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Oscillation criteria for second-order differential equations with superlinear neutral term

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Abstract. In this paper, some sufficient conditions for the oscillation of all solutions of a second-order nonlinear neutral differential equation with superlinear neutral term are obtained. By means of an inequality technique and an integral averaging method, some new oscillation criteria are presented which extend and complement those reported in the literature.

1 Introduction

Since neutral differential equations have wide applications in the fields of science, engineering, and technology, oscillation theory of such equations has attracted very great interest of mathematicians, and it has been studied extensively during the past few decades, see for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26, 27, 28] and the references cited therein.

From the review of literature, one can see that many results are available on the oscillation of second-order differential equations with linear neutral term, and very few results are available when

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the neutral term is nonlinear, see [1, 6, 7, 19, 22, 24, 25]. However, from the above references, one can see that all oscillation theorems are for second-order differential equations with sublinear neutral term, and to the best of our knowledge, no oscillation results are available for second-order differential equations with superlinear neutral term. Motivated by this observation, in this paper, we are concerned with the oscillatory behavior of the second-order neutral differential equation

$$(a(x+p(x^{\alpha}\circ\tau))')'(t)+q(t)f(x(\sigma(t)))=0, \quad t\geq t_0>0, \tag{1.1}$$

subject to the following conditions:

(H$_1$) $a\in \mathrm{C}^1([t_0,\infty),(0,\infty))$, $p,q\in \mathrm{C}([t_0,\infty),[0,\infty))$, q is not eventually zero on $[t_*,\infty)$ for $t_*\geq t_0$, and $p(t)\to\infty$ as $t\to\infty$;

(H$_2$) $f\in \mathrm{C}(\mathbb{R},\mathbb{R})$ and there exists $M>0$ such that $f(u)/u^{\beta}\geq M$ for all $u\neq 0$;

(H$_3$) $\alpha\geq 1$ and $\beta\in(0,\infty)$ are ratios of odd positive integers.

(H$_4$) $\tau,\sigma\in \mathrm{C}^1([t_0,\infty),\mathbb{R})$, $\tau(t)\leq t$, $\sigma(t)\leq t$, $\tau'(t)>0$, and

$$\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\sigma(t)=\infty.$$

By a solution of (1.1), we mean $x\in \mathrm{C}([T_x,\infty),\mathbb{R})$, $T_x\geq t_0$, with $a(x+p(x^{\alpha}\circ\tau))'\in \mathrm{C}^1([T_x,\infty),\mathbb{R})$ and such that (1.1) is satisfied on $[T_x,\infty)$. We consider only these solutions x of (1.1) which satisfy $\sup\{|x(t)|:t\geq T\}>0$ for all $T\geq T_x$, and we assume that (1.1) possesses such solutions. As usual, a solution of (1.1) is called oscillatory if it has zeros on $[T,\infty)$ for all $T\geq T_x$; otherwise it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

In [1, 7, 22, 24, 25], the authors considered (1.1) with $0<\alpha\leq 1$ and $f(u)=u^{\beta}$, $\beta\in(0,\infty)$, and they established conditions for oscillation of all solutions in the cases

$$\int_{t_0}^{\infty}\frac{\mathrm{d}t}{a(t)}=\infty \tag{1.2}$$

or

$$\int_{t_0}^{\infty}\frac{\mathrm{d}t}{a(t)}<\infty. \tag{1.3}$$

Therefore, in this paper, we obtain conditions for oscillation of (1.1) under the conditions (1.2) or (1.3) and $\alpha\geq 1$. Hence, the results presented here generalize and complement some of the results reported in [1, 6, 7, 19, 22, 24, 25].

2 Oscillation Results

In this section, we obtain sufficient conditions for oscillation of all solutions of (1.1). Due to the form of our equation, we only need to give proofs for the case of positive nonoscillatory solutions since the proofs for negative solutions are similar. For any $T\geq t_0$ and for any real-valued positive function μ which is decreasing and tending to zero as $t\to\infty$, we denote

$$z(t)=x(t)+p(t)x^{\alpha}(\tau(t)),\quad R(t)=\int_T^t\frac{\mathrm{d}s}{a(s)},\quad A(t)=\int_t^{\infty}\frac{\mathrm{d}s}{a(s)},$$

$$Q_1(t)=\frac{1-\frac{1}{\alpha p^{\frac{1}{\alpha}}(\tau^{-1}(\tau^{-1}(t)))}\left(\frac{R(\tau^{-1}(\tau^{-1}(t)))}{R(\tau^{-1}(t))}+\frac{\alpha-1}{\mu(\tau^{-1}(t))}\right)}{p(\tau^{-1}(t))}>0,$$

$$Q_2(t)=\frac{1-\frac{1}{\alpha p^{\frac{1}{\alpha}}(\tau^{-1}(\tau^{-1}(t)))}\left(1+\frac{\alpha-1}{A^2(\tau^{-1}(t))}\right)}{p(\tau^{-1}(t))}>0$$

for all $t \geq T$, where τ^{-1} is the inverse function of τ. Note that the last two conditions imply that $p(t) \to \infty$ as $t \to \infty$.

We begin with the following auxiliary result, which can be found in [12, Theorem 41, p. 39].

Lemma 2.1. If $a > 0$ and $0 < \gamma \leq 1$, then

$$a^{\gamma} \leq \gamma a + (1-\gamma). \tag{2.1}$$

Lemma 2.2. Assume (1.2). If x is a positive solution of (1.1), then the corresponding z satisfies

$$z > 0, \quad z' > 0, \quad \text{and} \quad (az')' \leq 0 \tag{2.2}$$

eventually.

Proof. The proof is similar to that of [28, Lemma 1], and hence the details are omitted.

Lemma 2.3. Let x be a positive solution of (1.1) such that the corresponding z satisfies (2.2). If there exists a positive function μ which is decreasing and tending zero, then

$$x^{\alpha}(t) \geq Q_1(t) z(\tau^{-1}(t)), \quad t \geq T. \tag{2.3}$$

Proof. From (H_1) and the definition of z, we have $z(t) \geq x(t)$ for all $t \geq t_1 \geq t_0$. Again from the definition of z, we have

$$\begin{aligned} x^{\alpha}(t) &= \frac{1}{p(\tau^{-1}(t))}\left(z(\tau^{-1}(t)) - x(\tau^{-1}(t))\right) \\ &\geq \frac{1}{p(\tau^{-1}(t))}\left(z(\tau^{-1}(t)) - \frac{z^{1/\alpha}(\tau^{-1}(\tau^{-1}(t)))}{p^{1/\alpha}(\tau^{-1}(\tau^{-1}))}\right). \end{aligned}$$

Since $\frac{1}{\alpha} \leq 1$, using Lemma 2.1 in the last inequality, we obtain

$$x^{\alpha}(t) \geq \frac{z(\tau^{-1}(t)) - \frac{1}{p^{1/\alpha}(\tau^{-1}(\tau^{-1}(t)))}\left(\frac{1}{\alpha} z(\tau^{-1}(\tau^{-1}(t))) + \frac{\alpha-1}{\alpha}\right)}{p(\tau^{-1}(t))}. \tag{2.4}$$

From (2.2), we have

$$z(t) = z(t_1) + \int_{t_1}^{t} \frac{a(s) z'(s)}{a(s)} \mathrm{d}s \geq R(t) a(t) z'(t), \tag{2.5}$$

and we deduce from (2.5) that for all $t \geq t_1$, we have

$$\left(\frac{z}{R}\right)'(t) \leq 0. \tag{2.6}$$

Using (2.6) and the condition $\tau^{-1}(t) \leq \tau^{-1}(\tau^{-1}(t))$, we conclude

$$x^{\alpha}(t) \geq \frac{z(\tau^{-1}(t)) - \frac{1}{\alpha p^{1/\alpha}(\tau^{-1}(\tau^{-1}(t)))}\left(\frac{R(\tau^{-1}(\tau^{-1}(t)))}{R(\tau^{-1}(t))} z(\tau^{-1}(t)) + \alpha - 1\right)}{p(\tau^{-1}(t))}. \tag{2.7}$$

Since z is increasing and μ is decreasing and tending to zero, there exists $T \geq t_1$ such that $z(t) \geq \mu(t)$ for all $t \geq T$. Substituting this in (2.7) and rearranging, we obtain (2.3). This completes the proof.

Lemma 2.4. Assume (1.3). If x is a positive solution of (1.1), then the corresponding z satisfies eventually one of the following two cases:

(1) $z > 0$, $z' > 0$, $(az')' \le 0$;
(2) $z > 0$, $z' < 0$, $(az')' \le 0$.

Proof. The proof is similar to that of [25, Lemma 2.1], and hence the details are omitted.

Lemma 2.5. Let x be a positive solution of (1.1) and suppose z satisfies Case (2) of Lemma 2.4. Then there exists $T \ge t_0$ such that

$$x^{\alpha}(t) \ge Q_2(t) z(\tau^{-1}(t)), \quad t \ge T. \tag{2.8}$$

Proof. From (H_1) and the definition of z, we have $z(t) \ge x(t)$ for all $t \ge T \ge t_0$. Further, for $s \ge t$, we have

$$z'(s) \le \frac{a(t)z'(t)}{a(s)},$$

and integrating this inequality from t to ℓ, we find

$$z(\ell) \le z(t) + a(t)z'(t) \int_t^{\ell} \frac{\mathrm{d}s}{a(s)}.$$

Letting $\ell \to \infty$, we get

$$0 \le z(t) + A(t)a(t)z'(t),$$

and we deduce from the last inequality

$$\left(\frac{z}{A}\right)'(t) \ge 0, \quad t \ge T. \tag{2.9}$$

Since $\tau^{-1}(t) \le \tau^{-1}(\tau^{-1}(t))$ and z is decreasing, we have from (2.4) that

$$x^{\alpha}(t) \ge \frac{z(\tau^{-1}(t)) - \frac{1}{\alpha p^{1/\alpha}(\tau^{-1}(\tau^{-1}(t)))}\left(z(\tau^{-1}(t)) + \alpha - 1\right)}{p(\tau^{-1}(t))} \tag{2.10}$$

holds. From (2.9), it follows that $\frac{z(t)}{A(t)}$ is increasing and $A(t)$ is decreasing and tending to zero, thus $\frac{z(t)}{A(t)} \ge A(t)$ for all $t \ge T$. Substituting this in (2.10) and rearranging, we get (2.8). This completes the proof.

Here is our first oscillation result.

Theorem 2.1. Assume (1.2) and $\beta \ge \alpha$. If $\sigma(t) \le \tau(t)$ for all $t \ge T$ and there exists a positive, nondecreasing and differentiable function ρ such that

$$\limsup_{t\to\infty} \int_T^t \left[\rho(s)q(s)Q_1^{\frac{\beta}{\alpha}}(\sigma(s)) \frac{R^{\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(s)))}{R^{\frac{\beta}{\alpha}}(s)} - \frac{a(s)(\rho'(s))^2}{4M_2\rho(s)}\right] \mathrm{d}s = \infty \tag{2.11}$$

for every constant $M_2 > 0$, then every solution of (1.1) is oscillatory.

Proof. Let x be an eventually positive solution of (1.1), such that $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for all $t \geq T \geq t_0$, where T is chosen so that (2.2) holds for $t \geq T$. From (1.1) and (2.3), we get

$$(az')'(t) + Mq(t)Q_1^{\frac{\beta}{\alpha}}(\sigma(t))z^{\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(t))) \leq 0, \quad t \geq T. \tag{2.12}$$

Since $\tau^{-1}(\sigma(t)) \leq t$ and $\frac{z(t)}{R(t)}$ is decreasing, we have

$$\frac{z(\tau^{-1}(\sigma(t)))}{R(\tau^{-1}(\sigma(t)))} \geq \frac{z(t)}{R(t)}, \quad t \geq T. \tag{2.13}$$

Using (2.13) in (2.12) yields

$$(az')'(t) + Mq(t)Q_1^{\frac{\beta}{\alpha}}(\sigma(t))\frac{R^{\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(t)))}{R^{\frac{\beta}{\alpha}}(t)}z^{\frac{\beta}{\alpha}}(t) \leq 0, \quad t \geq T. \tag{2.14}$$

Set

$$w(t) = \rho(t)\frac{a(t)z'(t)}{z(t)}, \quad t \geq T. \tag{2.15}$$

Then $w(t) > 0$ for $t \geq T$, and using (2.14), we have

$$\begin{aligned} w'(t) &\leq \rho(t)\frac{(az')'(t)}{z(t)} + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{w^2(t)}{a(t)\rho(t)} \\ &\leq -MK^{\frac{\beta}{\alpha}-1}\rho(t)q(t)Q_1^{\frac{\beta}{\alpha}}(\sigma(t))\frac{R^{\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(t)))}{R^{\frac{\beta}{\alpha}}(t)} + \frac{a(t)(\rho'(t))^2}{4\rho(t)}, \end{aligned}$$

where we have used $z(t) \geq K > 0$ and $\beta \geq \alpha$ for $t \geq T$. Integrating the last inequality from T to t, we obtain

$$\int_T^t \left[\rho(s)q(s)Q_1^{\frac{\beta}{\alpha}}(\sigma(s))\frac{R^{\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(s)))}{R^{\frac{\beta}{\alpha}}(s)} - \frac{a(s)(\rho'(s))^2}{4M_2\rho(s)}\right]\mathrm{d}s \leq w(T),$$

where $M_2 = MK^{\frac{\beta}{\alpha}-1}$. Letting $t \to \infty$ in the last inequality, we obtain a contradiction with (2.11). This completes the proof.

Theorem 2.2. Assume (1.2) and $\beta \leq \alpha$. If $\sigma(t) \leq \tau(t)$ for all $t \geq T$ and there exists a positive, nondecreasing and differentiable function ρ such that

$$\limsup_{t\to\infty}\int_T^t \left[\rho(s)q(s)Q_1^{\frac{\beta}{\alpha}}(\sigma(s))\frac{R^{\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(s)))}{R(s)} - \frac{a(s)(\rho'(s))^2}{4M_3\rho(s)}\right]\mathrm{d}s = \infty \tag{2.16}$$

for every constant $M_3 > 0$, then every solution of (1.1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 2.1, we have (2.12). Since $\frac{z(t)}{R(t)}$ is decreasing and $\beta < \alpha$, there exists $K_1 > 0$ such that

$$\frac{z^{\frac{\beta}{\alpha}-1}(t)}{R^{\frac{\beta}{\alpha}-1}(t)} \geq \frac{1}{K_1^{1-\frac{\beta}{\alpha}}}, \quad t \geq T.$$

Using this in (2.12), we obtain

$$(az')'(t) + M_3 q(t) Q_1^{\frac{\beta}{\alpha}}(\sigma(t)) \frac{R^{\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(t)))}{R(t)} z(t) \leq 0, \quad t \geq T,$$

where $M_3 = \frac{M}{K_1^{1-\frac{\beta}{\alpha}}}$. The rest of the proof is similar to that of Theorem 2.1, and the details are omitted. This completes the proof.

Our next theorems are for the case when (1.3) holds.

Theorem 2.3. Assume (1.3). If

$$\int_T^\infty q(t) Q_1^{\frac{\beta}{\alpha}}(\sigma(t)) \mathrm{d}t = \infty \tag{2.17}$$

and

$$\int_T^\infty \frac{1}{a(t)} \left(\int_T^t q(s) Q_2^{\frac{\beta}{\alpha}}(\sigma(s)) A^{\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(s))) \mathrm{d}s \right) \mathrm{d}t = \infty, \tag{2.18}$$

then every solution of (1.1) is oscillatory.

Proof. Let x be an eventually positive solution of (1.1) such that $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for all $t \geq T \geq t_0$, where T is chosen so that the cases of Lemma 2.4 hold for all $t \geq T$. We shall show that in each case we are led to a contradiction. Case (1). From (1.1) and (2.3), we obtain (2.12), and integrating this from T to t and using the fact that $z(t)$ is increasing, we have

$$\int_T^t q(s) Q_1^{\frac{\beta}{\alpha}}(\sigma(s)) \mathrm{d}s < \infty.$$

Letting $t \to \infty$ in the last inequality, we obtain a contradiction with (2.17). Case (2). From (1.1) and (2.9), we have

$$(az')'(t) + M q(t) Q_2^{\frac{\beta}{\alpha}}(\sigma(t)) z^{\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(t))) \leq 0, \quad t \geq T.$$

Integrating the last inequality from T to t, we obtain

$$M \int_T^t q(s) Q_2^{\frac{\beta}{\alpha}}(\sigma(s)) z^{\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(s))) \mathrm{d}s \leq -a(t) z'(t),$$

and hence

$$\frac{M K_2^{\frac{\beta}{\alpha}}}{a(t)} \int_T^t q(s) Q_2^{\frac{\beta}{\alpha}}(\sigma(s)) A^{\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(s))) \mathrm{d}s \leq -z'(t), \tag{2.19}$$

where we have used that $\frac{z(t)}{A(t)}$ is increasing and $\frac{z(t)}{A(t)} \geq K_2 > 0$ for all $t \geq T$. Integrating (2.19) from T to t and then letting $t \to \infty$, we obtain a contradiction with (2.18). This completes the proof.

Theorem 2.4. Assume (1.3) and (2.18). If $\sigma(t) < \tau(t)$ and the first-order delay differential inequality

$$w'(t) + M q(t) Q_1^{\frac{\beta}{\alpha}}(\sigma(t)) R^{\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(t))) w^{\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(t))) \leq 0, \quad t \geq T \tag{2.20}$$

has no positive solution, then every solution of (1.1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 2.3, we see that one of the two cases of Lemma 2.4 holds. Case (1). Proceeding as in the proof of Theorem 2.1, we have (2.12). Now using (2.5), it follows from (2.12) that

$$(az')'(t)+Mq(t)Q_1^{\frac{\beta}{\alpha}}(\sigma(t))R^{\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(t)))w^{\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(t)))\le 0, \quad t\ge T$$

holds. Let $w(t)=a(t)z'(t)>0$ for $t\ge T$. Then w is a positive solution of the inequality (2.20), which is a contradiction. The proof of Case (2) is similar to that of Case (2) of Theorem 2.3. This completes the proof.

Corollary 2.1. Assume (1.3), $\alpha=\beta$, and $\sigma(t)<\tau(t)$. If

$$\liminf_{t\to\infty}\int_{\tau^{-1}(\sigma(t))}^{t} q(s)Q_1(\sigma(s))R(\tau^{-1}(\sigma(s)))\mathrm{d}s>\frac{1}{eM} \tag{2.21}$$

and

$$\int_T^\infty \frac{1}{a(t)}\left(\int_T^s q(s)Q_3(\sigma(s))A(\tau^{-1}(\sigma(s)))\mathrm{d}s\right)\mathrm{d}t=\infty, \tag{2.22}$$

then every solution of (1.1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 2.3, we see that one of the two cases of Lemma 2.4 holds. Case (1). By (2.21) and [6, Theorem 2.1.1], (2.20) has no positive solution, which is a contradiction. The proof of Case (2) is the same as that of Case (2) of Theorem 2.3 with $\alpha=\beta$. This completes the proof.

Corollary 2.2. Assume (1.3), $\alpha>\beta$, and $\sigma(t)<\tau(t)$. If (2.18) holds and

$$\int_T^\infty q(t)Q_1^{\frac{\beta}{\alpha}}(\sigma(t))R^{\frac{\beta}{\alpha}}(\tau^{-1}(\sigma(t)))\mathrm{d}t=\infty, \tag{2.23}$$

then every solution of (1.1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 2.3, we see that one of the two cases of Lemma 2.4 holds. Case (1). By (2.23) and [6, Theorem 3.9.3], (2.20) has no positive solution, which is a contradiction. The proof of Case (2) is similar to that of Case (2) of Theorem 2.3. This completes the proof.

In the following corollary, we assume $\sigma(t)=t-k$ and $\tau(t)=t-m$, where $k>m>0$ are constants.

Corollary 2.3. Assume (1.3), $\alpha<\beta$, and $k>m$. If (2.18) holds and

$$\liminf_{t\to\infty}\left[\left(\frac{\beta}{\alpha}\right)^{-\frac{t}{k-m}}\log\left(q(t)Q^{\frac{\beta}{\alpha}}(t-k)R^{\frac{\beta}{\alpha}}(t-k+m)\right)\right]>0, \tag{2.24}$$

then every solution of (1.1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 2.4, we see that one of the two cases of Lemma 2.4 holds. Case (1). By [20, Lemma 2.2], the inequality (2.20) and the equation

$$w'(t)+Mq(t)Q_1^{\frac{\beta}{\alpha}}(t-k)R^{\frac{\beta}{\alpha}}(t-k+m)w^{\frac{\beta}{\alpha}}(t-k+m)=0, \quad t\ge T \tag{2.25}$$

has a positive solution. But by (2.24) and [20, Corollary 1.2], (2.25) has no positive solution, which is a contradiction. The proof of Case (2) is similar to that of Case (2) of Theorem 2.3. This completes the proof.

3 Examples

In this section, we present two examples to illustrate the main results.

Example 3.1. Consider the second-order neutral differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t\frac{\mathrm{d}}{\mathrm{d}t}\left(x(t)+\frac{t^3}{8}x^3\left(\frac{t}{2}\right)\right)\right)+\lambda t^2x^3\left(\frac{t}{3}\right)=0, \quad t\geq 1, \tag{3.1}$$

where $\lambda>0$ is a constant. Here,

$$M=1, \quad \alpha=\beta=3, \quad a(t)=t, \quad p(t)=\frac{t^3}{8},$$
$$q(t)=\lambda t^2, \quad \tau(t)=\frac{t}{2}, \quad \sigma(t)=\frac{t}{3}.$$

A simple calculation shows that $R(t)=\log t$, and by taking $\mu(t)=\frac{1}{4t}$, we see that

$$Q_1(t)=\frac{1}{t^3}\left[1-\frac{1}{6t}\left(\frac{\log 4t}{\log 2t}+t\right)\right]>0, \quad t\geq 1.$$

By taking $\rho(t)=1$, we see that (2.11) is satisfied for all $\lambda>0$. Hence by Theorem 2.1, every solution of (3.1) is oscillatory provided $\lambda>0$.

Example 3.2. Consider the second-order neutral differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t^2\frac{\mathrm{d}}{\mathrm{d}t}\left(x(t)+e^tx^3\left(\frac{t}{2}\right)\right)\right)+e^tx\left(\frac{t}{3}\right)=0, \quad t\geq 1. \tag{3.2}$$

Here,

$$M=1, \quad \alpha=3, \quad \beta=1, \quad a(t)=t^2, \quad p(t)=e^t,$$
$$q(t)=e^t, \quad \tau(t)=\frac{t}{2}, \quad \sigma(t)=\frac{t}{3}.$$

Then

$$R(t)=\frac{t-1}{t}, \quad A(t)=\frac{1}{t},$$
$$Q_1(t)=\frac{1}{3e^{2t}}\left[3-e^{-\frac{4}{3}t}\left(\frac{8t^2-2t}{2t-1}\right)\right]>0, \quad t\geq 1,$$

and

$$Q_2(t)=\frac{1}{3e^{2t}}\left[3-e^{-\frac{4}{3}t}(1+8t^2)\right]>0, \quad t\geq 1.$$

Now one can easily see that (2.17) and (2.18) are satisfied, and hence by Theorem 2.3, every solution of (3.2) is oscillatory.

References

[1] Ravi P. Agarwal, Martin Bohner, Tongxing Li, and Chenghui Zhang. Oscillation of second-order differential equations with a sublinear neutral term. *Carpathian J. Math.*, 30(1):1–6, 2014.

[2] Ravi P. Agarwal, Martin Bohner, and Wan-Tong Li. *Nonoscillation and oscillation: theory for functional differential equations*, volume 267 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 2004.

[3] Ravi P. Agarwal, Chenghui Zhang, and Tongxing Li. Some remarks on oscillation of second order neutral differential equations. *Appl. Math. Comput.*, 274:178–181, 2016.

[4] Blanka Baculíková and Jozef Džurina. Oscillation theorems for second order neutral differential equations. *Comput. Math. Appl.*, 61(1):94–99, 2011.

[5] Martin Bohner, Said R. Grace, and Irena Jadlovská. Oscillation criteria for second-order neutral delay differential equations. *Electron. J. Qual. Theory Differ. Equ.*, pages Paper No. 60, 12, 2017.

[6] Lynn H. Erbe, Qingkai Kong, and Binggen Zhang. *Oscillation theory for functional differential equations*, volume 190 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1995.

[7] Said R. Grace and John R. Graef. Oscillatory behavior of second order nonlinear differential equations with a sublinear neutral term. *Math. Model. Anal.*, 23(2):217–226, 2018.

[8] Said R. Grace and Bikkar S. Lalli. Oscillations of nonlinear second order neutral delay differential equations. *Rad. Mat.*, 3(1):77–84, 1987.

[9] John Graef, Sowdaiyan Murugadass, and Ethiraju Thandapani. Oscillation criteria for second order neutral delay differential equations with mixed nonlinearities. *Int. Electron. J. Pure Appl. Math.*, 2(1):85–99, 2010.

[10] Istvan Győri and Gerasimos Ladas. *Oscillation theory of delay differential equations*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1991. With applications, Oxford Science Publications.

[11] Jack Hale. *Theory of functional differential equations*. Springer-Verlag, New York-Heidelberg, second edition, 1977. Applied Mathematical Sciences, Vol. 3.

[12] Godfrey H. Hardy, John E. Littlewood, and George Pólya. *Inequalities*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988. Reprint of the 1952 edition.

[13] Mustafa Hasanbulli and Yuri V. Rogovchenko. Oscillation criteria for second order nonlinear neutral differential equations. *Appl. Math. Comput.*, 215(12):4392–4399, 2010.

[14] Gangaram S. Ladde, Vangipuram Lakshmikantham, and Binggen Zhang. *Oscillation theory of differential equations with deviating arguments*, volume 110 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1987.

[15] Tongxing Li, Ravi P. Agarwal, and Martin Bohner. Some oscillation results for second-order neutral differential equations. *J. Indian Math. Soc. (N.S.)*, 79(1-4):97–106, 2012.

[16] Tongxing Li, Zhenlai Han, Chenghui Zhang, and Hua Li. Oscillation criteria for second-order superlinear neutral differential equations. *Abstr. Appl. Anal.*, pages Art. ID 367541, 17, 2011.

[17] Tongxing Li, Zhenlai Han, Chenghui Zhang, and Shurong Sun. On the oscillation of second-order Emden-Fowler neutral differential equations. *J. Appl. Math. Comput.*, 37(1-2):601–610, 2011.

[18] Tongxing Li, Yuri V. Rogovchenko, and Chenghui Zhang. Oscillation of second-order neutral differential equations. *Funkcial. Ekvac.*, 56(1):111–120, 2013.

[19] Xiaoyan Lin and Xianhua Tang. Oscillation of solutions of neutral differential equations with a superlinear neutral term. *Appl. Math. Lett.*, 20(9):1016–1022, 2007.

[20] T. Sakamoto and Satoshi Tanaka. Eventually positive solutions of first order nonlinear differential equations with a deviating argument. *Acta Math. Hungar.*, 127(1-2):17–33, 2010.

[21] Samir H. Saker and Jelena V. Manojlović. Oscillation criteria for second order superlinear neutral delay differential equations. *Electron. J. Qual. Theory Differ. Equ.*, pages No. 10, 22, 2004.

[22] Samir H. Saker, Balakrishnan Sudha, Krishnan Thangavelu, and Ethiraju Thandapani. Oscillation criteria for second order differential equations with a sublinear neutral term. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, 2019. To appear.

[23] Shurong Sun, Tongxing Li, Zhenlai Han, and Chao Zhang. On oscillation of second-order nonlinear neutral functional differential equations. *Bull. Malays. Math. Sci. Soc. (2)*, 36(3):541–554, 2013.

[24] Sivaraj Tamilvanan, Ethiraju Thandapani, and Jozef Džurina. Oscillation of second order nonlinear differential equation with sub-linear neutral term. *Differ. Equ. Appl.*, 9(1):29–35, 2017.

[25] Sivaraj Tamilvanan, Ethiraju Thandapani, and Said R. Grace. Oscillation theorems for second-order non-linear differential equation with a non-linear neutral term. *Int. J. Dyn. Syst. Differ. Equ.*, 7(4):316–327, 2017.

[26] Ethiraju Thandapani and Renu Rama. Comparison and oscillation theorems for second order nonlinear neutral differential equations of mixed type. *Serdica Math. J.*, 39(1):1–16, 2013.

[27] Run Xu and Fanwei Meng. Oscillation criteria for second order quasi-linear neutral delay differential equations. *Appl. Math. Comput.*, 192(1):216–222, 2007.

[28] Chenghui Zhang, Mehmet Tamer Şenel, and Tongxing Li. Oscillation of second-order half-linear differential equations with several neutral terms. *J. Appl. Math. Comput.*, 44(1-2):511–518, 2014.