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Oscillation of second-order nonlinear difference equations with sublinear neutral term

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ABSTRACT. We establish some new criteria for the oscillation of second-order nonlinear difference equations with a sublinear neutral term. This is accomplished by reducing the involved nonlinear equation to a linear inequality.

1. INTRODUCTION

This paper deals with oscillatory behavior of all solutions of nonlinear second-order difference equations with a sublinear neutral term of the form

$$\Delta\left(a_n\Delta\left(x_n+p_nx_{n-k}^{\alpha}\right)\right)+q_nx_{n+1-m}^{\beta}=0. \tag{1}$$

We assume that

(H$_1$) $0<\alpha<1$ and $\beta>0$ are ratios of positive odd integers,

(H$_2$) $\{a_n\}$, $\{p_n\}$, $\{q_n\}$, $n\geq n_0$, are positive real sequences,

$$\lim_{n\to\infty}p_n=0 \quad \text{and} \quad \sum_{s=n_0}^{\infty}\frac{1}{a_s}<\infty,$$

(H$_3$) $k\in\mathbb{N}$ and $m\in\mathbb{N}_0$.

Let $\xi=\max\{k,m-1\}$. By a solution of (1), we mean a real sequence $\{x_n\}$ defined for all $n\geq n_0-\xi$ that satisfies (1) for $n\geq n_0$. A solution of (1) is said to be oscillatory if its terms are neither eventually positive nor eventually negative, and otherwise it is called nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

In recent years, there has been a great interest in establishing criteria for the oscillation and asymptotic behavior of solutions of various classes of second-order difference equations, see [1, 2, 4, 9–12, 15, 18, 20, 21, 24] and the references cited therein. However, it seems that there are no known results regarding the oscillation of second-order difference equations with positive

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sublinear neutral term. More exactly, the existing literature does not provide any criteria which ensure oscillation of all solutions of (1). In view of this motivation, our aim in this paper is to present sufficient conditions which ensure that all solutions of (1) are oscillatory. For related results concerning second-order differential equations with sublinear neutral term, we refer the reader to [3, 16, 17, 23]. Some related results concering second-order dynamic equations on time scales can be found in [6–8, 13, 14, 19, 22].

2. MAIN RESULTS

For $n \geq n_0^*$ for some $n_0^* \geq n_0$, we let

$$A_n = \sum_{s=n}^{\infty} \frac{1}{a_s}.$$

For convenience, for some $0 < \nu \leq 1$ and $n \geq n_0^*$, we set

$$y_n = x_n + p_n x_{n-k}^{\alpha},$$

$$P_n = 1 - p_n \frac{A_{n-k}^{\alpha}}{A_n^{1+(1-\alpha)\nu}} \geq 0$$

and

$$Q_n = q_n A_{n+1}^{(1+\nu)(\beta-1)} P_{n+1-m}^{\beta}.$$

In the following, we establish a new oscillation result for (1) when $\beta \geq 1$.

Theorem 2.1. *Let $\beta \geq 1$. Assume (H_1)–(H_3). If*

$$\limsup_{n\to\infty} \sum_{s=n_0^*}^{n} \left[Q_s A_{s+1} - \frac{1}{4a_s A_{s+1}} \right] = \infty, \tag{2}$$

then (1) *is oscillatory.*

Proof. Let x_n be a nonoscillatory solution of (1), say $x_n > 0$, $x_{n+1-m} > 0$, $x_{n-k} > 0$, and $y_n > 0$ for $n \geq n_1$ for some $n_1 \geq n_0^*$. It is easy to see that $y_n > 0$, $n \geq n_1$, and (1) becomes

$$\Delta\left(a_n \Delta y_n\right) + q_n x_{n+1-m}^{\beta} = 0. \tag{3}$$

Thus $\Delta\left(a_n \Delta y_n\right) \leq 0$ for $n \geq n_1$, which implies that y_n is bounded. Also, the decreasing nature of $a_n \Delta y_n$ implies that (I) $\Delta y_n > 0$ or (II) $\Delta y_n < 0$ for $n \geq n_1^* \geq n_1$. Therefore, y_n converges, and hence

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} (x_n + p_n x_{n-k}^{\alpha}) = \lim_{n\to\infty} x_n,$$

since $\lim_{n\to\infty} p_n = 0$. Now, we consider Case (I). Since y_n is a positive increasing sequence, there exist $n_2 \geq n_1^*$ and $d > 0$ such that

$$x_n > d \quad \text{for} \quad n \geq n_2. \tag{4}$$

Substituting (4) into (3), we get

$$\Delta\left(a_n \Delta y_n\right) + q_n d^\beta < 0 \quad \text{for} \quad n \geq n_2. \tag{5}$$

Summing (5) from n_2 to $n-1$, we obtain

$$a_n \Delta y_n - a_{n_2} \Delta y_{n_2} + d^\beta \sum_{s=n_2}^{n-1} q_s < 0 \quad \text{for} \quad n \geq n_2. \tag{6}$$

But (2) implies that

$$\sum_{n=n_2}^{\infty} q_n = \infty,$$

which together with (6) yields

$$\lim_{n\to\infty} a_n \Delta y_n = -\infty,$$

a contradiction due to the eventual positivity of $a_n \Delta y_n$.

Next, we consider Case (II). Define the sequence $\{v_n\}$ by

$$v_n = \frac{a_n \Delta y_n}{y_n} \quad \text{for} \quad n \geq n_1. \tag{7}$$

Then $v_n < 0$ for $n \geq n_1$. Also, the decreasing nature of $a_n \Delta y_n$ implies that

$$\Delta y_s \leq \frac{a_n}{a_s} \Delta y_n \quad \text{for} \quad s \geq n \geq n_1. \tag{8}$$

Summing (8) from n to $r-1 \geq n$, we obtain

$$y_r - y_n \leq a_n \Delta y_n \left(\sum_{s=n}^{r-1} \frac{1}{a_s} \right),$$

which, by letting $r \to \infty$, leads to

$$\frac{a_n \Delta y_n}{y_n} A_n \geq -1 \quad \text{for} \quad n \geq n_1, \tag{9}$$

i.e.,

$$v_n A_n \geq -1 \quad \text{for} \quad n \geq n_1. \tag{10}$$

On the other hand, we find from (9) that

$$\Delta\left(\frac{y_n}{A_n}\right) = \frac{A_n \Delta y_n - y_n \Delta A_n}{A_n A_{n+1}} = \frac{A_n \Delta y_n + \frac{y_n}{a_n}}{A_n A_{n+1}} \geq 0,$$

for $n \geq n_1$, and thus

$$\frac{y_n}{A_n} \geq \frac{y_{n-k}}{A_{n-k}} \quad \text{for} \quad n \geq n_1 + k. \tag{11}$$

Now,

$$x_n = y_n - p_n x_{n-k}^\alpha \geq y_n - p_n y_{n-k}^\alpha \quad \text{for} \quad n \geq n_1 + k,$$

and using (11), we obtain

$$x_n \geq y_n - p_n \frac{A_{n-k}^\alpha}{A_n^\alpha} y_n^\alpha = \left(1 - p_n \frac{A_{n-k}^\alpha}{A_n^\alpha} y_n^{\alpha-1}\right) y_n. \tag{12}$$

Since y_n/A_n is positive and increasing, we get

$$\frac{y_n}{A_n} \geq \frac{y_{n_1}}{A_{n_1}} =: \gamma > 0 \quad \text{for} \quad n \geq n_1. \tag{13}$$

Since $\{A_n\}$ is positive and converging to zero, there exists $n_3 \geq n_1 + k$ such that

$$0 < A_n^\nu \leq \gamma \quad \text{for all} \quad n \geq n_3. \tag{14}$$

Hence, by (13) and (14),

$$y_n \geq A_n^{1+\nu} \quad \text{for} \quad n \geq n_3. \tag{15}$$

Using (15) in (12), we get

$$x_n \geq \left(1 - p_n \frac{A_{n-k}^\alpha}{A_n^\alpha} A_n^{(1+\nu)(\alpha-1)}\right) y_n = P_n y_n \quad \text{for} \quad n \geq n_3. \tag{16}$$

By (16), from (3), we have

$$\begin{aligned} \Delta\left(a_n \Delta y_n\right) &= -\, q_n x_{n+1-m}^\beta \\ &\leq -\, q_n P_{n+1-m}^\beta y_{n+1-m}^\beta \\ &\leq -\, q_n P_{n+1-m}^\beta y_{n+1}^\beta \quad \text{for} \quad n \geq n_3, \end{aligned} \tag{17}$$

where we also used the decreasing nature of y_n in the last estimate. Now (17), in view of (15), leads to

$$\begin{aligned} \Delta\left(a_n \Delta y_n\right) &\leq -\, q_n A_{n+1}^{(1+\nu)(\beta-1)} P_{n+1-m}^\beta y_{n+1} \\ &= -\, Q_n y_{n+1} \quad \text{for} \quad n \geq n_3. \end{aligned} \tag{18}$$

Taking the difference of both sides of (7) and using the decreasing nature of $a_n \Delta y_n$, we get

$$\begin{aligned} \Delta v_n &= \frac{y_n \Delta(a_n \Delta y_n) - a_n (\Delta y_n)^2}{y_n y_{n+1}} \\ &= \frac{\Delta(a_n \Delta y_n)}{y_{n+1}} - \frac{y_n}{a_n y_{n+1}} v_n^2 \\ &\leq \frac{\Delta\left(a_n \Delta y_n\right)}{y_{n+1}} - \frac{v_n^2}{a_n} \quad \text{for} \quad n \geq n_3, \end{aligned} \tag{19}$$

where we have used again the decreasing nature of y_n. Combining (19) and (18), we have

$$\Delta v_n \leq -Q_n - \frac{v_n^2}{a_n} \quad \text{for} \quad n \geq n_3. \tag{20}$$

Using (20), we get

$$\begin{aligned}\Delta(A_n v_n) =& v_n \Delta A_n + A_{n+1}\Delta v_n \\ =& -\frac{v_n}{a_n} + A_{n+1}\Delta v_n \\ \le& -\frac{v_n}{a_n} - A_{n+1}Q_n - \frac{A_{n+1}v_n^2}{a_n} \\ \le& -A_{n+1}Q_n + \frac{1}{4a_n A_{n+1}},\end{aligned}$$

and summing this resulting inequality from n_3 to n and using (10) yields

$$\begin{aligned}\sum_{s=n_3}^{n}\left[Q_s A_{s+1} - \frac{1}{4a_s A_{s+1}}\right] \le& A_{n_3}v_{n_3} - A_{n+1}v_{n+1} \\ \le& 1 + A_{n_3}v_{n_3} < \infty \quad \text{for} \quad n \ge n_3,\end{aligned}$$

contradicting (2). This completes the proof. □

When $\beta = 1$, we have the following immediate corollary from Theorem 2.1.

Corollary 2.1. *Let* $\beta = 1$. *Assume* (H_1)–(H_3). *If*

$$\limsup_{n\to\infty}\sum_{s=n_0^*}^{n}\left[q_s P_{s+1-m}A_{s+1} - \frac{1}{4a_s A_{s+1}}\right] = \infty, \tag{21}$$

then (1) *is oscillatory.*

Next, we establish an oscillation result when $0 < \beta < 1$.

Theorem 2.2. *Let* $0 < \beta < 1$. *Assume* (H_1)–(H_3). *If*

$$\limsup_{n\to\infty}\sum_{s=n_0^*}^{n}\left[Lq_s P_{s+1-m}^{\beta}A_{s+1} - \frac{1}{4a_s A_{s+1}}\right] = \infty \quad \textit{for some} \quad L > 0, \tag{22}$$

then (1) *is oscillatory.*

Proof. Let x_n be a nonoscillatory solution of (1), say $x_n > 0$, $x_{n+1-m} > 0$, $x_{n-k} > 0$, and $y_n > 0$ for $n \ge n_1$ for some $n_1 \ge n_0^*$. Proceeding as in the proof of Theorem 2.1, we obtain the two cases (I) $\Delta y_n > 0$ or (II) $\Delta y_n < 0$ for $n \ge n_1$. Next, we consider only Case (II) as Case (I) can be treated similarly as in the proof of Theorem 2.1. Recall that y_n is positive and decreasing with $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n$. Then we have either $\lim_{n\to\infty} y_n = d_1 > 0$ or $\lim_{n\to\infty} y_n = 0$. The first case implies that $\lim_{n\to\infty} x_n = d_1$. Thus, there exist $d_2 > 0$ and $n_1^* \in \mathbb{N}$ such that $x_n \ge d_2$ for all $n \ge n_1^*$. Hence we can obtain a contradiction similarly as in Case (I). The other case implies that for $K := L^{1/(\beta-1)} > 0$, there exists $n_2^* \in \mathbb{N}$ such that

$$0 < y_n < K \quad \text{for all} \quad n \ge n_2^*. \tag{23}$$

Now proceeding as in the proof of Theorem 2.1, we obtain (17), which with (23) yields

$$\begin{aligned} 0 &\geq \Delta(a_n \Delta y_n) + q_n P_{n+1-m}^{\beta} y_{n+1}^{\beta} \\ &= \Delta(a_n \Delta y_n) + \frac{q_n P_{n+1-m}^{\beta} y_{n+1}}{y_{n+1}^{1-\beta}} \\ &\geq \Delta(a_n \Delta y_n) + \frac{q_n P_{n+1-m}^{\beta} y_{n+1}}{K^{1-\beta}} \\ &= \Delta(a_n \Delta y_n) + L q_n P_{n+1-m}^{\beta} y_{n+1} \quad \text{for} \quad n \geq n_3, \end{aligned}$$

with some $n_3 \geq n_2^*$. The remainder of the proof is similar to that of Theorem 2.1 and hence is omitted. □

3. EXAMPLES AND REMARKS

First, we give two examples for the case $\beta > 1$.

Example 3.1. Consider the second-order equation

$$\Delta\left(n(n+1)\Delta\left(x_n + \frac{x_{n-k}^{\alpha}}{n^2}\right)\right) + (n+1)^6 x_{n+1-m}^{\frac{5}{3}} = 0, \quad n \in \mathbb{N}. \tag{24}$$

Here, $0 < \alpha < 1$ is a ratio of positive odd integers, $\beta = 5/3$, the delays are $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$, and

$$a_n = n(n+1), \quad p_n = \frac{1}{n^2}, \quad \text{and} \quad q_n = (n+1)^6.$$

We let $\nu = 1$. It is easy to see that (H_2) holds. Also,

$$A_n = \frac{1}{n} \quad \text{and} \quad P_n = 1 - \frac{1}{n^{\alpha}(n-k)^{\alpha}}.$$

Moreover,

$$Q_n A_{n+1} - \frac{1}{4a_n A_{n+1}} = (n+1)^{\frac{11}{3}} \left[1 - \frac{1}{(n+1-m)^{\alpha}(n+1-m-k)^{\alpha}}\right]^{\frac{5}{3}} - \frac{1}{4n}.$$

Thus,

$$\lim_{n\to\infty} \left(Q_n A_{n+1} - \frac{1}{4a_n A_{n+1}}\right) = \infty.$$

Therefore, (2) of Theorem 2.1 is satisfied, and hence (24) is oscillatory.

Example 3.2. Consider the second-order equation

$$\Delta\left(n(n+1)\Delta\left(x_n + \frac{x_{n-k}^{\alpha}}{n^2}\right)\right) + (n+1)^2 x_{n+1-m}^{\frac{5}{3}} = 0, \quad n \in \mathbb{N}. \tag{25}$$

Here, all data are the same as in Example 3.1 except

$$q_n = (n+1)^2,$$

and therefore

$$Q_nA_{n+1} = \frac{1}{(n+1)^{\frac{1}{3}}}\left[1 - \frac{1}{(n+1-m)^\alpha(n+1-m-k)^\alpha}\right]^{\frac{5}{3}} \geq \frac{1}{n}\cdot\frac{1}{2},$$

for $n \geq N$ with some $N \in \mathbb{N}$, and thus

$$\begin{aligned}\sum_{n=N}^{M}\left(Q_nA_{n+1} - \frac{1}{4a_nA_{n+1}}\right) &= \sum_{n=N}^{M}\left(Q_nA_{n+1} - \frac{1}{4n}\right)\\ &\geq \sum_{n=N}^{M}\frac{1}{4n} \to \infty \quad \text{as} \quad M \to \infty.\end{aligned}$$

Hence, (2) of Theorem 2.1 is satisfied, and thus (25) is oscillatory.

Next, we give an example in the case $\beta = 1$.

Example 3.3. Consider the second-order equation

(26)
$$\Delta\left(n(n+1)\Delta\left(x_n + \sqrt[3]{\frac{(n-k)x_{n-k}}{8n^4}}\right)\right) + \frac{n+1}{n}x_{n+1-m} = 0, \quad n \in \mathbb{N}.$$

Here, $\alpha = 1/3$, $\beta = 1$, the delays are $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$, and

$$a_n = n(n+1), \quad p_n = \sqrt[3]{\frac{n-k}{8n^4}} \quad \text{and} \quad q_n = \frac{n+1}{n}.$$

We let $\nu = 1/2$. It is easy to see that (H_2) holds. Also,

$$A_n = \frac{1}{n} \quad \text{and} \quad P_n = \frac{1}{2}.$$

Moreover,

$$q_nP_{n+1-m}A_{n+1} - \frac{1}{4a_nA_{n+1}} = \frac{q_n}{2(n+1)} - \frac{1}{4n} = \frac{1}{4n}.$$

Therefore, (21) of Corollary 2.1 is satisfied, and hence (26) is oscillatory.

Finally, we present an example in the case $0 < \beta < 1$.

Example 3.4. Consider the second-order equation

$$(27) \qquad \Delta\frac{\Delta\left(x_n + 4^{(\alpha-1)(n-1)-(k\alpha+1)/2}x_{n-k}^\alpha\right)}{2^n} + n8^nx_{n+1-m}^\beta = 0, \quad n \in \mathbb{N}.$$

Here, $0 < \alpha < 1$ and $0 < \beta < 1$ are ratios of positive odd integers, the delays are $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$, and

$$a_n = \frac{1}{2^n}, \quad p_n = 4^{(\alpha-1)(n-1)-(k\alpha+1)/2} \quad \text{and} \quad q_n = n8^n.$$

We let $\nu = 1$. It is easy to see that (H_2) holds. Also,

$$A_n = \frac{1}{2^{n-1}} \quad \text{and} \quad P_n = \frac{1}{2}.$$

Moreover,

$$Lq_n P_{n+1-m}^{\beta} A_{n+1} - \frac{1}{4a_n A_{n+1}} = Ln2^{2n-\beta} - 2^{2n-2},$$

which tends to infinity for any constant $L > 0$. Therefore, (22) of Theorem 2.2 is satisfied, and hence (27) is oscillatory.

Remark 3.1. The results of this paper are presented in a form that makes it easy to study extensions to higher-order equations. It would also be of interest to use the approach here to study (1) with $\alpha > 1$, i.e., (1) with superlinear neutral term.

Remark 3.2. Another possibility for extension of the presented results would be to consider the time-scales [5, 8] analogue of (1).

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