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Oscillation criteria for first and second order forced difference equations with mixed nonlinearities

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Abstract

Some new criteria for the oscillation of certain difference equations with mixed nonlinearities are established. The main tool in the proofs is an inequality due to Hardy, Littlewood, and Pólya.

Keywords: Discrete; Oscillation; Nonoscillation; Mixed nonlinear; Nonhomogeneous

1. Introduction

We consider first and second order difference equations with mixed nonlinearities of type

$$\Delta x(n) + p(n)x(n+1) + q_1(n)x^{\lambda}(n+1) = q_2(n)x^{\mu}(n+1) + e(n) \tag{1.1}$$

and

$$\Delta(a(n)(\Delta x(n))^{\alpha}) + p(n)x(n+1) + q_1(n)x^{\lambda}(n+1) = q_2(n)x^{\mu}(n+1) + e(n), \tag{1.2}$$

where

(i) $\{a(n)\}$, $\{p(n)\}$, $\{q_i(n)\}$, $i = 1, 2$ and $\{e(n)\}$ are sequences of real numbers;
(ii) α, λ, μ are ratios of positive odd integers with $0 < \mu < 1$ and $\lambda > 1$.

By a solution of Eq. (1.1) (or (1.2)) we mean a nontrivial sequence $\{x(n)\}$ which is defined for $n \ge n_0 \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and satisfies Eq. (1.1) (or (1.2)). Such a solution is said to be oscillatory if for every $n_1 \in \mathbb{N}_0$ there exists $n \ge n_1$ such that $x(n)x(n+1) \le 0$; otherwise, it is called nonoscillatory. Any of Eq. (1.1) or (1.2) is said to be oscillatory if all its solutions are oscillatory.

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The problem of determining oscillation and nonoscillation of solutions of difference equations has been a very active area of research in the last two decades, and for a survey of recent results, we refer the reader to the monographs of Agarwal et al. [1–3,5] and the paper [4]. However, to study the oscillatory behavior of forced difference equations (1.1) or (1.2) with mixed nonlinearities, the known techniques either do not work or impose severe restrictions on the forcing term $e(n)$. Thus, in this paper we shall provide easily verifiable sufficiency criteria for the oscillation of solutions of (1.1) and (1.2). Then, we shall show that the obtained results can be extended rather easily to study oscillatory behavior of some neutral difference equations. In our results, the forcing term $e(n)$ in equations under consideration plays an important rôle for generating oscillations; in fact, results may not be applicable to these equations when $e(n) \equiv 0$. We shall also investigate the boundedness of solutions of (1.1) and (1.2).

In order to discuss our results in Sections 2 and 3, we shall need the following lemma from [6].

Lemma 1.1. *If X and Y are nonnegative, then*

(I) $X^{\lambda} - \lambda X Y^{\lambda-1} + (\lambda - 1)Y^{\lambda} \geq 0$ *for all* $\lambda > 1$;
(II) $X^{\mu} - \mu X Y^{\mu-1} - (1 - \mu)Y^{\mu} \leq 0$ *for all* $0 < \mu < 1$.

In the above inequalities, equality holds if and only if $X = Y$.

2. First order equations

In this section, we shall provide sufficient conditions for the oscillation of Eq. (1.1) and the special cases

$$\Delta x(n) - p(n)x(n+1) + q_1(n)x^{\lambda}(n+1) = e(n) \tag{2.1}$$

and

$$\Delta x(n) + p(n)x(n+1) = q_2(n)x^{\mu}(n+1) + e(n), \tag{2.2}$$

where $\{p(n)\}$, $\{q_1(n)\}$ and $\{q_2(n)\}$ are positive sequences of real numbers.

Theorem 2.1. *If*

$$\liminf_{n\to\infty} \sum_{j=n_0}^{n} e(j) = -\infty, \quad \limsup_{n\to\infty} \sum_{j=n_0}^{n} e(j) = \infty \tag{2.3}$$

and

$$\sum_{j=n_0}^{\infty} p^{\lambda/(\lambda-1)}(j) q_1^{1/(1-\lambda)}(j) < \infty, \tag{2.4}$$

then Eq. (2.1) is oscillatory.

Proof. Let $\{x(n)\}$ be an eventually positive solution of Eq. (2.1). It follows from Eq. (2.1) that

$$\Delta x(n) = e(n) + [p(n)x(n+1) - q_1(n)x^{\lambda}(n+1)]. \tag{2.5}$$

Set

$$X = q_1^{1/\lambda}(n)x(n+1) \quad \text{and} \quad Y = \left(\frac{1}{\lambda} p(n) q_1^{-1/\lambda}(n)\right)^{1/(\lambda-1)}$$

and apply Lemma 1.1(I) in Eq. (2.5) to obtain

$$\Delta x(n) \leq e(n) + (\lambda - 1)\lambda^{\lambda/(1-\lambda)} p^{\lambda/(\lambda-1)}(n) q_1^{1/(1-\lambda)}(n), \quad n \geq n_0 \geq 0. \tag{2.6}$$

Summing both sides of Eq. (2.6) from n_0 to $n - 1 \geq n_0$ and taking the lim inf on both sides of the resulting inequality as $n \to \infty$, we see that

$$0 \leq \liminf_{n\to\infty} x(n) = -\infty,$$

a contradiction. □

Next, we have the following oscillation result for Eq. (2.2).

Theorem 2.2. *If condition (2.3) holds and*

$$\sum_{j=n_0}^{\infty} p^{\mu/(\mu-1)}(j)q_2^{1/(1-\mu)}(j) < \infty, \tag{2.7}$$

then Eq. (2.2) is oscillatory.

Proof. Let $\{x(n)\}$ be an eventually positive solution of Eq. (2.2). From Eq. (2.2), we see that

$$\Delta x(n) = e(n) + [q_2(n)x^{\mu}(n+1) - p(n)x(n+1)]. \tag{2.8}$$

Set

$$X = q_2^{1/\mu}(n)x(n+1) \quad \text{and} \quad Y = \left(\frac{1}{\mu}p(n)q_2^{-1/\mu}(n)\right)^{1/(\mu-1)}, \quad n \geq n_0 \geq 0$$

and apply Lemma 1.1(II) in (2.8) to obtain

$$\Delta x(n) \leq e(n) + (1-\mu)\mu^{\mu/(1-\mu)} p^{\mu/(\mu-1)}(n)q_2^{1/(1-\mu)}(n), \quad n \geq n_0.$$

The rest of the proof is similar to that of Theorem 2.1 and hence is omitted. □

The following example is illustrative.

Example 2.3. Each of the two equations

$$\Delta x(n) - \frac{1}{(n+1)^3}x(n+1) + \frac{1}{(n+1)^5}x^3(n+1) = (2n+1)(-1)^{n+1} \tag{2.9}$$

and

$$\Delta x(n) + \frac{1}{(n+1)^3}x(n+1) = \frac{1}{(n+1)^{7/3}}x^{1/3}(n+1) + (2n+1)(-1)^{n+1} \tag{2.10}$$

has an oscillatory solution $x(n) = n(-1)^n$. It is easy to check that all conditions of Theorems 2.1 and 2.2 are satisfied for Eqs. (2.9) and (2.10), respectively. Thus, we conclude that both Eqs. (2.9) and (2.10) are oscillatory.

In Theorems 2.1 and 2.2, if any of the conditions fail, then we may apply the following corollaries, which follow from the proofs of Theorems 2.1 and 2.2.

Corollary 2.4. *If*

$$\liminf_{n\to\infty} \sum_{j=n_0}^{n} [e(j) + f_1(j)] = -\infty \quad \textit{and} \quad \limsup_{n\to\infty} \sum_{j=n_0}^{n} [e(j) - f_1(j)] = \infty, \tag{2.11}$$

where

$$f_1(n) = (\lambda - 1)\lambda^{\lambda/(1-\lambda)} p^{\lambda/(\lambda-1)}(n)q_1^{1/(1-\lambda)}(n), \tag{2.12}$$

then Eq. (2.1) is oscillatory.

Corollary 2.5. *If*

$$\liminf_{n\to\infty} \sum_{j=n_0}^{n} [e(j) + f_2(j)] = -\infty \quad \textit{and} \quad \limsup_{n\to\infty} \sum_{j=n_0}^{n} [e(j) - f_2(j)] = \infty, \tag{2.13}$$

where

$$f_2(n) = (1-\mu)\mu^{\mu/(1-\mu)} p^{\mu/(\mu-1)}(n)q_2^{1/(1-\mu)}(n), \tag{2.14}$$

then Eq. (2.2) is oscillatory.

Example 2.6. Each of the two equations

$$\Delta x(n) - \frac{3\sqrt{3}}{4(n+1)}x(n+1) + \frac{3\sqrt{3}}{4(n+1)^3}x^3(n+1) = (2n+1)(-1)^{n+1} \tag{2.15}$$

and

$$\Delta x(n) + \frac{3\sqrt{3}}{4(n+1)}x(n+1) = \frac{3\sqrt{3}}{4(n+1)^{1/3}}x^{1/3}(n+1) + (2n+1)(-1)^{n+1} \tag{2.16}$$

has an oscillatory solution $x(n) = n(-1)^n$. It is easy to check that (2.4) and (2.7) are not satisfied for (2.15) and (2.16), respectively. However, (2.11) and (2.13) are readily seen to hold, respectively. Hence, by Corollaries 2.4 and 2.5 we conclude that both Eqs. (2.15) and (2.16) are oscillatory.

The following result is concerned with the oscillation of Eq. (1.1) when $\{p(n)\}$, $\{q_i(n)\}$, $i = 1, 2$ are positive sequences.

Theorem 2.7. *If there exists a constant $\gamma > 0$ such that*

$$\liminf_{n\to\infty} \sum_{j=n_0}^{n} [e(j) + E(j)] = -\infty \quad \textit{and} \quad \limsup_{n\to\infty} \sum_{j=n_0}^{n} [e(j) - E(j)] = \infty, \tag{2.17}$$

where

$$E(n) = \gamma^{\lambda/(\lambda-1)} f_1(n) + (1+\gamma)^{\mu/(\mu-1)} f_2(n) \tag{2.18}$$

with f_1 and f_2 defined in (2.12) and (2.14), then Eq. (1.1) is oscillatory.

Proof. Let $\{x(n)\}$ be an eventually positive solution of Eq. (1.1). From Eq. (1.1), it follows that

$$\Delta x(n) = e(n) + [\gamma p(n)x(n+1) - q_1(n)x^{\lambda}(n+1)] + [q_2(n)x^{\mu}(n+1) - (1+\gamma)p(n)x(n+1)].$$

The rest of the proof is similar to the proofs of Theorems 2.1 and 2.2 and hence is omitted. □

The following corollary is immediate.

Corollary 2.8. *If conditions (2.3), (2.4) and (2.7) hold, then Eq. (1.1) is oscillatory.*

The following theorem deals with the oscillation of a special case of Eq. (1.1), namely, the equation

$$\Delta x(n) + q_1(n)x^{\lambda}(n+1) = e(n) + q_2(n)x^{\mu}(n+1), \tag{2.19}$$

where $\{q_i(n)\}$, $i = 1, 2$ are positive sequences.

Theorem 2.9. *If there exists a positive sequence $\{p(n)\}$ such that*

$$\liminf_{n\to\infty} \sum_{j=n_0}^{n} [e(j) + F(j)] = -\infty \quad \textit{and} \quad \limsup_{n\to\infty} \sum_{j=n_0}^{n} [e(j) - F(j)] = \infty, \tag{2.20}$$

where

$$F(n) = f_1(n) + f_2(n) \tag{2.21}$$

with f_1 and f_2 defined in (2.12) and (2.14), then Eq. (2.19) is oscillatory.

Proof. Let $\{x(n)\}$ be an eventually positive solution of Eq. (2.19). From Eq. (2.19), we have

$$\Delta x(n) = e(n) + [p(n)x(n+1) - q_1(n)x^{\lambda}(n+1)] + [q_2(n)x^{\mu}(n+1) - p(n)x(n+1)].$$

The rest of the proof is similar to the proofs of Theorems 2.1 and 2.7 and hence is omitted. □

Corollary 2.10. *If there exists a positive sequence $\{p(n)\}$ such that conditions (2.3), (2.4) and (2.7) hold, then Eq. (2.19) is oscillatory.*

The following example is illustrative.

Example 2.11. Consider the equation

$$\Delta x(n) + \frac{1}{(n+1)^5} x^3(n+1) = \frac{1}{(n+1)^{7/3}} x^{1/3}(n+1) + (2n+1)(-1)^{n+1}, \quad n \in \mathbb{N}_0. \tag{2.22}$$

Here, $e(n) = (2n+1)(-1)^{n+1}$, $q_1(n) = (n+1)^{-5}$ and $q_2(n) = (n+1)^{-7/3}$, $n \in \mathbb{N}_0$. Applying Theorem 2.9 with $p(n) = 1/(n+1)^3$ for $n \in \mathbb{N}_0$, we see that

$$F(n) = \frac{4}{3\sqrt{3}(n+1)^2}, \quad n \in \mathbb{N}_0,$$

and hence condition (2.20) is satisfied. Thus, we conclude that Eq. (2.22) is oscillatory. One such oscillatory solution of Eq. (2.22) is $x(n) = n(-1)^n$, $n \in \mathbb{N}_0$.

From the above proofs, one can easily establish criteria for the boundedness of all solutions of the equations under consideration, and as examples we give the following results.

Theorem 2.12. *If condition (2.4) holds and*

$$\sum_{j=n_0}^{\infty} |e(j)| < \infty, \tag{2.23}$$

then all nonoscillatory solutions of Eq. (2.1) are bounded.

Proof. As in the proof of Theorem 2.1, we obtain

$$|x(n)| \le |x(n_0)| + \sum_{j=n_0}^{n-1} \left[|e(j)| + (\lambda - 1)\lambda^{\lambda/(1-\lambda)} p^{\lambda/(\lambda-1)}(j) q_1^{1/(1-\lambda)}(j) \right].$$

The conclusion now follows by applying conditions (2.4) and (2.23). □

Theorem 2.13. *If condition (2.23) holds and there exists a positive sequence $\{p(n)\}$ of real numbers such that conditions (2.4) and (2.7) hold, then all solutions of Eq. (2.19) are bounded.*

Example 2.14. The equation

$$\Delta x(n) + \frac{1}{(n+1)^3} x^3(n+1) = \frac{2}{(n+1)^{1/3}} x^{1/3}(n+1), \quad n \in \mathbb{N}_0$$

has an unbounded solution $x(n) = n$. If we let $p(n) = 1/(n+1)^2$ for $n \in \mathbb{N}_0$, we see that all conditions of Theorem 2.13 are fulfilled except condition (2.7).

We observe that the above results may be applied to neutral equations of the type

$$\Delta(x(n) + c(n)x(n-\tau)) + p(n)x(n-\sigma+1) + q_1(n)x^{\lambda}(n-\sigma+1) = e(n) + q_2(n)x^{\mu}(n-\sigma+1), \tag{2.24}$$

where $\{c(n)\}$, $\{p(n)\}$ are nonnegative sequences, $\{q_i(n)\}$, $i = 1, 2$ are positive sequences, $\{e(n)\}$ is a sequence of real numbers, τ and σ are nonnegative integers, and μ and λ are as in Eq. (1.1), without additional conditions. As an example, we consider a special case of Eq. (2.24), namely, the neutral equation

$$\Delta(x(n) + c(n)x(n-\tau)) + q_1(n)x^{\lambda}(n-\sigma+1) = e(n) + q_2(n)x^{\mu}(n-\sigma+1) \tag{2.25}$$

and obtain the following result.

Theorem 2.15. *If there exists a positive sequence $\{p(n)\}$ such that condition (2.20) holds, then Eq. (2.25) is oscillatory.*

Proof. This follows as in the proof of Theorem 2.9. □

Example 2.16. The neutral equation

$$\Delta(x(n)+x(n-2))+\frac{1}{(n-1)^2}x^3(n-1)=(n-1)^{2/3}x^{1/3}(n-1)+(4n-2)(-1)^{n+1},\quad n>1$$

has an oscillatory solution $x(n)=n(-1)^n$. It is easy to check that with $p(n)=1/(n-1)^3$ for $n>1$ all conditions of Theorem 2.15 are satisfied, and hence this equation is oscillatory.

Also, as in Theorem 2.13, we have the following result.

Theorem 2.17. *If condition (2.23) holds and there exists a positive sequence $\{p(n)\}$ such that conditions (2.4) and (2.7) hold, then all nonoscillatory solutions of Eq. (2.25) are bounded.*

3. Second order equations

In this section we shall establish sufficient conditions for the oscillation of Eq. (1.2) and the special cases

$$\Delta(a(n)(\Delta x(n))^{\alpha})-p(n)x(n+1)+q_1(n)x^{\lambda}(n+1)=e(n) \tag{3.1}$$

and

$$\Delta(a(n)(\Delta x(n))^{\alpha})+p(n)x(n+1)=q_2(n)x^{\mu}(n+1)+e(n), \tag{3.2}$$

where $\{a(n)\}$, $\{p(n)\}$, $\{q_i(n)\}$, $i=1,2$ are positive sequences.

Theorem 3.1. *If conditions (2.3) and (2.4) hold, and for all $N\ge n_0$,*

$$\liminf_{n\to\infty}\sum_{j=N}^{n}\left[\frac{1}{a(j)}\sum_{s=N}^{j-1}e(s)\right]^{1/\alpha}=-\infty\quad\text{and}\quad\limsup_{n\to\infty}\sum_{j=N}^{n}\left[\frac{1}{a(j)}\sum_{s=N}^{j-1}e(s)\right]^{1/\alpha}=\infty, \tag{3.3}$$

then Eq. (3.1) is oscillatory.

Proof. Let $\{x(n)\}$ be an eventually positive solution of Eq. (3.1). From (3.1), we have

$$\Delta(a(n)(\Delta x(n))^{\alpha})=e(n)+[p(n)x(n+1)-q_1(n)x^{\lambda}(n+1)].$$

Proceeding as in the proof of Theorem 2.1, we have

$$\Delta(a(n)(\Delta x(n))^{\alpha})\le e(n)+(\lambda-1)\lambda^{\lambda/(1-\lambda)}p^{\lambda/(\lambda-1)}(n)q_1^{1/(1-\lambda)}(n),\quad n\ge n_0. \tag{3.4}$$

Summing both sides of (3.4) from $n_0\ge 0$ to $n-1\ge n_0$, we get

$$a(n)(\Delta x(n))^{\alpha}\le a(n_0)(\Delta x(n_0))^{\alpha}+\sum_{s=n_0}^{n-1}\left[e(s)+(\lambda-1)\lambda^{\lambda/(1-\lambda)}p^{\lambda/(\lambda-1)}(s)q_1^{1/(1-\lambda)}(s)\right].$$

Because of (2.3) and (2.4), there exists $n_1\ge n_0$ such that

$$a(n_1)(\Delta x(n_1))^{\alpha}\le-(\lambda-1)\lambda^{\lambda/(1-\lambda)}\sum_{s=n_0}^{\infty}p^{\lambda/(\lambda-1)}(s)q_1^{1/(1-\lambda)}(s). \tag{3.5}$$

Summing both sides of (3.4) again from n_1 to $n-1\ge n_1$ and using (3.5), we get

$$\begin{aligned}a(n)(\Delta x(n))^{\alpha}&\le a(n_1)(\Delta x(n_1))^{\alpha}+\sum_{s=n_1}^{n-1}\left[e(s)+(\lambda-1)\lambda^{\lambda/(1-\lambda)}p^{\lambda/(\lambda-1)}(s)q_1^{1/(1-\lambda)}(s)\right]\\&\le a(n_1)(\Delta x(n_1))^{\alpha}+\sum_{s=n_1}^{n-1}e(s)+(\lambda-1)\lambda^{\lambda/(1-\lambda)}\sum_{s=n_0}^{\infty}p^{\lambda/(\lambda-1)}(s)q_1^{1/(1-\lambda)}(s)\\&\le\sum_{s=n_1}^{n-1}e(s)\end{aligned}$$

so that

$$\Delta x(n) \le \left[\frac{1}{a(n)}\sum_{s=n_1}^{n-1} e(s)\right]^{1/\alpha}, \quad n \ge n_1$$

and therefore

$$x(n) \le x(n_1) + \sum_{j=n_1}^{n-1}\left[\frac{1}{a(j)}\sum_{s=n_1}^{j-1} e(s)\right]^{1/\alpha},$$

a contradiction with (3.3). □

We note that if any of the conditions (2.3), (2.4) or (3.3) fails to apply, then we may replace them by

$$\begin{cases} \liminf_{n\to\infty} \sum_{j=N}^{n-1}\left(\frac{c}{a(j)} + \frac{1}{a(j)}\sum_{s=N}^{j-1}[e(s)+f_1(s)]\right)^{1/\alpha} = -\infty, \\ \limsup_{n\to\infty} \sum_{j=N}^{n-1}\left(\frac{c}{a(j)} + \frac{1}{a(j)}\sum_{s=N}^{j-1}[e(s)-f_1(s)]\right)^{1/\alpha} = \infty \end{cases} \tag{3.6}$$

for all $N \ge n_0$ and all $c \in \mathbb{R}$ (with f_1 defined in (2.12)).

Theorem 3.2. *If conditions (2.3), (2.7) and (3.3) hold, then Eq. (3.2) is oscillatory.*

Proof. Let $\{x(n)\}$ be an eventually positive solution of Eq. (3.2). Then we have

$$\Delta(a(n)(\Delta x(n))^\alpha) = e(n) + [q_2(n)x^\mu(n+1) - p(n)x(n+1)].$$

Proceeding exactly as in the proof of Theorem 2.2, we obtain

$$\Delta(a(n)(\Delta x(n))^\alpha) \le e(n) + (1-\mu)\mu^{\mu/(1-\mu)} p^{\mu/(\mu-1)}(n) q_2^{1/(1-\mu)}(n), \quad n \ge n_0.$$

The rest of the proof is similar to that of Theorem 3.1 and hence is omitted. □

Once again if any of the conditions (2.3), (2.7) or (3.3) fails, then we may replace them by

$$\begin{cases} \liminf_{n\to\infty} \sum_{j=N}^{n-1}\left(\frac{c}{a(j)} + \frac{1}{a(j)}\sum_{s=N}^{j-1}[e(s)+f_2(s)]\right)^{1/\alpha} = -\infty, \\ \limsup_{n\to\infty} \sum_{j=N}^{n-1}\left(\frac{c}{a(j)} + \frac{1}{a(j)}\sum_{s=N}^{j-1}[e(s)-f_2(s)]\right)^{1/\alpha} = \infty \end{cases} \tag{3.7}$$

for all $N \ge n_0$ and all $c \in \mathbb{R}$ (with f_2 defined in (2.14)).

The following example is illustrative.

Example 3.3. Each of the equations

$$\Delta(n\Delta x(n)) - \frac{1}{(n+1)^3}x(n+1) + \frac{1}{(n+1)^5}x^3(n+1) = (4n^2+6n+3)(-1)^n \tag{3.8}$$

and

$$\Delta(n\Delta x(n)) + \frac{1}{(n+1)^3}x(n+1) = \frac{1}{(n+1)^{7/3}}x^{1/3}(n+1) + (4n^2+6n+3)(-1)^n \tag{3.9}$$

has an oscillatory solution $x(n) = n(-1)^n$. Conditions (3.6) and (3.7) are satisfied for Eqs. (3.8) and (3.9), respectively. For example, for Eq. (3.8) we have

$$\sum_{j=N}^{n-1}\left(\frac{c}{a(j)} + \frac{1}{a(j)}\sum_{s=N}^{j-1}[e(s)+f_1(s)]\right)^{1/\alpha} \le n(-1)^n + c_1 + c_2\sum_{j=1}^{n}\frac{1}{j}$$

with $c_1 = N(-1)^{N+1}$ and $c_2 = c + N(2N+1)(-1)^N + \pi^2/(9\sqrt{3})$. Thus, we conclude that both Eqs. (3.8) and (3.9) are oscillatory.

The following result is concerned with the oscillation of Eq. (1.2) when $\{p(n)\}$, $\{q_i(n)\}$, $i = 1, 2$ are positive sequences.

Theorem 3.4. *If there exists a constant $\gamma > 0$ such that condition (2.17) holds and*

$$\begin{cases} \liminf_{n\to\infty} \sum_{j=N}^{n} \left(\frac{1}{a(j)} \sum_{s=N}^{j-1} [e(s) + E(s)] \right)^{1/\alpha} = -\infty, \\ \limsup_{n\to\infty} \sum_{j=N}^{n} \left(\frac{1}{a(j)} \sum_{s=N}^{j-1} [e(s) - E(s)] \right)^{1/\alpha} = \infty \end{cases} \tag{3.10}$$

for all $N \geq n_0$, with $E(n)$ defined in (2.18), then Eq. (1.2) is oscillatory.

Proof. This follows from the proofs of Theorems 2.7 and 3.1. □

The following result deals with the oscillation of a special case of Eq. (1.2), namely,

$$\Delta(a(n)(\Delta x(n))^\alpha) + q_1(n)x^\lambda(n+1) = q_2(n)x^\mu(n+1) + e(n). \tag{3.11}$$

Theorem 3.5. *If there exists a positive sequence $\{p(n)\}$ such that (2.20) holds and*

$$\begin{cases} \liminf_{n\to\infty} \sum_{j=N}^{n} \left(\frac{1}{a(j)} \sum_{s=N}^{j-1} [e(s) + F(s)] \right)^{1/\alpha} = -\infty, \\ \limsup_{n\to\infty} \sum_{j=N}^{n} \left(\frac{1}{a(j)} \sum_{s=N}^{j-1} [e(s) - F(s)] \right)^{1/\alpha} = \infty \end{cases} \tag{3.12}$$

for all $N \geq n_0$, with $F(n)$ defined in (2.21), then Eq. (3.11) is oscillatory.

Proof. This follows from the proofs of Theorems 2.9 and 3.1. □

As in Theorem 2.12, boundedness of all nonoscillatory solutions of Eq. (3.11), say, can be established in a similar way. In particular, we state the following result.

Theorem 3.6. *If there exists a positive sequence $\{p(n)\}$ so that for all $N \geq n_0$ and $c \in \mathbb{R}$,*

$$\sum_{j=N}^{\infty} \left(\frac{c}{a(j)} + \frac{1}{a(j)} \sum_{s=N}^{j-1} [|e(s)| + |F(s)|] \right)^{1/\alpha} < \infty$$

with $F(n)$ defined in (2.21), then all nonoscillatory solutions of Eq. (3.11) are bounded.

Remark 3.7. 1. The results of this section are new even when $\alpha = 1$.

2. As in the above results of Section 2, one can easily see that the results of Section 3 may be applied to neutral equations. For example, we can consider the equation

$$\Delta(a(n)(\Delta(x(n) + c(n)x(n-\tau+1)))^\alpha) + q_1(n)x^\lambda(n-\sigma+1) = e(n) + q_2(n)x^\mu(n-\sigma+1),$$

where $c(n)$, τ, σ are as in Eq. (2.24), without any additional conditions. The details are left to the reader.

3. The results of this paper can be easily extended to higher order forced difference equations with mixed nonlinearities, say,

$$\Delta^m x(n) + p(n)x(n+1) + q_1(n)x^\lambda(n+1) = e(n) + q_2(n)x^\mu(n+1),$$

where $m \in \mathbb{N}$, $\{e(n)\}$, $\{p(n)\}$, $\{q_i(n)\}$, $i = 1, 2$, λ and μ are as in Eq. (1.1). The details are left to the reader.

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