx\frac{g'(x)}{g(x)} = \lambda, \tag{4.5}$$

and hence (4.4) appears again if $g(x) = x^s$. Of course, for other functions g, e.g., $g(x) = e^x$, the last equation in (4.5) is not satisfied for any λ. Then separation of variables will not work. However, our Theorem 2.1 will work.

Remark 4.4. The method and technique presented in this work can be easily applied to the case of Eq. (1.1) for nonhomogeneous boundary value problems of the form

$$u_t + ax^2u_{xx} + bxu_x - ru = f(t,x),$$

where f is a known function which can be computed by integral approximation. The details are left to the reader.

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