Oscillation of second-order differential equations with a sublinear neutral term

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ABSTRACT. This paper is concerned with oscillation of a certain class of second-order differential equations with a sublinear neutral term. Two oscillation criteria and two illustrative examples are included. In particular, the results obtained improve those reported in the literature.

1. INTRODUCTION

The neutral differential equations find numerous applications in natural sciences and technology. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines; see Hale [15].

In this paper, we restrict our attention to oscillation of a class of second-order differential equations of the form

\[(r(t) (x(t) + p(t)x(\tau(t))))' + q(t)x(\sigma(t)) = 0, \quad t \geq t_0.\]

Throughout, we use the notation \(z(t) := x(t) + p(t)x(\tau(t))\) and always assume that the following assumptions are satisfied.

\[(A_1)\quad 0 < \alpha \leq 1 \text{ is a ratio of odd positive integers};\]
\[(A_2)\quad r \in C^1([t_0, \infty), (0, \infty)), \ p, q \in C([t_0, \infty), [0, \infty)), \text{ and } q \text{ is not eventually zero on any half line } [t_*, \infty) \text{ for } t_* \geq t_0;\]
\[(A_3)\quad \tau \in C([t_0, \infty), \mathbb{R}), \ \sigma \in C^1([t_0, \infty), \mathbb{R}), \ \tau(t) \leq t, \ \sigma(t) \leq t, \ \sigma'(t) > 0, \text{ and } \lim_{t \to \infty} \tau(t) = \lim_{t \to \infty} \sigma(t) = \infty.\]

We consider only those solutions \(x\) of (1.1) which satisfy \(\sup\{|x(t)| : t \geq T\} > 0\) for all \(T \geq t_0\) and assume that (1.1) possesses such solutions. As usual, a solution of (1.1) is called oscillatory if it has arbitrarily large zeros on \([t_0, \infty)\); otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

In the last few years, there has been much research activity concerning oscillatory behavior of various classes of differential equations. We refer the reader to [1–14, 16–32] and the references cited therein. For oscillation of second-order neutral differential equations, Agarwal et al. [1–5], Baculíková and Džurina [9], Grace and Lalli [14], Han et al. [16], Hasanbulli and Rogovchenko [17], Karpuz et al. [18], Li et al. [20–24, 26], and Zafer [28] studied (1.1) when \(\alpha = 1\). Lin and Tang [27] considered a first-order neutral differential equation with a superlinear neutral term

\[|x(t) - px^\alpha(t - \tau)|' + q(t) \prod_{j=1}^m |x(t - \sigma_j)|^\beta_j \text{sgn}[x(t - \sigma_j)] = 0, \quad \alpha > 1.\]
As yet, there are few results for oscillation of (1.1) in the case \( \alpha \neq 1 \). Hence, we will investigate equation (1.1) under the assumption that \( 0 < \alpha \leq 1 \). In Section 2, we consider two cases

\[
\int_{t_0}^{\infty} \frac{dt}{r(t)} = \infty
\]

and

\[
\int_{t_0}^{\infty} \frac{dt}{r(t)} < \infty.
\]

2. Main results

In what follows, all functional inequalities are assumed to hold eventually, that is, they are satisfied for all \( t \) large enough. Without loss of generality, we can deal only with the positive solutions of (1.1) in the proofs of our main theorems.

**Theorem 2.1.** Assume \((A_1)-(A_3)\) and (1.2). If there exists a positive function \( \rho \in C^1([t_0, \infty), \mathbb{R}) \) such that

\[
\limsup_{t \to \infty} \int_{t_0}^{t} \left( 1 - \left( \alpha 2^{1-\alpha} + \frac{2^{1-\alpha} - 1}{M} \right) \rho(\sigma(s)) \rho(s)q(s) - \frac{r(\sigma(s))(\rho'(s))^2}{4\rho(s)\sigma'(s)} \right) ds = \infty
\]

holds for all constants \( M > 0 \), then (1.1) is oscillatory.

**Proof.** Suppose to the contrary that equation (1.1) has an eventually positive solution \( x \), i.e., there exists a \( t_1 \geq t_0 \) such that \( x(t) > 0, x(\tau(t)) > 0 \), and \( x(\sigma(t)) > 0 \) for \( t \geq t_1 \). From (1.1), we have

\[
(rz')' (t) = -q(t)x(\sigma(t)) \leq 0.
\]

By condition (1.2), we see that \( z' > 0 \). It follows from the definition of \( z \), [7, Lemma 2.2], and Bernoulli inequality that

\[
x(t) = z(t) - p(t)x(\tau(t)) = z(t) - p(t)(1 + x(\tau(t))) + p(t) \\
\geq z(t) - 2^{1-\alpha}p(t)(1 + x(\tau(t))) + p(t) \\
\geq z(t) - 2^{1-\alpha}p(t)(1 + \alpha x(\tau(t))) + p(t) \\
\geq z(t) - \alpha 2^{1-\alpha}p(t)z(\tau(t)) + (1 - 2^{1-\alpha})p(t) \\
\geq (1 - \alpha 2^{1-\alpha}p(t))z(t) + (1 - 2^{1-\alpha})p(t).
\]

In view of (2.5) and (2.6), we have

\[
(rz')' (t) \leq - (1 - \alpha 2^{1-\alpha}p(\sigma(t)))q(t)z(\sigma(t)) - (1 - 2^{1-\alpha})q(t)p(\sigma(t)).
\]

Define the function

\[
\omega(t) := \rho(t) \frac{r(t)z'(t)}{z(\sigma(t))}, \quad t \geq t_1.
\]

Then \( \omega(t) > 0 \) for \( t \geq t_1 \) and

\[
\omega' (t) = \rho'(t) \frac{r(t)z'(t)}{z(\sigma(t))} + \rho(t) \frac{(rz')'(t)}{z(\sigma(t))} - \rho(t) \frac{r(t)z'(t)z'(\sigma(t))\sigma'(t)}{z^2(\sigma(t))}.
\]
By (2.5) and \( \sigma(t) \leq t \), we get
\[
(2.10) \quad r(t)z'(t) \leq r(\sigma(t))z'(\sigma(t)).
\]
From \( z' > 0 \), there exists a constant \( M > 0 \) such that \( z(t) \geq M \) for all \( t \) large enough. It follows from (2.7), (2.8), (2.9), and (2.10) that
\[
\omega'(t) \leq -\left(1 - \left(\alpha 2^{1-\alpha} + \frac{2^{1-\alpha} - 1}{M}\right)p(\sigma(t))\right)\rho(t)q(t) + \rho'(t)\omega(t) - \frac{\sigma'(t)\omega^2(t)}{r(\sigma(t))\rho(t)}.
\]
Thus, we obtain
\[
\omega'(t) \leq -\left(1 - \left(\alpha 2^{1-\alpha} + \frac{2^{1-\alpha} - 1}{M}\right)p(\sigma(t))\right)\rho(t)q(t) + \frac{r(\sigma(t))(\rho'(t))^2}{4\rho(s)\sigma'(s)}.
\]
Integrating the last inequality from \( t_1 \) to \( t \), we have
\[
\int_{t_1}^t \left[\left(1 - \left(\alpha 2^{1-\alpha} + \frac{2^{1-\alpha} - 1}{M}\right)p(\sigma(s))\right)\rho(s)q(s) - \frac{r(\sigma(s))(\rho'(s))^2}{4\rho(s)\sigma'(s)}\right] ds \leq \omega(t_1),
\]
which contradicts (2.4). This completes the proof. □

Next we establish an oscillation criterion for (1.1) in the case where (1.3) holds.

**Theorem 2.2.** Let \( (A_1)-(A_3), (1.3), \) and \( 1 - \alpha 2^{1-\alpha}p(t)\delta(\tau(t))/\delta(t) > 0 \) hold. Assume there exists a positive function \( \rho \in C^1([t_0, \infty), \mathbb{R}) \) such that (2.4) holds for all constants \( M > 0 \). If
\[
\limsup_{t \to \infty} \int_{t_0}^t \left[\left(1 - \left(\alpha 2^{1-\alpha} + \frac{2^{1-\alpha} - 1}{M}\right)p(\sigma(s))\right)\delta(\sigma(s)) + \frac{2^{1-\alpha} - 1}{K\delta(s)}\right] q(\sigma(s))\delta(s) - \frac{1}{4r(s)\delta(s)}\right] ds = \infty
\]
holds for all constants \( K > 0 \), where
\[
\delta(t) := \int_t^\infty \frac{ds}{r(s)},
\]
then (1.1) is oscillatory.

**Proof.** Suppose to the contrary that equation (1.1) has an eventually positive solution \( x \), i.e., there exists a \( t_1 \geq t_0 \) such that \( x(t) > 0 \), \( x(\tau(t)) > 0 \), and \( x(\sigma(t)) > 0 \) for \( t \geq t_1 \). From (1.1), we have that (2.5) holds. Then we get either \( z' > 0 \) or \( z' < 0 \) eventually. Assume \( z' > 0 \). Proceeding as in the proof of Theorem 2.1, we can get a contradiction to (2.4).

Suppose \( z' < 0 \). Define the function \( u \) by
\[
(2.12) \quad u(t) := \frac{r(t)z'(t)}{z(t)}, \quad t \geq t_1.
\]
Then \( u(t) < 0 \) for \( t \geq t_1 \). From (2.5), we have
\[
z'(s) \leq \frac{r(t)}{r(s)}z'(t), \quad s \geq t.
\]
Hence we get
\[
z(l) - z(t) \leq r(t)z'(t) \int_t^l \frac{ds}{r(s)},
\]
which implies that
\[
(2.13) \quad \frac{r(t)z'(t)}{z(t)} \delta(t) \geq -1,
\]
that is,
\begin{equation}
(2.14) \quad u(t)\delta(t) \geq -1.
\end{equation}
On the other hand, we get by (2.13) that
\begin{equation}
(2.15) \quad \left(\frac{z}{\delta}\right)'(t) \geq 0.
\end{equation}
It follows from the definition of $z$, (2.15), [7, Lemma 2.2], and Bernoulli inequality that
\begin{equation}
(2.16) \quad x(t) = z(t) - p(t)x^\alpha(\tau(t)) = z(t) - p(t)(1 + x^\alpha(\tau(t))) + p(t) \\
\geq z(t) - 2^{1-\alpha}p(t)(1 + x(\tau(t))) + p(t) \\
\geq z(t) - 2^{1-\alpha}p(t)(1 + \alpha x(\tau(t))) + p(t) \\
\geq z(t) - \alpha 2^{1-\alpha}p(t)z(\tau(t)) + (1 - 2^{1-\alpha})p(t) \\
\geq \left(1 - \alpha 2^{1-\alpha}p(t)\frac{\delta(\tau(t))}{\delta(t)}\right)z(t) + (1 - 2^{1-\alpha})p(t).
\end{equation}
By virtue of (2.5) and (2.16), we have
\begin{equation}
(2.17) \quad (rz')'(t) \leq -\left(1 - \alpha 2^{1-\alpha}p(\sigma(t))\frac{\delta(\sigma(t))}{\delta(\sigma(t))}\right)q(t)z(\sigma(t)) - (1 - 2^{1-\alpha})q(t)p(\sigma(t)).
\end{equation}
Differentiating (2.12), we obtain
\begin{equation}
(2.18) \quad u'(t) = \frac{(rz')'(t)}{z(t)} - \frac{u^2(t)}{r(t)}.
\end{equation}
Combining (2.15), (2.17), and (2.18), we get
\begin{equation}
(2.19) \quad u'(t) \leq -\left(1 - \alpha 2^{1-\alpha}\frac{\delta(\tau(\sigma(\sigma(t))))}{\delta(\tau(\sigma(\sigma(t))))} + \frac{2^{1-\alpha} - 1}{K\delta(t)}\right)q(t) - \frac{u^2(t)}{r(t)}
\end{equation}
for some constant $K > 0$. Multiplying (2.19) by $\delta(t)$ and integrating the resulting inequality from $t_1$ to $t$, we see that
\begin{align*}
\delta(t)u(t) - \delta(t_1)u(t_1) + \int_{t_1}^{t} \left(1 - \alpha 2^{1-\alpha}\frac{\delta(\tau(\sigma(\sigma(t))))}{\delta(\tau(\sigma(\sigma(t))))} + \frac{2^{1-\alpha} - 1}{K\delta(s)}\right)q(s)\delta(s)ds \\
+ \int_{t_1}^{t} \frac{u(s)}{r(s)}ds + \int_{t_1}^{t} \frac{u^2(s)}{r(s)}\delta(s)ds \leq 0,
\end{align*}
which yields
\begin{align*}
\int_{t_1}^{t} \left[\left(1 - \alpha 2^{1-\alpha}\frac{\delta(\tau(\sigma(\sigma(t))))}{\delta(\tau(\sigma(\sigma(t))))} + \frac{2^{1-\alpha} - 1}{K\delta(s)}\right)q(s)\delta(s) - \frac{1}{4r(s)\delta(s)}\right]ds \leq 1 + \delta(t_1)u(t_1)
\end{align*}
when using (2.14), this contradicts (2.11). The proof is complete. 

3. EXAMPLES AND DISCUSSIONS

In the following, we illustrate possible applications with two examples.

**Example 3.1.** Consider a second-order neutral differential equation
\begin{equation}
(3.20) \quad \left(t\left(x(t) + \frac{1}{t}x^\alpha(\tau(t))\right)' + \lambda x(\sigma(t)) = 0, \quad t \geq 1,
\right.
\end{equation}
where $\lambda > 0$ is a constant. Set $\rho(t) = 1$. Then (3.20) is oscillatory by Theorem 2.1.
Example 3.2. Consider a second-order neutral differential equation

\[(3.21) \quad \left(t^2 \left(x(t) + \frac{1}{t^2} x^\alpha \left(\frac{t}{2}\right)\right)\right)' + \lambda x \left(\frac{t}{2}\right) = 0, \quad t \geq 1,\]

where \(\lambda > 0\) is a constant. Let \(\rho(t) = 1\). Then (3.21) is oscillatory by Theorem 2.2.

Remark 3.1. Using the Riccati transformation technique and some inequalities, two sufficient conditions are presented that can be used in oscillation problem of neutral differential equation (1.1) under the assumption that \(0 < \alpha \leq 1\). Note that when \(\alpha = 1\), Theorem 2.2 obtained improves [16, Theorem 2.1, Theorem 2.2, Theorem 3.1, and Theorem 3.2] in the sense that we remove the restrictive conditions that \(p' \geq 0, \tau(t) = t - \tau_0 \leq t, \sigma(t) \leq t - \tau_0, \) and \(\tau \circ \sigma = \sigma \circ \tau, \) where \(\tau_0 \geq 0\) is a constant. One can easily see that results reported in [1–5, 9, 14–16, 18–21, 24, 26–28] cannot be applied in equations (3.20) and (3.21) when \(\alpha < 1\). However, to achieve these results for the case \(\alpha < 1\), we are forced to impose some assumptions on the coefficient \(p; \) e.g., \(\lim_{t \to \infty} p(t) = 0.\) It would be interesting to find other methods to investigate (1.1) that can deal with different \(p.\)

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References


