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Oscillation Criteria for Perturbed Nonlinear Dynamic Equations

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Abstract—In this paper, we discuss the oscillatory behavior of a certain nonlinear perturbed dynamic equation on time scales. We establish some new oscillation criteria for such dynamic equations and supply examples.

Keywords—Oscillation, Second-order nonlinear dynamic equation, Time scale, Riccati transformation technique, Positive solution.

1. INTRODUCTION

The theory of time scales, which has recently received a lot of attention, was introduced by Hilger in his Ph.D. thesis [1] in order to unify continuous and discrete analysis. Not only can this theory of so-called "dynamic equations" *unify* the theories of differential equations and of difference equations, but also it is able to *extend* these classical cases to cases "in between", e.g., to so-called q-difference equations. A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to plenty of applications, among them the study of population dynamic models (see [2]). A book on the subject of time scales by Bohner and Peterson [2] summarizes and organizes much of the time scale calculus (see also [3]). For the notions used below, we refer to [2] and to the next section, where we recall some of the main tools used in the subsequent sections of this paper.

While oscillation theories for differential equations and for difference equations (see, e.g., [4]) are well established, the discrepancies in some of the results in these two theories are not well understood. In the last years there has been much research activity concerning the oscillation and nonoscillation of solutions of some dynamic equations on time scales, and we refer the reader to

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the papers [5–13]. Following this trend, in this paper we shall provide some sufficient conditions for oscillation of second-order nonlinear perturbed dynamic equations of the form

$$\left(\alpha(t)\left(x^{\Delta}\right)^{\gamma}\right)^{\Delta}+F(t,x^{\sigma})=G\left(t,x^{\sigma},x^{\Delta}\right),\qquad \text{for } t\in[a,b], \tag{1.1}$$

where γ is a positive odd integer and α is a positive, real-valued rd-continuous function defined on the time scales interval $[a,b]$ (throughout $a,b\in\mathbb{T}$ with $a<b$). Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above, i.e., it is a time scales interval of the form $[a,\infty)$. By a solution of (1.1) we mean a nontrivial real-valued function x satisfying (1.1) for $t\geq a$. A solution x of (1.1) is said to be *oscillatory* if it is neither eventually positive nor eventually negative, otherwise it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory. Our attention is restricted to those solutions of (1.1) which exist on some half line $[t_x,\infty)$ and satisfy $\sup\{|x(t)|:t>t_0\}>0$ for any $t_0\geq t_x$.

In this paper, we obtain some oscillation criteria for (1.1). The paper is organized as follows. In the next section, we present some basic definitions concerning the calculus on time scales. In Section 3, we give some sufficient conditions for oscillation of (1.1) by using elementary calculus on time scales. In Section 4, we will use Riccati transformation techniques to give some sufficient conditions in terms of the coefficients which guarantee that every solution of (1.1) is oscillatory or converges to zero. To the best of our knowledge, nothing is known regarding the qualitative behavior of (1.1) on time scales up to now.

2. SOME PRELIMINARIES ON TIME SCALES

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. In this paper, we only consider time scales that are unbounded above. On $\mathbb{T}$ we define the *forward jump operator* σ and the *graininess* μ by

$$\sigma(t):=\inf\{s\in\mathbb{T}:s>t\}\qquad\text{and}\qquad \mu(t):=\sigma(t)-t.$$

A point $t\in\mathbb{T}$ with $\sigma(t)=t$ is called right-dense while t is referred to as being right-scattered if $\sigma(t)>t$. The backward jump operator ρ and left-dense and left-scattered points are defined in a similar way. A function $f:\mathbb{T}\to\mathbb{R}$ is said to be *rd-continuous* if it is continuous at each right-dense point and if there exists a finite left limit in all left-dense points. The (delta) *derivative* f^{Δ} of f is defined by

$$f^{\Delta}(t)=\lim_{\substack{s\to t\\ s\in U(t)}}\frac{f(\sigma(t))-f(s)}{\sigma(t)-s},\qquad \text{where } U(t)=\mathbb{T}\setminus\{\sigma(t)\}.$$

The derivative and the forward jump operator are related by the useful formula

$$f^{\sigma}=f+\mu f^{\Delta},\qquad \text{where } f^{\sigma}:=f\circ\sigma. \tag{2.1}$$

We will also make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma}\neq 0$) of two differentiable functions f and g:

$$(fg)^{\Delta}=f^{\Delta}g+f^{\sigma}g^{\Delta}\qquad\text{and}\qquad \left(\frac{f}{g}\right)^{\Delta}=\frac{f^{\Delta}g-fg^{\Delta}}{gg^{\sigma}}. \tag{2.2}$$

By using the product rule from (2.2), the derivative of $f(t)=(t-\alpha)^m$ for $m\in\mathbb{N}$ and $\alpha\in\mathbb{R}$ can be calculated (see [2, Theorem 1.24]) as

$$f^{\Delta}(t)=\sum_{\nu=0}^{m-1}(\sigma(t)-\alpha)^{\nu}(t-\alpha)^{m-\nu-1}. \tag{2.3}$$

For $a, b \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t)\,\Delta t = f(b) - f(a).$$

The integration by parts formula follows from (2.2) and reads

$$\int_a^b f^{\Delta}(t)g(t)\,\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^{\sigma}(t)g^{\Delta}(t)\,\Delta t, \tag{2.4}$$

and infinite integrals are defined as

$$\int_a^{\infty} f(s)\,\Delta s = \lim_{t\to\infty}\int_a^t f(s)\,\Delta s.$$

Note that rd-continuous functions possess antiderivatives and, hence, are integrable.

EXAMPLE 2.1. In case $\mathbb{T} = \mathbb{R}$, we have

$$\sigma(t) = \rho(t) = t, \quad \mu(t) \equiv 0, \qquad f^{\Delta} = f', \qquad \text{and} \qquad \int_a^b f(t)\,\Delta t = \int_a^b f(t)\,dt,$$

and in case $\mathbb{T} = \mathbb{Z}$, we have

$$\sigma(t) = t+1, \quad \rho(t) = t-1, \quad \mu(t) \equiv 1, \qquad f^{\Delta} = \Delta f, \qquad \text{and} \qquad \int_a^b f(t)\,\Delta t = \sum_{t=a}^{b-1} f(t).$$

3. OSCILLATION CRITERIA

In this section, we give some oscillation criteria for (1.1). Throughout this paper, we shall assume that

(H_1) $\alpha : \mathbb{T} \to \mathbb{R}$ is a positive and rd-continuous function;
(H_2) $\gamma \in \mathbb{N}$ is odd;
(H_3) $p, q : \mathbb{T} \to \mathbb{R}$ are rd-continuous functions such that $q(t) - p(t) > 0$ for all $t \in \mathbb{T}$;
(H_4) $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and nondecreasing such that

$$uf(u) > 0, \qquad \text{for all } u \in \mathbb{R}\setminus\{0\};$$

(H_5) $F : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$ and $G : \mathbb{T}\times\mathbb{R}^2 \to \mathbb{R}$ are functions such that

$$uF(t,u) > 0 \quad \text{and} \quad uG(t,u,v) > 0, \qquad \text{for all } u \in \mathbb{R}\setminus\{0\}, \quad v \in \mathbb{R}, \quad t \in \mathbb{T};$$

(H_6) $F(t,u)/f(u) \ge q(t)$ and $G(t,u,v)/f(u) \le p(t)$ for all $u, v \in \mathbb{R}\setminus\{0\}$ and all $t \in \mathbb{T}$.

For simplicity, we list the conditions used in the main results as follows ($t_0 \ge a$):

$$\int_{t_0}^{\infty} \frac{\Delta t}{(\alpha(t))^{1/\gamma}} = \infty, \tag{3.1}$$

$$\int_{t_0}^{\infty} \frac{\Delta t}{(\alpha(t))^{1/\gamma}} < \infty, \tag{3.2}$$

$$\int_{t_0}^{\infty} [q(t) - p(t)]\,\Delta t = \infty, \tag{3.3}$$

$$\lim_{t\to\infty}\int_{t_0}^{t}\left\{\frac{1}{\alpha(s)}\int_s^{\infty}[q(\tau) - p(\tau)]\,\Delta\tau\right\}^{1/\gamma}\Delta s = \infty, \tag{3.4}$$

$$\int_{t_0}^{\infty} [q(t) - p(t)]\,\Delta t > 0, \tag{3.5}$$

$$\int_{t_0}^{\infty}\left\{\frac{M}{\alpha(s)} - \frac{1}{\alpha(s)}\int_{t_0}^{s}[q(t) - p(t)]\,\Delta t\right\}\Delta s = -\infty, \qquad \text{for all } M > 0, \tag{3.6}$$

$$\int_{t_0}^{\infty}\left\{\frac{1}{\alpha(s)}\int_{t_0}^{s}[q(t) - p(t)]\,\Delta t - \frac{M}{\alpha(s)}\right\}^{1/\gamma}\Delta s = \infty, \qquad \text{for all } M > 0. \tag{3.7}$$

THEOREM 3.1. *Assume (H₁)–(H₆). Suppose that (3.1) and (3.3) hold. Then every solution of (1.1) is oscillatory on $[a,\infty)$.*

PROOF. Let x be a nonoscillatory solution of (1.1), say, $x(t)>0$ for $t\ge t_0$ for some $t_0\ge a$. We consider only this case, because the proof for the case that x is eventually negative is similar. From (1.1), (2.2), and the chain rule [2, Theorem 1.87], we have for $t\ge t_0$

$$\left(\frac{\alpha\left(x^{\Delta}\right)^{\gamma}}{f\circ x}\right)^{\Delta}(t)=\frac{G\left(t,x^{\sigma}(t),x^{\Delta}(t)\right)}{f(x^{\sigma}(t))}-\frac{F(t,x^{\sigma}(t))}{f(x^{\sigma}(t))}-\frac{f'(x(\xi))\alpha(t)\left(x^{\Delta}(t)\right)^{\gamma+1}}{f(x(t))f(x(\sigma(t)))},$$

where ξ is a number in the real interval $[t,\sigma(t)]$. In view of (H_2), (H_4), (H_5), and (H_6), we have for all $t\ge t_0$

$$\left(\frac{\alpha\left(x^{\Delta}\right)^{\gamma}}{f\circ x}\right)^{\Delta}(t)\le p(t)-q(t). \tag{3.8}$$

Because of (H_6) and (H_3), from (1.1) we obtain for all $t\ge t_0$

$$\left(\alpha\left(x^{\Delta}\right)^{\gamma}\right)^{\Delta}(t)\le -f(x(\sigma(t)))[q(t)-p(t)]<0, \tag{3.9}$$

which implies that $\alpha(x^{\Delta})^{\gamma}$ is decreasing on $[t_0,\infty)$. We claim that $x^{\Delta}(t)\ge 0$ for all $t\ge t_1\ge t_0$. If not, then there exists $t_2\ge t_1$ such that $\alpha(t)(x^{\Delta}(t))^{\gamma}\le\alpha(t_2)(x^{\Delta}(t_2))^{\gamma}=:c<0$. Hence,

$$x^{\Delta}(t)\le\frac{c^{1/\gamma}}{(\alpha(t))^{1/\gamma}}. \tag{3.10}$$

Integrating (3.10) from t_2 to t provides

$$x(t)\le x(t_2)+c^{1/\gamma}\int_{t_2}^{t}\frac{\Delta s}{(\alpha(s))^{1/\gamma}}\overset{(3.1)}{\longrightarrow}-\infty,\qquad \text{as } t\to\infty, \tag{3.11}$$

while the left-hand side of (3.11), i.e., $x(t)$, is eventually positive. This contradiction implies that $x^{\Delta}(t)\ge 0$ for all $t\ge t_1$. Then, integrating (3.8) from t_1 to t gives

$$\frac{\alpha(t)\left(x^{\Delta}(t)\right)^{\gamma}}{f(x(t))}\le\frac{\alpha(t_1)\left(x^{\Delta}(t_1)\right)^{\gamma}}{f(x(t_1))}-\int_{t_1}^{t}[q(s)-p(s)]\,\Delta s\overset{(3.3)}{\longrightarrow}-\infty \tag{3.12}$$

as $t\to\infty$, while the left-hand side of (3.12) is always nonnegative, a contradiction. Therefore, every solution of (1.1) oscillates. The proof is complete. ∎

EXAMPLE 3.2. If $\mathbb{T}=\mathbb{R}$, then $\sigma(t)=t$ and $\mu(t)\equiv 0$. Then (3.1) and (3.3) become (the Leighton-Wintner-type criteria)

$$\int_{t_0}^{\infty}\frac{dt}{(\alpha(t))^{1/\gamma}}=\infty\qquad\text{and}\qquad\int_{t_0}^{\infty}[q(t)-p(t)]\,dt=\infty.$$

If $\mathbb{T}=\mathbb{Z}$, then $\sigma(t)=t+1$ and $\mu(t)\equiv 1$. Then (3.1) and (3.3) become (the discrete analogue of Leighton-Wintner-type criteria)

$$\sum_{t=t_0}^{\infty}\frac{1}{(\alpha(t))^{1/\gamma}}=\infty\qquad\text{and}\qquad\sum_{t=t_0}^{\infty}[q(t)-p(t)]=\infty.$$

EXAMPLE 3.3. Let $\mathbb{T}\subset[1,\infty)$ be any time scale that is unbounded above. Some of the examples included are $\mathbb{T}=[1,\infty)$, $\mathbb{T}=\mathbb{N}$, and $\mathbb{T}=\{2^k: k\in\mathbb{N}_0\}$. On $\mathbb{T}$, we consider the perturbed nonlinear dynamic equation

$$\left(tx^{\Delta}\right)^{\Delta}+x^{\sigma}\left(\frac{1}{t}+\frac{1}{t^2}+t^2(x^{\sigma})^2\right)=\frac{(x^{\sigma})^5}{2t((x^{\sigma})^4+1)((x^{\Delta})^2+1)}. \tag{3.13}$$

Let

$$\alpha(t) = t, \qquad \gamma = 1, \qquad f(u) = u, \qquad p(t) = \frac{1}{2t}, \qquad q(t) = \frac{1}{t},$$

and

$$F(t,u) = u\left(\frac{1}{t} + \frac{1}{t^2} + t^2u^2\right), \qquad G(t,u,v) = \frac{u^5}{2t(u^4+1)(v^2+1)}.$$

Then (3.13) is in the form (1.1) and Conditions (H_1), (H_2), (H_4), and (H_5) are clearly satisfied. In [14, Theorem 5.11], it was shown that for an unbounded time scale $\mathbb{T} \subset [1,\infty)$ with $a \in \mathbb{T}$, we have

$$\int_a^\infty \frac{1}{t}\,\Delta t = \infty. \tag{3.14}$$

Hence, (3.1) is satisfied, and because of $q(t) - p(t) = 1/(2t) > 0$ and (3.14), (H_3) and (3.3) are satisfied as well. Finally, (H_6) follows from

$$\frac{F(t,u)}{f(u)} = \frac{1}{t} + \frac{1}{t^2} + t^2u^2 \geq \frac{1}{t} = q(t)$$

and

$$\frac{G(t,u,v)}{f(u)} = \frac{u^4}{2t(u^4+1)(v^2+1)} \leq \frac{1}{2t}\frac{u^4}{u^4+1} \leq \frac{1}{2t} = p(t).$$

It follows from Theorem 3.1 that all solutions of (3.13) are oscillatory on $[1,\infty)$. Note that the same statement is also true for the equation

$$\left(t^3\left(x^\Delta\right)^3\right)^\Delta + x^\sigma\left(\frac{1}{t} + \frac{1}{t^2} + t^2(x^\sigma)^2\right) = \frac{(x^\sigma)^5}{2t((x^\sigma)^4+1)((x^\Delta)^2+1)}.$$

THEOREM 3.4. *Assume (H_1)–(H_6). Suppose that (3.1) and (3.4) hold. Then any bounded solution of (1.1) is oscillatory on $[a,\infty)$.*

PROOF. Suppose that x is a bounded nonoscillatory solution of (1.1), say, $x(t) > 0$ for $t \geq t_0$ for some $t_0 \geq a$. As in the proof of Theorem 3.1, since (3.1) holds, we have $x^\Delta(t) \geq 0$ for all $t \geq t_1 \geq t_0$ and the inequality in (3.12) holds. Since the left-hand side of (3.12) is nonnegative, we find

$$\int_{t_1}^t [q(s) - p(s)]\,\Delta s \leq \frac{\alpha(t_1)\left(x^\Delta(t_1)\right)^\gamma}{f(x(t_1))},$$

and therefore, for $t \geq t_1$,

$$\int_t^\infty [q(s) - p(s)]\,\Delta s \leq \frac{\alpha(t)\left(x^\Delta(t)\right)^\gamma}{f(x(t))}. \tag{3.15}$$

Integrating (3.15) from t_1 to t, we get

$$\int_{t_1}^t \left\{\frac{1}{\alpha(s)}\int_s^\infty [q(\tau) - p(\tau)]\,\Delta\tau\right\}^{1/\gamma} \Delta s \leq \int_{t_1}^t \frac{x^\Delta(s)\Delta s}{(f(x(s)))^{1/\gamma}}. \tag{3.16}$$

In view of (H_4), we find that $f(x(t)) \geq f(x(t_1))$ for all $t \geq t_1$. Hence, it follows from (3.16) that

$$\begin{aligned}\int_{t_1}^t \left\{\frac{1}{\alpha(s)}\int_s^\infty [q(\tau) - p(\tau)]\Delta\tau\right\}^{1/\gamma} \Delta s &\leq \int_{t_1}^t \frac{x^\Delta(s)\Delta s}{(f(x(s)))^{1/\gamma}} \\ &\leq \int_{t_1}^t \frac{x^\Delta(s)\Delta s}{(f(x(t_1)))^{1/\gamma}} = \frac{x(t) - x(t_1)}{(f(x(t_1)))^{1/\gamma}}.\end{aligned}$$

By (3.4), the left-hand side of the above inequality tends to ∞ as $t \to \infty$, while the right-hand side is bounded, a contradiction. Therefore, every bounded solution of (1.1) oscillates on $[a,\infty)$. ∎

THEOREM 3.5. *Assume* (H_1)–(H_6). *Suppose that (3.2) and (3.5)–(3.7) hold. Then every solution of (1.1) is oscillatory or converges to zero on* $[a,\infty)$.

PROOF. Suppose that x is a nonoscillatory solution of (1.1) that does not converge to zero, say, $x(t)>0$ for $t\ge t_0$ for some $t_0\ge a$. From (3.9), we have that $\alpha(x^\Delta)^\gamma$ is a decreasing function on $[t_0,\infty)$ and x^Δ is monotone and of one sign.

CASE 1. Suppose that $x^\Delta(t)\ge 0$ for all $t\ge t_1\ge t_0$. As in the proof of Theorem 3.1, we get the inequality in (3.12). Let

$$M=\frac{\alpha(t_1)\left(x^\Delta(t_1)\right)^\gamma}{f(x(t_1))}.$$

Then it follows from the inequality in (3.12) that for all $t\ge t_1$

$$\frac{\left(x^\Delta(t)\right)^\gamma}{f(x(t))}\le\frac{M}{\alpha(t)}-\frac{1}{\alpha(t)}\int_{t_1}^t[q(s)-p(s)]\,\Delta s. \tag{3.17}$$

Integrating (3.17) from t_1 to t, we obtain

$$\int_{t_1}^t\frac{\left(x^\Delta(s)\right)^\gamma}{f(x(s))}\,\Delta s\le\int_{t_1}^t\left\{\frac{M}{\alpha(s)}-\frac{1}{\alpha(s)}\int_{t_1}^s[q(\tau)-p(\tau)]\,\Delta\tau\right\}\Delta s. \tag{3.18}$$

By (3.6), the right-hand side of (3.18) tends to $-\infty$ as $t\to\infty$, whereas the left-hand side is nonnegative, a contradiction.

CASE 2. Suppose that $x^\Delta(t)<0$ for all $t\ge t_1\ge t_0$. Hence, $x(t)\to N>0$ as $t\to\infty$, and by (H_4), $f(x(t))\ge f(N)>0$ for all $t\ge t_1$. From (3.17), it follows that

$$\begin{aligned}\left(x^\Delta(t)\right)^\gamma&\le-\left\{\frac{1}{\alpha(t)}\int_{t_1}^t[q(\tau)-p(\tau)]\Delta\tau-\frac{M}{\alpha(t)}\right\}f(x(t))\\&\le-f(N)\left\{\frac{1}{\alpha(t)}\int_{t_1}^t[q(\tau)-p(\tau)]\Delta\tau-\frac{M}{\alpha(t)}\right\}.\end{aligned}$$

Hence,

$$x^\Delta(t)\le-\left(f(N)\right)^{1/\gamma}\left\{\frac{1}{\alpha(t)}\int_{t_1}^t[q(\tau)-p(\tau)]\Delta\tau-\frac{M}{\alpha(t)}\right\}^{1/\gamma}. \tag{3.19}$$

Integrating (3.19) from t_1 to t, we have

$$x(t)\le x(t_1)-\left(f(N)\right)^{1/\gamma}\int_{t_1}^t\left\{\frac{1}{\alpha(s)}\int_{t_1}^s[q(\tau)-p(\tau)]\Delta\tau-\frac{M}{\alpha(s)}\right\}^{1/\gamma}\Delta s. \tag{3.20}$$

By (3.7), the right-hand side of (3.20) tends to $-\infty$ as $t\to\infty$, but this contradicts the fact that $x(t)$ is positive. This contradiction completes the proof. ∎

THEOREM 3.6. *Assume* (H_1)–(H_6). *Suppose that (3.2), (3.3), and*

$$\int_{t_0}^\infty\left\{\frac{1}{\alpha(t)}\int_{t_0}^t[q(s)-p(s)]\Delta s\right\}^{1/\gamma}\Delta t=\infty \tag{3.21}$$

hold. Then every solution of (1.1) is oscillatory or converges to zero on $[a,\infty)$.

PROOF. Let x be a nonoscillatory solution of (1.1), say, $x(t)>0$ for $t\ge t_0$ for some $t_0\ge a$. As in the proof of Theorem 3.1, we see that x^Δ is either eventually positive or eventually negative. If x^Δ is eventually positive, we can derive a contradiction as in the proof of Theorem 3.1, since (3.3) holds. If $x^\Delta(t)$ is eventually negative, then $\lim_{t\to\infty}x(t)=:N$ exists. We prove that $N=0$.

If not, then $N > 0$, from which by (H_4) we have $f(x(\sigma(t))) \geq f(N) > 0$ for all $t \geq t_1$. Hence, it follows from (1.1) and (H_6) that

$$\left(\alpha\left(x^{\Delta}\right)^{\gamma}\right)^{\Delta}(t) + [q(t) - p(t)]f(N) \leq 0. \tag{3.22}$$

Define the function $u = \alpha(x^{\Delta})^{\gamma}$. Then from (3.22), for $t \geq t_1$, we obtain

$$u^{\Delta}(t) \leq -[q(t) - p(t)]f(N).$$

Hence, for $t \geq t_1$, we have

$$u(t) \leq u(t_1) - f(N)\int_{t_1}^{t}[q(s) - p(s)]\,\Delta s < -f(N)\int_{t_1}^{t}[q(s) - p(s)]\,\Delta s, \tag{3.23}$$

where $u(t_1) = \alpha(t_1)(x^{\Delta}(t_1))^{\gamma} < 0$. From (3.23), we find

$$\int_{t_1}^{t} x^{\Delta}(s)\,\Delta s \leq -(f(N))^{1/\gamma}\int_{t_1}^{t}\left(\frac{1}{\alpha(s)}\int_{t_1}^{s}[q(\tau) - p(\tau)]\,\Delta\tau\right)^{1/\gamma}\Delta s \overset{(3.21)}{\longrightarrow} -\infty$$

as $t \to \infty$, and so $x(t) \to -\infty$ as $t \to \infty$, a contradiction to the fact that $x(t) > 0$ for $t \geq t_0$. Thus, $N = 0$ and then $x(t) \to 0$ as $t \to \infty$. ∎

THEOREM 3.7. *Assume (H_1)–(H_6). Suppose that (3.2), (3.6), and (3.21) hold. Then every solution of (1.1) is oscillatory or converges to zero on $[a, \infty)$.*

PROOF. Again suppose that x is a nonoscillatory solution of (1.1), say, $x(t) > 0$ for $t \geq t_0$ for some $t_0 \geq a$. Since (3.2) holds, we can see from the proof of Theorem 3.4 that x^{Δ} is either eventually positive or eventually negative. If x^{Δ} is eventually positive, we can derive a contradiction as in Case 1 of the proof of Theorem 3.5, since (3.6) holds. If $x^{\Delta}(t)$ is eventually negative, we can prove as in Theorem 3.6 that $x(t)$ converges to zero, and this completes the proof. ∎

4. OSCILLATION CRITERIA BY RICCATI TECHNIQUES

By means of Riccati transformation techniques, we establish some new oscillation criteria for (1.1) in terms of the coefficients.

Throughout this section we shall assume besides (H_1)–(H_6) that

(H_7) there exists $K > 0$ such that $f(u) \geq Ku$ for all $u \in \mathbb{R}$.

THEOREM 4.1. *Assume (H_1)–(H_7). Suppose that (3.1) holds. Furthermore, assume that there exists a differentiable function z such that for all constants $M > 0$,*

$$\limsup_{t\to\infty}\int_{a}^{t}\left\{K[q(s) - p(s)](z^{\sigma}(s))^2 - \frac{(\alpha(s))^{1/\gamma}}{M^{1-1/\gamma}}\left(z^{\Delta}(s)\right)^2\right\}\Delta s = \infty. \tag{4.1}$$

Then every solution of (1.1) is oscillatory on $[a, \infty)$.

PROOF. Suppose that x is a solution of (1.1) with $x(t) \neq 0$ for all t and make the *Riccati substitution*

$$w = z^2\frac{\alpha\left(x^{\Delta}\right)^{\gamma}}{x}. \tag{4.2}$$

We use the rules (2.2) to find

$$\begin{aligned}
-w^{\Delta} &= -z^{\Delta}z\frac{\alpha\left(x^{\Delta}\right)^{\gamma}}{x} - z^{\sigma}\left\{z^{\sigma}\left(\frac{\alpha\left(x^{\Delta}\right)^{\gamma}}{x}\right)^{\Delta} + z^{\Delta}\frac{\alpha\left(x^{\Delta}\right)^{\gamma}}{x}\right\} \\
&= -z^{\Delta}z\frac{\alpha\left(x^{\Delta}\right)^{\gamma}}{x} - (z^{\sigma})^2\left\{\frac{\left[\alpha\left(x^{\Delta}\right)^{\gamma}\right]^{\Delta}}{x^{\sigma}} - \frac{\alpha\left(x^{\Delta}\right)^{\gamma+1}}{xx^{\sigma}}\right\} - z^{\sigma}z^{\Delta}\frac{\alpha\left(x^{\Delta}\right)^{\gamma}}{x} \\
&= (z^{\sigma})^2\left\{\frac{F(t,x^{\sigma})}{x^{\sigma}} - \frac{G\left(t,x^{\sigma},x^{\Delta}\right)}{x^{\sigma}}\right\} + (z^{\sigma})^2\frac{\alpha\left(x^{\Delta}\right)^{\gamma+1}}{xx^{\sigma}} - z^{\Delta}z\frac{\alpha\left(x^{\Delta}\right)^{\gamma}}{x} - z^{\sigma}z^{\Delta}\frac{\alpha\left(x^{\Delta}\right)^{\gamma}}{x} \\
&= (z^{\sigma})^2\left\{\frac{F(t,x^{\sigma})}{x^{\sigma}} - \frac{G\left(t,x^{\sigma},x^{\Delta}\right)}{x^{\sigma}}\right\} + \alpha zz^{\sigma}\left(x^{\Delta}\right)^{\gamma-1}\left\{\frac{z^{\sigma}}{z}\frac{\left(x^{\Delta}\right)^2}{xx^{\sigma}} - \frac{z^{\Delta}}{z^{\sigma}}\frac{x^{\Delta}}{x} - \frac{z^{\Delta}}{z}\frac{x^{\Delta}}{x}\right\}.
\end{aligned}$$

We put

$$r = \frac{x^\Delta}{x} \qquad \text{and} \qquad s = \frac{z^\Delta}{z},$$

and recall [2] the definitions $\ominus r = -r/(1+\mu r)$, $r^{②} = (-r)(\ominus r)$, and $r \ominus s = (r-s)/(1+\mu s)$. Then

$$\begin{aligned} \frac{z^\sigma}{z}\frac{\left(x^\Delta\right)^2}{xx^\sigma} - \frac{z^\Delta}{z^\sigma}\frac{x^\Delta}{x} - \frac{z^\Delta}{z}\frac{x^\Delta}{x} &= \frac{z+\mu z^\Delta}{z} r^{②} + (\ominus s)r - sr \\ &= r^{②} + \mu s r^{②} - sr + (\ominus s)r \\ &= r^{②} + s\left(\mu r^{②} - r\right) + (\ominus s)r \\ &= r^{②} + s(\ominus r) + (\ominus s)r \\ &= (r \ominus s)^{②} - s^{②} \\ &= (r\ominus s)^{②} - \frac{\left(z^\Delta\right)^2}{zz^\sigma}. \end{aligned}$$

Altogether, we have shown now that

$$-w^\Delta = (z^\sigma)^2 \left\{ \frac{F(t,x^\sigma)}{x^\sigma} - \frac{G\left(t,x^\sigma,x^\Delta\right)}{x^\sigma} \right\} + \alpha z z^\sigma \left(x^\Delta\right)^{\gamma-1} (r\ominus s)^{②} - \alpha\left(z^\Delta\right)^2 \left(x^\Delta\right)^{\gamma-1}.$$

Hence, if $xx^\sigma > 0$, we can estimate (apply (H_1)–(H_7))

$$-w^\Delta \geq K(z^\sigma)^2(q-p) - \alpha\left(z^\Delta\right)^2\left(x^\Delta\right)^{\gamma-1}. \tag{4.3}$$

Using these preliminaries, we now may start the actual proof of the theorem. Assume that x is a solution of (1.1) which is positive on $[t_0,\infty)$ for some $t_0 \geq a$ (a similar proof applies to the case that x is eventually negative). Define

$$y = \alpha\left(x^\Delta\right)^\gamma. \tag{4.4}$$

Then for $t \geq t_0$, $x(\sigma(t)) > 0$, $f(x^\sigma(t)) > 0$, and

$$y^\Delta(t) = G\left(t, x^\sigma(t), x^\Delta(t)\right) - F(t, x^\sigma(t)) \leq f(x^\sigma(t))[p(t)-q(t)] < 0,$$

and therefore, y is strictly decreasing on $[t_0,\infty)$. If there exists $t_1 \geq t_0$ with $y(t_1) =: c < 0$, then

$$\alpha(s)\left(x^\Delta(s)\right)^\gamma = y(s) \leq y(t_1) = c, \qquad \text{for all } s \geq t_1,$$

and so

$$\left(x^\Delta(s)\right)^\gamma \leq \frac{c}{\alpha(s)}, \qquad \text{for all } s \geq t_1.$$

Therefore,

$$x^\Delta(s) \leq \frac{c^{1/\gamma}}{(\alpha(s))^{1/\gamma}}, \qquad \text{for all } s \geq t_1.$$

Integrating from t_1 to $t \geq t_1$ provides

$$x(t) - x(t_1) = \int_{t_1}^t x^\Delta(s)\,\Delta s \leq c^{1/\gamma}\int_{t_1}^t \frac{\Delta s}{(\alpha(s))^{1/\gamma}}$$

for all $t \geq t_1$ so that

$$x(t) \leq x(t_1) + c^{1/\gamma}\int_{t_1}^t \frac{\Delta s}{(\alpha(s))^{1/\gamma}} \xrightarrow{(3.1)} -\infty,$$

contradicting the positivity of x on $[t_0, \infty)$. Therefore, $y(t) > 0$ for all $t \geq t_0$, and hence, $x^\Delta(t) > 0$ for all $t \geq t_0$. Now, since y is positive and decreasing on $[t_0, \infty)$, we find $0 < y(t) \leq y(t_0)$ for all $t \geq t_0$. Let $M = 1/y(t_0)$. Then

$$x^\Delta(t) \leq \frac{1}{(\alpha(t)M)^{1/\gamma}}, \qquad \text{and hence,} \qquad \left(x^\Delta(t)\right)^{\gamma-1} \leq \frac{1}{(\alpha(t)M)^{1-1/\gamma}}$$

for all $t \geq t_0$. Using this in (4.3), we obtain

$$-w^\Delta \geq K(z^\sigma)^2(q-p) - \frac{\alpha^{1/\gamma}}{M^{1-1/\gamma}}\left(z^\Delta\right)^2. \tag{4.5}$$

Integrating (4.5) from t_0 to $t \geq t_0$ provides (note that $w(t) > 0$ for all $t \geq t_0$ by (4.2))

$$w(t_0) \geq \int_{t_0}^{t}\left\{K(z^\sigma(s))^2[q(s)-p(s)] - \frac{(\alpha(s))^{1/\gamma}}{M^{1-1/\gamma}}\left(z^\Delta(s)\right)^2\right\}\Delta s \overset{(4.1)}{\longrightarrow} \infty,$$

which is impossible. The proof is therefore complete. ∎

We remark that in case $\gamma = 1$, $M^{1-1/\gamma} = 1$ so that (4.1) is independent from the number M. Similar remarks also hold for the results that follow.

COROLLARY 4.2. *Assume (H$_1$)–(H$_7$). Suppose that (3.1) holds. Furthermore, assume that there exists a positive differentiable function δ such that for all constants $M > 0$,*

$$\limsup_{t\to\infty}\int_a^t\left\{K[q(s)-p(s)]\delta^\sigma(s) - \frac{(\alpha(s))^{1/\gamma}}{M^{1-1/\gamma}}\left(\frac{\delta^\Delta(s)}{\sqrt{\delta(s)}+\sqrt{\delta^\sigma(s)}}\right)^2\right\}\Delta s = \infty. \tag{4.6}$$

Then every solution of (1.1) is oscillatory on $[a, \infty)$.

PROOF. Define $z = \sqrt{\delta}$ and note that

$$z^\Delta = \frac{\delta^\Delta}{\sqrt{\delta}+\sqrt{\delta^\sigma}}.$$

If (4.6) holds for δ, then (4.1) holds for $z = \sqrt{\delta}$. Thus, the claim follows from Theorem 4.1. ∎

From Theorem 4.1 and Corollary 4.2, we can obtain different conditions for oscillation of all solutions of (1.1) by different choices of $\delta(t)$. For instance, let $\delta(t) \equiv 1$ or $\delta(t) = t$. By Corollary 4.2, we then have the following two results.

COROLLARY 4.3. *Assume (H$_1$)–(H$_7$). Suppose that (3.1) holds. If*

$$\limsup_{t\to\infty}\int_a^t [q(s)-p(s)]\,\Delta s = \infty, \tag{4.7}$$

then every solution of (1.1) is oscillatory on $[a, \infty)$.

COROLLARY 4.4. *Assume (H$_1$)–(H$_7$). Suppose that (3.1) holds. If for all constants $M > 0$,*

$$\limsup_{t\to\infty}\int_a^t\left\{K[q(s)-p(s)]\sigma(s) - \frac{(\alpha(s))^{1/\gamma}}{M^{1-1/\gamma}\left(\sqrt{s}+\sqrt{\sigma(s)}\right)^2}\right\}\Delta s = \infty, \tag{4.8}$$

then every solution of (1.1) is oscillatory on $[a, \infty)$.

EXAMPLE 4.5. Again let $\mathbb{T} \subset [1, \infty)$ be a time scale which is unbounded above. On $\mathbb{T}$, we consider the perturbed nonlinear dynamic equation

$$x^{\Delta\Delta} + x^\sigma\left(\frac{1}{t\sigma(t)} + \frac{1}{t^2} + (x^\sigma)^2\right) = \frac{(x^\sigma)^3}{2t\sigma(t)\left((x^\sigma)^2 + (x^\Delta)^2 + 1\right)}. \tag{4.9}$$

Let

$$\alpha(t) \equiv 1, \qquad \gamma = 1, \qquad f(u) = u, \qquad K = 1, \qquad p(t) = \frac{1}{2t\sigma(t)}, \qquad q(t) = \frac{1}{t\sigma(t)},$$

and

$$F(t,u) = u\left(\frac{1}{t\sigma(t)} + \frac{1}{t^2} + u^2\right), \qquad G(t,u,v) = \frac{u^3}{2t\sigma(t)(u^2+v^2+1)}.$$

Then (4.9) is in the form (1.1) and Conditions (H_1), (H_2), (H_4), (H_5), (H_7), and (3.1) are clearly satisfied. Because of $q(t) - p(t) = 1/(2t\sigma(t)) > 0$, ($H_3$) is satisfied as well. Next, (H_6) follows from

$$\frac{F(t,u)}{f(u)} = \frac{1}{t\sigma(t)} + \frac{1}{t^2} + u^2 \geq \frac{1}{t\sigma(t)} = q(t)$$

and

$$\frac{G(t,u,v)}{f(u)} = \frac{u^2}{2t\sigma(t)(u^2+v^2+1)} \leq \frac{1}{2t\sigma(t)} = p(t).$$

Finally, (4.8) follows from the estimate

$$\begin{aligned}\int_a^t \left\{[q(s)-p(s)]\sigma(s) - \frac{1}{\left(\sqrt{s}+\sqrt{\sigma(s)}\right)^2}\right\}\Delta s &= \int_a^t \left\{\frac{1}{2s} - \frac{1}{\left(\sqrt{s}+\sqrt{\sigma(s)}\right)^2}\right\}\Delta s \\ &\geq \int_a^t \left\{\frac{1}{2s} - \frac{1}{(\sqrt{s}+\sqrt{s})^2}\right\}\Delta s \\ &= \frac{1}{4}\int_a^\infty \frac{1}{s}\Delta s \overset{(3.14)}{\longrightarrow} \infty.\end{aligned}$$

By Corollary 4.4, every solution of (4.9) oscillates. We remark that the same statement is also true for the equation

$$x^{\Delta\Delta} + x^\sigma\left(\frac{c}{t\sigma(t)} + \frac{1}{t^2} + (x^\sigma)^2\right) = \frac{d(x^\sigma)^3}{t\sigma(t)\left((x^\sigma)^2 + (x^\Delta)^2 + 1\right)},$$

provided $d > 0$ and $c > d + 1/4$.

THEOREM 4.6. *Assume (H_1)–(H_7). Suppose that (3.1) holds. Furthermore, assume that there exists a differentiable function z such that for all constants $M > 0$,*

$$\limsup_{t\to\infty} \frac{1}{t^m}\int_a^t (t-s)^m\left\{K[q(s)-p(s)]\left(z^\sigma(s)\right)^2 - \frac{(\alpha(s))^{1/\gamma}}{M^{1-1/\gamma}}\left(z^\Delta(s)\right)^2\right\}\Delta s = \infty, \tag{4.10}$$

where $m \in \mathbb{N}$ is odd. Then every solution of (1.1) is oscillatory on $[a,\infty)$.

PROOF. We proceed as in the proof of Theorem 4.1. We may assume that (1.1) has a nonoscillatory solution x such that $x(t) > 0$, $x^\Delta(t) \geq 0$, $(\alpha(x^\Delta)^\gamma)^\Delta(t) \leq 0$ for $t \geq t_0$. Define w by (4.2) as before, then we have $w(t) > 0$ and (4.5) holds. Then from (4.5) we have, using integration by parts (2.4) and (2.3),

$$\begin{aligned}&\int_{t_0}^t (t-s)^m\left\{K(z^\sigma(s))^2[q(s)-p(s)] - \frac{(\alpha(s))^{1/\gamma}}{M^{1-1/\gamma}}\left(z^\Delta(s)\right)^2\right\}\Delta s \\ &\leq -\int_{t_0}^t (t-s)^m w^\Delta(s)\,\Delta s \\ &= (t-t_0)^m w(t_0) - (-1)^{m+1}\int_{t_0}^t \sum_{\nu=0}^{m-1} (\sigma(t)-s)^\nu (t-s)^{m-\nu-1} w(\sigma(s))\Delta s \\ &< (t-t_0)^m w(t_0).\end{aligned}$$

Hence,

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_{t_0}^{t}(t-s)^m\left\{K(z^\sigma(s))^2[q(s)-p(s)]-\frac{(\alpha(s))^{1/\gamma}}{M^{1-1/\gamma}}\left(z^\Delta(s)\right)^2\right\}\Delta s\le w(t_0),$$

which contradicts (4.10). ∎

Note that when $z(t)\equiv 1$, then (4.10) reduces to

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_a^t(t-s)^m[q(s)-p(s)]\,\Delta s=\infty, \tag{4.11}$$

which can be considered as an extension of Kamenev-type oscillation criteria for second-order differential equations. When $\mathbb{T}=\mathbb{R}$, then (4.11) becomes

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_a^t(t-s)^m[q(s)-p(s)]\,ds=\infty,$$

and when $\mathbb{T}=\mathbb{Z}$, then (4.11) becomes

$$\limsup_{t\to\infty}\frac{1}{t^m}\sum_{s=a}^{t-1}(t-s)^m[q(s)-p(s)]=\infty.$$

Next, we give some sufficient conditions when (3.1) does not hold, which guarantee that every solution of (1.1) oscillates or converges to zero in $[a,\infty)$.

THEOREM 4.7. *Assume* (H_1)–(H_7). *Suppose that (3.2) and (3.21) hold. Assume there exists a differentiable function z such that (4.1) holds for all constants $M>0$. Then every solution of (1.1) is oscillatory or converges to zero in $[a,\infty)$.*

PROOF. We proceed as in Theorem 4.1 and assume that (1.1) has a nonoscillatory solution such that $x(t)>0$ for $t\ge t_0>a$. From the proof of Theorem 4.1, we see that there exist two possible cases of the sign of $x^\Delta(t)$. The proof when x^Δ is eventually positive is similar to the proof of Theorem 4.1, and hence, is omitted. Now suppose that $x^\Delta(t)<0$ for $t\ge t_1$. Then x is decreasing and $\lim_{t\to\infty}x(t)=b\ge 0$. We assert that $b=0$. If not, then $x(\sigma(t))>b>0$ for $t\ge t_2>t_1$. Then there exists $t_3>t_2$ such that $f(x(\sigma(t)))\ge Kb$ for $t\ge t_3$. Define the function y by (4.4). Then from (4.1), for $t\ge t_3$, we obtain

$$y^\Delta(t)\le-[q(t)-p(t)]f(x(\sigma(t)))\le-Kb[q(t)-p(t)].$$

Hence, for $t\ge t_3$, we have

$$y(t)\le y(t_3)-Kb\int_{t_3}^t[q(s)-p(s)]\Delta s<-Kb\int_{t_3}^t[q(s)-p(s)]\,\Delta s,$$

where $y(t_3)=\alpha(t_3)(x^\Delta(t_3))^\gamma<0$. Integrating the last inequality from t_3 to t, we have

$$\int_{t_3}^t x^\Delta(s)\,\Delta s\le-(Kb)^{1/\gamma}\int_{t_3}^t\left(\frac{1}{\alpha(s)}\int_{t_3}^s[q(\tau)-p(\tau)]\,\Delta\tau\right)^{1/\gamma}\Delta s.$$

By (3.21), we get

$$x(t)\to-\infty,\qquad\text{as } t\to\infty,$$

a contradiction to the fact that $x(t)>0$ for $t\ge t_0$. Thus, $b=0$ and $x(t)\to 0$ as $t\to\infty$. ∎

As in the proof of Theorem 4.7, we can prove the following theorem.

THEOREM 4.8. *Assume* (H_1)–(H_7). *Suppose that (3.2) and (3.21) hold. Assume there exists a differentiable function z such that (4.10) holds for all constants $M>0$. Then every solution of (1.1) is oscillatory or converges to zero in $[a,\infty)$.*

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