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Oscillation of solutions of second-order linear differential equations and corresponding difference equations

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In this paper, we present conditions ensuring that solutions of linear second-order differential equations oscillate, provided solutions of corresponding difference equations oscillate. We also establish the converse result, namely, when oscillation of solutions of difference equations implies oscillation of solutions of corresponding differential equations.

Keywords: oscillation; zeros of solutions; difference equation; finite difference; amplitude of oscillation

1. Introduction

Difference equations are important objects of study from both theoretical and practical points of view. Difference schemes arise in numerical integration of differential equations. Besides, they are convenient mathematical models of objects whose evolution has discrete character. A good example of such a model is the model of a financial market with the change in prices of risky assets at discrete points of time (see [20]). In the simplest case, the function that describes the total capital of an investor at this market satisfies a linear difference equation. Since the change in value of shares (risky asset) is of oscillatory nature, so is the evolution of the total capital. Therefore, oscillatory solutions become especially important in such models.

Oscillatory properties of solutions of difference equations were studied by numerous authors, e.g. [1,3,13,14,18,17], to name only a few. For corresponding equations on time scales, the notion of a generalized zero of a solution and oscillation of solutions were investigated in, e.g. [2,4,6,7,16], again to name only a few.

The qualitative properties of solutions of ordinary differential equations and corresponding difference equations, provided the step size $h > 0$ goes to zero, are of particular interest (see, e.g. [9] and references therein). The works [8,10] investigate the relation between existence of attractors in systems of differential equations and corresponding difference equations.

The work [11] generalizes the Runge–Kutta scheme by constructing a hybrid system for an autonomous system of differential equations. The authors investigate the question of uniform global asymptotic stability of the trivial solution of this hybrid system. They show that the main condition for such stability is uniform global asymptotic and local exponential stability of the trivial solution of the corresponding autonomous system of differential equations.

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The papers [12,22] establish the existence of bounded solutions of differential equations on the axis, provided the corresponding difference equations have such solutions, and vice versa.

It is well known (see, e.g. [9, p. 114]) that on finite time intervals, solutions of difference equations behave essentially the same as solutions of corresponding differential equations for small step sizes $h > 0$, and the error at the nodal points is proportional to h. However, this error estimate does not guarantee that oscillatory properties of solutions are preserved.

The question of the relation between oscillation of the solutions of linear difference and the corresponding differential equations was considered in the works [5,21]. The paper [21] established existence of oscillation of solutions of linear second-order difference equations for sufficiently small step sizes $h > 0$, provided that solutions of corresponding differential equations have this property. The converse result was obtained in [5].

The aforementioned works study oscillation of a fixed solution of some Cauchy problem for a difference equation, given that the solution of the Cauchy problem with the same initial data for the corresponding differential equation has such property, and vice versa. In this approach, the step size $h > 0$ depends on the initial data, and the coefficients have certain smoothness requirements which are somewhat artificial for such equations. The natural question which arises is whether there exists a universal step size $h > 0$, independent of the choice of the initial conditions, which would guarantee the oscillation properties uniformly in the initial data.

In this paper, our main result establishes the existence of such step size h. Besides, we provide several generalizations of the results in [21]. In particular, we study oscillation properties of linear *functional* second-order difference and differential equations. We also remove the technical smoothness conditions on the coefficients, replacing them with a more natural Lipschitz condition.

This paper consists of this introduction and two further sections. Section 2 provides the formulation of the problem and some preliminary results. In our opinion, these results are of separate interest themselves. The main results of this paper are given in Section 3.

2. Problem statements and auxiliary results

Consider the linear second-order differential equation

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0. \tag{2.1}$$

The following equations are called the *functional difference equation* and the *difference equation*, corresponding to (2.1), respectively:

$$\Delta^2 x(t) + hp(t)\Delta x(t) + h^2 q(t)x(t) = 0, \tag{2.2}$$

$$\Delta_k^2 x(t_0) + hp(t_0 + kh)\Delta_k x(t_0) + h^2 q(t_0 + kh)x(t_0 + kh) = 0. \tag{2.3}$$

Here

$$\Delta x(t) = x(t+h) - x(t), \quad \Delta^2 x(t) = \Delta(\Delta x(t)) = x(t+2h) - 2x(t+h) + x(t),$$
$$\Delta_k x(t_0) = x(t_0 + (k+1)h) - x(t_0 + kh), \quad \Delta_k^2 x(t_0) = \Delta_k(\Delta_k x(t_0)).$$

Denote by $x_k^h = x(t_k)$ the solution of (2.3), where $t_k = t_0 + kh$.

Definition 2.1. (*See* [21]) *We say that the solution* x_k^h *of* (2.3) changes sign *at* t_k *if either one of the following conditions holds:*

(1) $x_k^h x_{k+1}^h < 0$;
(2) $x_k^h = 0$ *and* $x_{k-1}^h x_{k+1}^h < 0$.

Definition 2.2. (*See* [21]) *A solution* x_k^h *of* (2.3) *is called* oscillatory *on some interval if it has at least two changes of signs on this interval.*

We study (2.2) under conditions that ensure continuity of its solutions. Thus, we have the usual concept of a zero for solutions of (2.2), and the notion of oscillation of its solutions is essentially the same as for solutions of (2.1).

In this paper, we give conditions ensuring the existence of a (universal) step size $h > 0$, for which oscillation of solutions of (2.1) follows from oscillation of solutions of (2.2) and (2.3). We also obtain the converse result.

We start with some preliminary results. Consider the system of functional difference equations in $\mathbb{R}^d$

$$x^h(t+h) = x^h(t) + hX(t, x^h(t)), \tag{2.4}$$

where $h > 0$ is the step size and $X(\cdot, x)$ is defined and continuous on $[0, \infty)$, $x \in D$ is a set in $\mathbb{R}^d$. For fixed $h > 0$, any solution of (2.4) may be extended uniquely to (h, ∞) with the initial function φ defined on $[0, h]$, i.e. $x^h(t) = \varphi(t)$ for $t \in [0, h]$. In this case, the *coherence condition*

$$\varphi(h) = \varphi(0) + hX(0, \varphi(0)) \tag{2.5}$$

clearly holds. If φ is continuous on $[0, h]$ and (2.5) is satisfied, then x^h is a continuous function defined on $[0, \infty)$ as long as $x^h(t-h) \in D$. Consider now (2.4) for $t = t_0 + kh$, where t_0 is fixed:

$$x_{k+1}^h = x_k^h + hX(t_0 + kh, x_k^h), \tag{2.6}$$

$k \in \mathbb{N}_0$, $h > 0$, $x_k^h = x^h(t_0 + kh)$. System (2.6) is a system of difference equations. Its solutions may be extended uniquely to the right, using the initial data $x_0^h = x^h(t_0)$ for $k > 0$ as long as $x_{k-1}^h \in D$.

Let I_{x^h} be the maximal interval for which the solution x_h of (2.4) may be extended to the right. Similarly, let $I_{x_k^h}$ be the maximal interval of extension of the solution x_k^h of (2.6). The following auxiliary result holds.

Lemma 2.3. *Let* x^h *be the solution of* (2.4) *with the given initial function* $\varphi \in \mathrm{C}([0, h])$ *satisfying* (2.5). *Then, for each* $t \in I_{x^h}$, *there exists a unique* $t_0 \in [0, h]$ *and* $k = k(t)$ *such that* $x^h(t) = x_k^h$, *where* x_k^h *is the solution of the initial value problem*

$$\begin{cases} x_{k+1}^h = x_k^h + hX(t_0 + kh, x_k^h), \\ x_0^h = \varphi(t_0). \end{cases} \tag{2.7}$$

Proof. The proof immediately follows from the definitions of solutions of (2.4) and (2.6). $\square$

By Lemma 2.3, every solution of (2.4) with the initial function φ satisfying (2.5) comprises solutions of the initial value problem (2.7) with the initial condition $x_0^h = \varphi(t_0)$, where $t_0 \in [0, h]$.

We now consider (2.1) for $t \in [0, a]$, $a > 0$, and $p, q \in \mathrm{C}([0, a])$. We study the solution x of (2.1) with the initial data $x(t_0) = x_0$, $\dot{x}(t_0) = x_1$, where $t_0 \in [0, \bar{h}]$,

$$x_0^2 + x_1^2 = 1, \tag{2.8}$$

and $\bar{h} \in (0, a)$ is fixed. It follows from [19] that x exists and is unique on the entire interval $[0, a]$. If this solution oscillates on $(0, a)$, then it has at least two zeros on this interval. Let t_k, t_{k+1} be two consecutive zeros of x on $(0, a)$. Consider the (finite) sequence of amplitudes of oscillations of the solution x on the interval $(0, a)$

$$M_k^x := \max_{t \in [t_k, t_{k+1}]} |x(t)|.$$

The following auxiliary result gives a *uniform* lower bound on the amplitudes of solutions.

Lemma 2.4. *Assume $p, q \in \mathrm{C}([0, a])$. Then there exists $\Delta > 0$ such that for an arbitrary oscillatory solution of* (2.1) *with the initial data* (2.8)*, we have*

$$M_k^x \geq \Delta. \tag{2.9}$$

Proof. We argue by contradiction. Assume (2.9) does not hold. Then there exists an infinite sequence of oscillatory solutions $\{x_n\}_{n \in \mathbb{N}}$ with the initial data described by $t_n \in [0, \bar{h}]$ and x_{0n}, x_{1n} satisfying (2.8), and such that for each n from the sequence of amplitudes of these solutions, we can choose an amplitude $M_{k(n)}^{x_n}$ such that the sequence $\{M_{k(n)}^{x_n}\}$ formed from these numbers satisfies the condition

$$M_{k_n}^{x_n} \to 0 \quad \text{as} \quad n \to \infty, \tag{2.10}$$

where $M_{k_n}^{x_n} = \max_{t \in [t_{k_n}^n, t_{k_n+1}^n]} |x_n(t)|$. Let t_n^* be the point at which this maximum is attained. Then $\dot{x}_n(t_n^*) = 0$ and $|x_n(t_n^*)| = M_{k(n)}^{x_n}$. Since the set of the initial data (2.8) is compact, there exists a convergent subsequence of (t_n, x_{0n}, x_{1n}), which is still denoted by (t_n, x_{0n}, x_{1n}), such that

$$(t_n, x_{0n}, x_{1n}) \to (t_0, x_0, x_1) \quad \text{as } n \to \infty, \tag{2.11}$$

where $t_0 \in [0, \bar{h}]$, $x_0^2 + x_1^2 = 1$. Similarly, $\{t_n^*\}_{n \in \mathbb{N}}$ also has a convergent subsequence, which is still denoted by $\{t_n^*\}$. Thus, $t_n^* \to t^* \in [0, a]$ as $n \to \infty$.

Let x be the solution of (2.1) with the initial data $x(t_0) = x_0$, $\dot{x}(t_0) = x_1$. Due to (2.8), x is not identically zero. Using the continuous dependence of solutions of the Cauchy problem on the finite interval on the initial data, together with the inequality

$$|x_n(t_n^*) - x(t^*)| \leq |x_n(t_n^*) - x(t_n^*)| + |x(t_n^*) - x(t^*)|,$$

we get

$$x_n(t_n^*) \to x(t^*) \quad \text{and} \quad \dot{x}_n(t_n^*) \to \dot{x}(t^*).$$

On the other hand, $x_n(t_n^*) \to 0$ as $n \to \infty$ and $\dot{x}_n(t_n^*) = 0$ for every n. Thus, x must be the trivial solution, a contradiction that completes the proof. $\square$

Let x be an oscillatory solution of (2.1) on $(0, a)$, t_k its zeros on this interval, and t_k^* the points between its zeros, in which the modulus of x attains its maximum. Let $I_{\Delta,k}^x$ be a symmetric and closed neighbourhood of t_k^* such that for every $t \in I_{\Delta,k}^x$, we have $|x(t)| \geq \Delta/2$. Denote by $|I_{\Delta,k}^x|$ the length of this interval.

Lemma 2.5. *Assume the conditions of Lemma* 2.4 *hold. Then there exists* $\delta > 0$ *such that for an arbitrary oscillatory solution of* (2.1) *on* $(0, a)$ *with initial data* (2.8)*, we have*

$$|I_{\Delta,k}^x| \geq 2\delta. \tag{2.12}$$

Proof. We argue by contradiction. Assume (2.12) is not satisfied. Then, similarly as in the argument of the proof of Lemma 2.4, there exists an infinite sequence of oscillatory solutions $x_n(t)$ on $(0, a)$ with initial data $t_n \in [0, h]$, x_{0n}, x_{1n} which satisfy (2.8), and there exists $k(n)$ such that

$$\left| I_{\Delta,k_n}^{x_n} \right| \to 0 \quad \text{as } n \to \infty. \tag{2.13}$$

Since the intervals $I_{\Delta,k_n}^{x_n}$ are symmetric, (2.13) means that at least one of the one-sided neighbourhoods of each of them becomes arbitrarily small. Assume this is the right neighbourhood.

We can assume that on the interval $[t_k^n, t_{k+1}^n]$, the solution x_n is non-negative. Let t_n^* be the midpoint of the interval $I_{\Delta,k_n}^{x_n}$. By construction, $x_n(t_n^*) = \max_{t \in [t_k^n, t_{k+1}^n]} x(t)$. Let $t_{\Delta,k}^{(n)}$ be right-end point of the neighbourhood $I_{\Delta,k(n)}^{x_n}$. Then

$$x_n\left(t_{\Delta,k}^{(n)}\right) = \frac{\Delta}{2}.$$

By virtue of Lemma 2.4,

$$x_n(t_n^*) \geq \Delta.$$

Using Lagrange's formula, we have

$$\left| x_n(t_n^*) - x_n(t_{\Delta,k}^{(n)}) \right| = |\dot{x}_n(\theta_n)| \left| t_{\Delta,k}^{(n)} - t_n^* \right|, \tag{2.14}$$

where $\theta_n \in (t_n^*, t_{\Delta,k}^{(n)})$. The left-hand part of (2.14) is at least $\Delta/2$. Since $|t_{\Delta,k}^{(n)} - t_n^*| \to 0$, we obtain $|\dot{x}_n(\theta_n)| \to \infty$ as $n \to \infty$. On the other hand, $\dot{x}_n = \dot{x}(\cdot, t_n, x_{0n}, x_{1n})$ is continuous on the compact interval $[0, a]$ and thus is, due to (2.8), bounded on $[0, a]$. Since $\theta_n \in [0, a]$ for all $n \in \mathbb{N}$, we thus cannot have $|\dot{x}_n(\theta_n)| \to \infty$ as $n \to \infty$. This contradiction completes the proof. $\square$

Along with (2.4), we consider the corresponding system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X(t, x) \tag{2.15}$$

for $t \geq 0$, $x \in D$, where D is a set (possibly closed) in $\mathbb{R}^d$.

Definition 2.6. *The solutions x and x_k^h of* (2.15) *and* (2.6) *are called* corresponding *if* $x(t_0) = x_0^h = x_0 \in D$.

The following lemma holds for corresponding solutions.

Lemma 2.7. *Assume $X \in C([0, a] \times D)$ satisfies*

(1) *there exists $M > 0$ such that $|X(t, x)| \le M$, $t \in [0, a]$, $x \in D$;*
(2) *there exists $L > 0$ such that for arbitrary $t_1, t_2 \in [0, a]$, $x_1, x_2 \in D$, we have*

$$|X(t_1, x_1) - X(t_2, x_2)| \le L\big(|t_1 - t_2| + |x_1 - x_2|\big).$$

If the corresponding solutions of (2.6) *and* (2.15) *are defined on the interval $[t_0, t_0 + T]$, then the estimate*

$$|x(t_0 + kh) - x_k^h| \le Ch \tag{2.16}$$

holds, where C depends only on M, L and T.

Proof. The proof is obtained by a slight modification of the scheme of proof of [9, Lemma 5.1.2, p. 114], taking into account [9, Proposition 5.2.2, p. 118]. □

The following result applies to (2.15) and (2.6), namely, to systems of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x \tag{2.17}$$

and

$$x_{k+1}^h = x_k^h + hA(t_0 + kh)x_k^h. \tag{2.18}$$

If A is continuous on $[0, \infty)$, then all solutions of (2.17) and (2.18) exist on the entire right semi-axis. We consider the solutions with the initial data

$$t_0 \in [0, \bar{h}], \quad |x_0| = 1, \tag{2.19}$$

where $\bar{h}$ is chosen in (2.8). Let $M(T) := \max_{[t_0, t_0+T]} \|A(t)\|$, where $T > 0$ is fixed.

Lemma 2.8. *Let x and x_k^h be solutions of the Cauchy problems for* (2.17) *and* (2.18), *respectively, with the initial data* (2.19). *Then there exists $R > 0$, which depends only on T and M, such that for $t \in [t_0, t_0 + T]$, $t_0 + kh \in [t_0, t_0 + T]$, we have*

$$|x(t)| \le R, \quad |x_k^h| \le R. \tag{2.20}$$

Proof. The first inequality in (2.20) is a simple consequence of the properties of linear systems of differential equations. The second inequality in (2.20) is a consequence of similar properties for systems of difference equations (see, for example, [15, p.35]). □

Remark 2.9. Note that R, which appears in Lemma 2.8, does not depend on h.

We now consider the converse problem: under which conditions does oscillation of solutions of differential equations implies oscillation of solutions of corresponding difference equations? It is well known (see, e.g. [19, p. 207]) that if p is smooth on $[0, a]$, the linear substitution $z = \varphi(t)x$ eliminates the first derivative in (2.1). Since the zeros of the solutions remain unchanged under this substitution, without loss of generality, we focus our attention on the oscillatory properties of the solutions of equations in the form

$$\ddot{x} + p(t)x = 0. \tag{2.21}$$

Assume that the following conditions hold:

$$p(t) \geq 0, \quad t \in [0, a] \tag{2.22}$$

and

$$p \text{ is Lipschitz on } [0, a]. \tag{2.23}$$

The difference equation corresponding to (2.21) is

$$\Delta_k^2 x + h^2 p(kh)x(kh) = 0. \tag{2.24}$$

We now study conditions under which oscillation of solutions of (2.24) implies oscillation of corresponding solutions of (2.21). We start with the following preliminary results. Rewrite (2.24) as a system

$$\begin{cases} x_{k+1}^h = x_k^h + hy_k^h, \\ y_{k+1}^h = y_k^h - hp(kh)x_k^h \end{cases} \tag{2.25}$$

with the initial conditions

$$x_0^h = x_0, \quad y_0^h = y_0, \quad \text{where } x_0^2 + y_0^2 = 1. \tag{2.26}$$

Now oscillation of x_k^h of (2.24) on $[0, a]$ is equivalent to oscillation of the first component of (2.25) on the same interval. Let (x_k^h, y_k^h) be an oscillatory solution of (2.25) on $[0, a]$ with the initial conditions (2.26). By Definition 2, x_k^h has at least two changes of sign on $[0, a]$. Let t_p and t_m be two consecutive points at which x_k^h changes sign. Introduce

$$M_p^x(h) := \max_{k \in [p+1, m]} |x_k^h|,$$

which is the amplitude of oscillation of x_k^h between t_p and t_m.

LEMMA 2.10. *Assume p is continuous on $[0, a]$ and satisfies (2.12). Then there exists $\Delta(h) > 0$ such that for any oscillatory solution of (2.25) with the initial data (2.26), we have*

$$M_p^x(h) \geq \Delta(h). \tag{2.27}$$

Proof. Assume (2.27) fails to hold. Then there exist an infinite sequence of oscillatory solutions $(x_k^h(n), y_k^h(n))$ of (2.25) on $[0, a]$, with the initial data (x_{0n}, y_{0n}) satisfying (2.26), and sequences $\{p(n)\}$ and $\{m(n)\}$ (consecutive points of sign changes of $x_k^h(n)$) such that

$$M_{p(n)}^{x_n}(h) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty, \tag{2.28}$$

where $M_{p(n)}^{x_n}(h) = \max_{k \in [p(n)+1, m(n)]} \left| x_k^h(n) \right|$. Let t_n be the point of the interval $[(p(n)+1)h, mh]$ where this maximum is attained. Since the set of the initial data (2.26) is compact, we can choose a convergent subsequence, still denoted by (x_{0n}, y_{0n}), such that

$$(x_{0n}, y_{0n}) \longrightarrow (x_0, y_0) \quad \text{as} \quad n \longrightarrow \infty \tag{2.29}$$

and $x_0^2 + y_0^2 = 1$. We know that all $M_{p(n)}^{x_n}(h) > 0$ as otherwise we would have considered solutions with opposite sign. Obviously, there is a point $k_0 h \in (0, a)$ at which for an infinite number of solutions from the sequence $(x_k^h(n), y_k^h(n))$, we have

$$x_{k_0}^h(n) = M_{p(n)}^{x_n}(h). \tag{2.30}$$

Let us consider this subsequence of solutions. Denote it by $(x_k^h(r), y_k^h(r))$ and their initial data by (x_{0r}, y_{0r}). Consider now the solution $(\bar{x}_k^h, \bar{y}_k^h)$ of (2.25) with initial data (x_0, y_0) from (2.29). Clearly, it is not identically zero. Using uniqueness and the continuous dependence of solutions on the initial data, we obtain

$$\bar{x}_{k_0}^h = 0, \quad \bar{x}_{k_0-1}^h < 0, \quad x_{k_0+1}^h < 0. \tag{2.31}$$

Note that by Definition 2.1, $k_0 h$ is an interior point of the interval $[0, a]$. By (2.25), we have

$$\begin{cases} \bar{x}_{k_0}^h - x_{k_0-1}^h = h y_{k_0-1}^h, \\ \bar{y}_{k_0}^h - \bar{y}_{k_0-1}^h = -h p(k_0 h) \bar{x}_{k_0-1}. \end{cases} \tag{2.32}$$

Using (2.31) and (2.32), we have that $y_{k_0-1}^h > 0$. Similarly, $\bar{x}_{k_0+1}^h - \bar{x}_{k_0}^h = h y_{k_0}^h$, and thus $y_{k_0}^h < 0$, which leads to a contradiction to the second equation in (2.32). □

Lemma 2.10 allows the possibility that $\Delta(h)$ may depend on h. The following result shows that this number may be chosen independently of h.

Lemma 2.11. *Assume p satisfies* (2.22) *and* (2.23). *Then there exist $h_0 > 0$ and $B_0 > 0$ such that for all $h \in (0, h_0]$, we have*

$$\Delta(h) \geq B_0. \tag{2.33}$$

Proof. By contradiction, assume there exists a sequence of step sizes $\{h_n\}$, $h_n > 0$, $h_n \longrightarrow 0$ as $n \longrightarrow \infty$ such that

$$\Delta(h_n) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{2.34}$$

Therefore, there exists an oscillatory solution $(x_k^{h_n}, y_k^{h_n})$ of (2.25) on $[0, a]$ with the initial data (x_{0n}, y_{0n}) satisfying (2.26), and all of its amplitudes satisfy

$$M_{p(n)}^{x_n}(h_n) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{2.35}$$

Up to a subsequence, still denoted by (x_{0n}, y_{0n}), we have

$$(x_{0n}, y_{0n}) \longrightarrow (x_0, y_0) \quad \text{as} \quad n \longrightarrow \infty. \tag{2.36}$$

Choose $t_n \in [0, a]$, $t_n = k_n h_n$, such that $x_{k_n}^{h_n} = M_{p(n)}^{x_n}(h_n)$, where, as before, we assume that $x_{k_n}^{h_n} > 0$. Thus, (2.35) now reads

$$x_{k_n}^{h_n} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{2.37}$$

Passing to a further subsequence, still denoted by $\{t_n\}$, we have $t_n \longrightarrow t_0$ as $n \longrightarrow \infty$, $t_0 \in [0, a]$. Furthermore, it follows from Lemma 2.8 that $\{y_{k_n}^{h_n}\}$ is uniformly bounded in n, and thus, up to a subsequence,

$$y_{k_n}^{h_n} \longrightarrow y^0 \quad \text{as} \quad n \longrightarrow \infty. \tag{2.38}$$

Due to Definition 2.1, both t_{n-1} and t_n are in $[0, a]$, and thus

$$\lim_{n \to \infty} t_{n-1} = \lim_{n \to \infty} t_n = t_0. \tag{2.39}$$

By the definition of k_n, we have

$$x_{k_n-1}^{h_n} \leq x_{k_n}^{h_n} \quad \text{and} \quad x_{k_n}^{h_n} \geq x_{k_n+1}^{h_n}. \tag{2.40}$$

Now, rewrite (2.25) as

$$\begin{cases} x_{k_n+1}^{h_n} - x_{k_n}^{h_n} = h_n y_{k_n}^{h_n}, \\ y_{k_n+1}^{h_n} - y_{k_n}^{h_n} = -h_n p(k_n h_n) x_{k_n}^{h_n} \end{cases} \tag{2.41}$$

and

$$\begin{cases} x_{k_n}^{h_n} - x_{k_n-1}^{h_n} = h_n y_{k_n-1}^{h_n}, \\ y_{k_n}^{h_n} - y_{k_n-1}^{h_n} = -h_n p((k_n - 1)h_n) x_{k_n-1}^{h_n}. \end{cases} \tag{2.42}$$

Since, by Lemma 2.8, $x_k^{h_n}$ and $y_k^{h_n}$ are uniformly bounded, the right-hand sides in (2.41) and (2.42) go to zero as $n \longrightarrow \infty$. Therefore, using (2.37) and (2.38), we obtain

$$\lim_{n \to \infty} x_{k_n+1}^{h_n} = \lim_{n \to \infty} x_{k_n-1}^{h_n} = 0 \tag{2.43}$$

and

$$\lim_{n \to \infty} y_{k_n+1}^{h_n} = \lim_{n \to \infty} y_{k_n-1}^{h_n} = y^0. \tag{2.44}$$

Now, applying (2.40) to the first equations in (2.41) and (2.42), we find

$$y_{k_n-1}^{h_n} \geq 0 \quad \text{and} \quad y_{k_n}^{h_n} \leq 0. \tag{2.45}$$

Combining (2.38), (2.44) and (2.45), we get

$$y^0 = 0. \tag{2.46}$$

Let $x(\cdot, x_0, y_0)$ be the solution of (2.21) with initial data (2.36). Clearly, it is not identically zero. Let $x(\cdot, x_{0n}, y_{0n})$ be the solution of (2.21) with the initial data (x_{0n}, y_{0n}). By continuity

$$x(t_n, x_0, y_0) \to x(t_0, x_0, y_0) \quad \text{as} \quad n \to \infty. \tag{2.47}$$

Moreover, the continuous dependence of solutions on initial data implies that

$$|x(t_n, x_{0n}, y_{0n}) - x(t_n, x_0, y_0)| \to 0 \quad \text{as} \quad n \to \infty. \tag{2.48}$$

Lemma 2.8 implies that all solutions of the Cauchy problem (2.21) and the difference system (2.25) with the initial data (2.26) are uniformly bounded on $[0, a]$. Consequently, it follows from Lemma 2.7 that (2.16) holds uniformly at the nodal points. In other words, C from Lemma 2.7 and R from Lemma 2.8 depend only on a and on the maximum of $|p(t)|$ on $[0, a]$. Therefore,

$$\left| x(t_n, x_{0n}, y_{0n}) - x_{k_n}^{h_n} \right| \to 0 \quad \text{as} \quad n \to \infty. \tag{2.49}$$

Due to (2.37), we have by (2.47), (2.48) and (2.49) that

$$x(t_0, x_0, y_0) = 0 \tag{2.50}$$

holds. In a similar way, we obtain

$$\dot{x}(t_0, x_0, y_0) = 0 \tag{2.51}$$

which, due to (2.50), contradicts the fact that $x(t, x_0, y_0)$ is non-trivial. □

3. Main results

In this section, we present the main results about the relation between oscillation of solutions of (2.1), (2.2), (2.3), (2.21) and (2.24). These equations are equivalent to the following systems:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -p(t)y - q(t)x, \end{cases} \tag{3.1}$$

$$\begin{cases} x(t+h) = x(t) + hy(t), \\ y(t+h) = y(t) - h(p(t)y(t) + q(t)x(t)), \end{cases} \tag{3.2}$$

$$\begin{cases} x_{k+1}^h = x_k^h + hx_k^h, \\ y_{k+1}^h = y_k^h - h(p(t_0 + kh)y_k^h + q(t_0 + kh)x_k^h). \end{cases} \tag{3.3}$$

Systems (3.2) and (3.3) are of the form (2.4) and (2.6), respectively. Therefore, the solutions of (3.2) are uniquely determined by the initial functions $x = \varphi(t)$, $y = \psi(t)$,

$t \in [0, h]$ which satisfy the coherence condition

$$\begin{cases} \varphi(h) = \varphi(0) + h\psi(0), \\ \psi(h) = \psi(0) - h\big(p(0)\psi(0) + q(0)\varphi(0)\big). \end{cases} \tag{3.4}$$

In what follows, we assume that $\varphi, \psi \in \mathrm{C}([0, h])$. The solutions of (3.3) are uniquely determined by the initial data

$$x_0^h(t_0) = x_0, \quad y_0^h(t_0) = y_0.$$

For the statements of our following main results, recall from Definition 2.6 that the solutions x and x_k^h of (2.15) and (2.6) are called corresponding if $x(t_0) = x_0^h = x_0 \in D$.

THEOREM 3.1. *Let p and q in (2.1) be Lipschitz on $[0, a]$. Then there exists $h_0 > 0$ such that for all $h \in (0, h_0]$, the following assertion holds: if x is a solution of (2.1) which starts at $t_0 \in [0, h]$ and has at least three zeros on $[t_0, a)$, then the corresponding solution of the difference equation (2.3) oscillates on $[t_0, a]$.*

Proof. Consider (2.3) for $h \le \bar{h}$, where $\bar{h}$ is chosen from (2.8). It is convenient to treat (2.3) as a system (3.3) in order to track the sign change of x_k^h. Equation (2.1) and systems (3.1) and (3.3) satisfy all conditions of Lemmas 2.4, 2.5, 2.7 and 2.8. Fix $\rho \in (0, \Delta/2)$, where Δ is given in Lemma 2.4, and let $h_1 := \min\{\bar{h}, \delta\}$, where δ is defined in Lemma 2.5. It follows from Lemma 2.8 that all solutions of (3.1) and (3.3) with the initial data given at $t_0 \in [0, h]$ are bounded by some $R > 0$ for $t \in [t_0, a]$, $t_0 + kh \in [t_0, a]$ and $h \le h_1$. Note that M in Lemma 2.8 depends only on the maxima of functions $|p|$ and $|q|$ on $[0, a]$; R also depends only on a and M. Choosing $D = B_0(R)$ in Lemma 2.7, we may conclude that for the corresponding solutions of (3.1) and (3.3), (2.16) holds with C independent of a, R and M. Finally, choose $h_0 \in (0, h_1]$ such that for $0 < h \le h_0$, the right-hand side of (2.16) satisfies the condition

$$Ch \le \rho. \tag{3.5}$$

For such h, let x be an arbitrary non-trivial solution of (2.1) with the initial data $x(t_0) = x_0$, $\dot{x}(t_0) = x_1$, $t_0 \in [0, h]$, which has at least three zeros on (t_0, a). We would like to show that the corresponding solution (x_k^h, y_k^h) of (3.3) has at least two changes of sign of x_k^h on (t_0, a). To this end, define $\tau_0 := \sqrt{x_0^2 + x_1^2}$. Due to linearity,

$$\frac{1}{\tau_0}(x(t), \dot{x}(t)) = (z(t), \xi(t)) \quad \text{and} \quad \frac{1}{\tau_0}(x_k^h, y_k^h) = (z_k^h, \xi_k^h)$$

also satisfy (3.1) and (3.3), respectively, while the component $z(t)$ has the same zeros as $x(t)$, and z_k^h has the same changes of sign as x_k^h. The initial data of the solution (z, ξ) satisfy (2.8). Consider a ρ-neighbourhood of $(z(t), \xi(t))$ for $t \in [t_0, a]$. The function z has at least two amplitudes of oscillations on (t_0, a). Moreover, for an appropriate choice of h_0 and $n \in \mathbb{N}$, $t_0 + nh \in I_{\Delta,k}^z$, appearing in Lemma 2.5. Thus, it follows from (3.5) that at $t_0 + nh \in I_{\Delta,k}^z$ and at t_0, z_k^h has the same sign as $z(t)$, and, therefore has at least two changes of sign. □

Consider now (2.2), or the equivalent system (3.2). The following result follows from Theorem 3.1 and Lemma 2.3.

THEOREM 3.2. *Let p and q in (2.2) be Lipschitz on $[0, a]$. Then there exists $h_0 > 0$ such that for all $h \in (0, h_0]$, the following statement holds: every solution of (3.2) with the initial functions $\varphi, \psi \in C([0, h])$ satisfying (3.4) has oscillatory first component on $(0, a)$, provided that there exists $t_0 \in [0, h]$ such that the solution of (2.1) with the initial data*

$$X(t_0) = \varphi(t_0), \quad \dot{x}(t_0) = \psi(t_0)$$

has at least three zeros on (t_0, a).

Consider now (2.21), the corresponding functional difference equation

$$\Delta^2 x(t) + h^2 p(t)x(t) = 0, \tag{3.6}$$

and the difference equation

$$\Delta_k^2 x(t_0) + h^2 p(t_0 + kh)x(t_0 + kh) = 0 \tag{3.7}$$

with p satisfying the Lipschitz condition on $[0, a]$. Let

$$m = \min_{t \in [0,a]} p(t) \quad \text{and} \quad M = \max_{t \in [0,a]} p(t).$$

Assume

$$m > 0 \quad \text{and} \quad a > \frac{3\pi}{\sqrt{m}}. \tag{3.8}$$

If

$$a - \bar{h} > \frac{3\pi}{\sqrt{m}}, \tag{3.9}$$

then all solutions of (2.21) with the initial data $t_0 \in [0, \bar{h}]$ have at least three zeros on $[t_0, a)$. Taking into account this fact, from Theorem 3.1 and Theorem 3.2, we can obtain the following two corollaries about oscillation of the solutions of (3.6) and (3.7).

COROLLARY 3.3. *Assume p is Lipschitz on $[0, a]$ and (3.8) and (3.9) hold. Then there exists $h_0 > 0$ such that for all $h \in (0, h_0]$, all solutions of (3.7) with the initial data given at $t_0 \in [0, h]$ oscillate on $[t_0, a)$.*

COROLLARY 3.4. *Assume p is Lipschitz on $[0, a]$ and (3.8) and (3.9) hold. Then there exists $h_0 > 0$ such that for all $h \in (0, h_0]$, all solutions of the system*

$$\begin{cases} x(t+h) = x(t) + hy(t), \\ y(t+h) = y(t) - hp(t)x(t) \end{cases}$$

with the initial functions $\varphi, \psi \in \mathrm{C}([0, h])$ *satisfying the coherence condition*

$$\begin{cases} \varphi(h) = \varphi(0) + h\psi(0), \\ \psi(h) = \psi(0) - hp(0)\varphi(0)) \end{cases}$$

have an oscillatory first component on $(0,\ a)$.

Example 3.5. If we take in (3.6)

$$p(t) = t + 1 \quad \text{and} \quad a = 4\pi,$$

then all conditions of Corollary 3.3 and Corollary 3.4 are satisfied.

The following theorem describes the relation between oscillation of the solutions of (2.1) and (2.24).

THEOREM 3.6. *Let* p *satisfy* (2.22) *and* (2.23). *Then there exists* h_0 *such that for all* $h \in (0, h_0]$, *the following assertion holds: if* x_k^h *is a solution of* (2.24) *which has at least three changes of sign on* $[0, a]$, *then the corresponding solution of the differential equation* (2.2) *oscillates on* $[0, a]$.

Proof. Rewrite (2.21) as

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -p(t)x. \end{cases} \tag{3.10}$$

As noted in the proof of Lemma 2.11, there is a uniform estimate (2.16) at the nodal points of the solutions of (2.25) and (3.10) with the initial data (x_0, y_0), $x_0^2 + y_0^2 = 1$, given at $t_0 = 0$, where C depends only on a and on the maximum of $|p|$ over $[0, a]$. Choose

$$Ch_1 \leq \frac{B_0}{2}, \tag{3.11}$$

where B_0 is defined in Lemma 2.11. Then for all

$$0 < h \leq \min\{h_0, h_1\}, \tag{3.12}$$

(3.11) and the assertion of Lemma 2.11 hold. Fix h which satisfies (3.12), and let (x_k^h, y_k^h) be an arbitrary non-trivial solution of (2.25) such that its first component x_k^h has at least three changes of sign on $[0, a]$. Let (x_0, y_0) be its initial data. We want to show that the solution of (2.21) with initial data $x(0) = x_0$, $\dot{x}(0) = y_0$ oscillates on $[0, a]$. Define $r_0 := \sqrt{x_0^2 + y_0^2} \neq 0$. Clearly,

$$\frac{1}{r_0}(x(t), \dot{x}(t)) = (z(t), \xi(t)) \quad \text{and} \quad \frac{1}{r_0}(x_k^h, y_k^h) = (z_k^h, \xi_k^h)$$

solve (2.25) and (3.10), respectively. Then $z(t)$ has the same zeros as $x(t)$, z_k^h has the same changes of sign as x_k^h and the initial conditions for $(z(t), \xi(t))$ and (z_k^h, ξ_k^h) satisfy (2.26).

Now, consider a ρ-neighbourhood of (z_k^h, ξ_k^h) for $kh \in [0, a]$, where $\rho \leq (B_0/2)$. Since z_k^h has at least three changes of sign, z_k^h has at least two amplitudes of oscillation. Let $k_0 h$ and $p_0 h$ be the points at which these amplitudes are attained. Then $z_{k_0}^h$ and $z_{p_0}^h$ must have different signs. Inequalities (2.16), (3.11) and Lemma 2.11 imply that $z(t)$ at $t = 0$, $t = k_0 h$, $t = p_0 h$ has the same sign as z_k^h, and, therefore, has at least two zeros on $[0, a]$. Consequently, $x(t)$ has at least two zeros on $[0, a]$, and this means that it is oscillatory. □

Remark 3.7. We can consider our results as results on the particular time scale $h\mathbb{Z}$ (see [6]). Corresponding results for more general time scales such as, e.g. $q^{\mathbb{N}_0}$, will be presented in a forthcoming paper of the authors.

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