Oscillation of third-order nonlinear damped delay differential equations

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A B S T R A C T
This paper is concerned with the oscillation of certain third-order nonlinear delay differential equations with damping. We give new characterizations of oscillation of the third-order equation in terms of oscillation of a related, well-studied, second-order linear differential equation without damping. We also establish new oscillation results for the third-order equation by using the integral averaging technique due to Philos. Numerous examples are given throughout.

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1. Introduction
In this article, we consider nonlinear third-order functional differential equations of the form

\[(r_2(r_1(y'))')'(t) + p(t)(y'(t))^\alpha + q(t)f(y(g(t))) = 0, \quad t \geq t_0 > 0,\]

where \(\alpha \geq 1\) is the ratio of positive odd integers. We assume that

(i) \(r_1, r_2, p, q \in C(I, \mathbb{R}^+), \) where \(I = [t_0, \infty)\) and \(\mathbb{R}^+ = (0, \infty)\);
(ii) \(g \in C^1(I, \mathbb{R}), \) \(g'(t) \geq 0, \) and \(g(t) \to \infty \) as \(t \to \infty;\)
(iii) \(f \in C(\mathbb{R}, \mathbb{R})\) such that \(xf(x) > 0 \) for \(x \neq 0\) and \(f(x)x^{\beta} \geq k > 0,\) where \(\beta\) is a ratio of positive odd integers.

A function \(y\) is called a solution of (1.1) if

\[y, r_1(y'), r_2(r_1(y'))' \in C^1([t_y, \infty), \mathbb{R}),\]

and \(y\) satisfies (1.1) on \([t_y, \infty)\) for some \(t_y \geq t_0.\)

We restrict our attention to those solutions of (1.1) which exist on \([t_0, \infty)\) and satisfy the condition 
\[\sup\{|y(t)| : t \leq t_0 < \infty\} > 0 \quad \text{for} \quad t_1 \in I.\]
Such a solution is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. Eq. (1.1) is said to be oscillatory if all of its solutions are oscillatory.

Determining oscillation criteria for particular second-order differential equations has received a great deal of attention in the last few years. Compared to second-order differential equations, the study of oscillation and asymptotic behavior of third-order differential equations has received considerably less attention in the literature. For some classical and recent

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results on third-order equations, the reader can refer to [1–4,7–9,15,16,20] and the references contained therein. It is interesting to note that there are third-order delay differential equations which have only oscillatory solutions or have both oscillatory and nonoscillatory solutions. For example,

$$y'''(t) + 2y'(t) - y\left(t - \frac{3\pi}{2}\right) = 0$$

admits the oscillatory solution $y_1(t) = \sin t$ and a nonoscillatory solution $y_2(t) = e^{\lambda t}$, where $\lambda > 0$ is such that $\lambda^3 + 2\lambda = e^{-3\pi/2}$ (see [14, p. 6]). On the other hand, all solutions of

$$y'''(t) + y(t - \tau) = 0, \quad \tau > 0$$

are oscillatory if and only if $\tau e > 3$ (see [13]). But the corresponding ordinary differential equation

$$y'''(t) + y(t) = 0$$

admits the nonoscillatory solution $y_1(t) = e^{-t}$ and oscillatory solutions $y_2(t) = e^{\pi/2}\sin(\sqrt{3}t/2)$ and $y_3(t) = e^{\pi/2}\cos(\sqrt{3}t/2)$.

In the literature, there are some papers and books (see, e.g., [5,6,11,17,19,21,22] and the references therein) which deal with the oscillatory and asymptotic behavior of solutions of functional differential equations. In [21], the authors used a generalized Riccati transformation and an integral averaging technique in order to establish some sufficient conditions which ensure that any solution of (1.1) oscillates or converges to zero. The purpose of this paper is to improve and unify the results in [21] and present some new sufficient conditions which ensure that any solution of (1.1) oscillates when the related second-order linear ordinary differential equation without delay

$$(r_2z')' + \left(\frac{p(t)}{r_1(t)}\right)z(t) = 0 \quad (2.1)$$

is nonoscillatory or oscillatory. We also apply our results to equations of the form

$$a_3(t)y'''(t) + a_2(t)y''(t) + a_1(t)y'(t) + q(t)f(y(g(t))) = 0, \quad (1.3)$$

where $a_1, a_2, a_3, q \in C(I, \mathbb{R}^+)$.  

2. Auxiliary results

For the sake of brevity, we define

$$L_0y = y, \quad L_1y = r_1((L_0y)')', \quad L_2y = r_2(L_1y)', \quad L_3y = (L_2y)'$$

on $I$. Hence (1.1) can be written as

$$L_3y(t) + \left(\frac{p(t)}{r_1(t)}\right)L_1y(t) + q(t)f(y(g(t))) = 0.$$

Remark 2.1. If $y$ is a solution of (1.1), then $z = -y$ is a solution of the equation

$$L_3z(t) + \left(\frac{p(t)}{r_1(t)}\right)L_1z(t) + q(t)f^*(z(g(t))) = 0,$$

where $f^*(z) = -f(-z)$ and $z^* > 0$ for $z \neq 0$. Thus, concerning nonoscillatory solutions of (1.1), we can restrict our attention only to solutions which are positive for all large $t$.

Define the functions

$$R_1(t, t_1) = \int_{t_1}^{t} \frac{ds}{(r_1(s))^{1/\alpha}}, \quad R_2(t, t_1) = \int_{t_1}^{t} \frac{ds}{r_2(s)},$$

and

$$R^*(t, t_1) = \int_{t_1}^{t} \left(\frac{R_2(s, t_1)}{r_1(s)}\right)^{1/\alpha} ds$$

for $t_0 \leq t_1 \leq t < \infty$. We assume that

$$R_1(t, t_0) \rightarrow \infty \quad \text{as} \quad t \rightarrow \infty \quad (2.1)$$

and

$$R_2(t, t_0) \rightarrow \infty \quad \text{as} \quad t \rightarrow \infty. \quad (2.2)$$

In this section, we state and prove the following lemmas which we will use in the proof of our main results.

Lemma 2.2. Suppose that (1.2) is nonoscillatory. If $y$ is a nonoscillatory solution of (1.1) on $[t_1, \infty)$, $t_1 \geq t_0$, then there exists $t_2 \in [t_1, \infty)$ such that $y(t)L_1y(t) > 0$ or $y(t)L_1y(t) < 0$ for $t \geq t_2$. 

Lemma 2.3. Let $y$ be a nonoscillatory solution of (1.1) on $[t_1, \infty)$, say $y(t) > 0$ and $y'(t) > 0$ for $t \geq t_1 \geq t_0$. Set $x = -L_1 y$. From (1.1), $x$ satisfies the equation

$$(r_2 x')' + \left(\frac{p(t)}{r_1(t)}\right)x(t) = q(t)f(y(g(t))) > 0, \quad t \geq t_1.$$ 

Let $u$ be any solution of (1.2), say $u(t) > 0$ for $t \geq t_1 \geq t_0$. Assume that $x$ is oscillatory. Hence $x$ has consecutive zeros at $a$ and $b$ ($t_1 < a < b$) such that $x'(a) \geq 0$, $x'(b) \leq 0$, and $x(t) \geq 0$ for $t \in (a, b)$. Thus

$$0 < \int_a^b \left(\frac{r_2 x'}{r_1 x}\right)(t)u(t)\,dt
= \int_a^b r_2 x'\left|\frac{b}{a}\right|u\,dt + \int_a^b \left(\frac{p}{r_1} u\right)(t)\,dt
= \int_a^b r_2 x'\left|\frac{b}{a}\right|u\,dt + \int_a^b \left(\frac{r_2 u'}{r_1 u}\right)(t)x(t)\,dt + \int_a^b \left(\frac{p}{r_1} u\right)(t)\,dt
= \int_a^b r_2 x'\left|\frac{b}{a}\right|u\,dt + \int_a^b \left(\frac{r_2 u'}{r_1 u}\right)(t)x(t)\,dt = \int_a^b r_2 x'\left|\frac{b}{a}\right|u \leq 0.$$ 

This completes the proof. □

Lemma 2.3. Let $y$ be a nonoscillatory solution of (1.1) with $y(t)L_1 y(t) > 0$ for $t \geq t_1 \geq t_0$. Then

$L_1 y(t) \geq R_2(t, t_1)L_2 y(t)$ for all $t \geq t_1$ (2.3)

and

$y(t) \geq R^*(t, t_1)L_2^{1/\alpha} y(t)$ for all $t \geq t_1$. (2.4)

Proof. Let $y$ be a nonoscillatory solution of (1.1), say $y(t) > 0$, $y'(t) > 0$, and $L_1 y(t) > 0$ for $t \geq t_1 \geq t_0$. Since

$L_3 y(t) = -\left(\frac{P_1}{r_1}\right)y(t) - q(t)f(y(g(t))) \leq 0.$

we have that $L_2 y$ is nonincreasing on $[t_1, \infty)$, and hence

$L_1 y(t) = L_1 y(t_1) + \int_{t_1}^t L_1 y'(s)\,ds \geq \int_{t_1}^t L_1 y'(s)\,ds
= \int_{t_1}^t L_2^1 y(s)\,ds \geq \int_{t_1}^t L_2^1 y(t)\,ds
= R_2(t, t_1)L_2^1 y(t).$

This implies

$$y'(t) \geq \left(\frac{R_2(t, t_1)}{R_1(t_1)}\right)^{1/\alpha} L_2^{1/\alpha} y(t).$$

Now, integrating this inequality from $t_1$ to $t$ and using the fact that $L_2 y$ is nonincreasing, we find

$$y(t) = y(t_1) + \int_{t_1}^t y'(s)\,ds \geq \int_{t_1}^t y'(s)\,ds
\geq \int_{t_1}^t \left(\frac{R_2(s, t_1)}{R_1(s)}\right)^{1/\alpha} L_2^{1/\alpha} y(s)\,ds
\geq \int_{t_1}^t \left[\frac{R_2(s, t_1)}{R_1(s)}\right]^{1/\alpha} L_2^{1/\alpha} y(t)\,ds
= R^*(t, t_1)L_2^{1/\alpha} y(t)$$

for $t \geq t_1$. This completes the proof. □

In the following two lemmas, we consider the second-order delay differential equation

$$(r_2 x')' + Q(t)x(h(t)), \quad t \to \infty,$$

where the function $r_2$ is as in (1.1), $Q \in C(I, R^+)$, and $h \in C^1(I, R)$ is such that $h(t) \leq t$, $h'(t) \geq 0$ for $t \geq t_0$, and $h(t) \to \infty$ as $t \to \infty$.

Lemma 2.4. If

$$\lim_{t \to \infty} \sup_{t \geq t_0} \int_{h(t)}^t Q(s)R_2(h(t), h(s))\,ds > 1,$$

then all bounded solutions of (2.5) are oscillatory.
Lemma 2.5. If \( r_2 x' \) is strictly increasing on \([t_1, \infty)\). Hence, for any \( t_2 \geq t_1 \), we have

\[
x(t) = x(t_2) + \int_{t_1}^{t} x'(s) \, ds = x(t_2) + \int_{t_1}^{t} \frac{r_2(s) x'(s)}{r_2(s)} \, ds
\]

so \( x'(t_2) < 0 \) as otherwise (2.2) would imply \( x(t) \to \infty \) as \( t \to \infty \), a contradiction to the boundedness of \( x \). Altogether,

\[
x > 0, \quad x' < 0, \quad \text{and} \quad (r_2 x')' > 0 \quad \text{on} \quad [t_1, \infty).
\]  

(2.7)

Now, for \( v \geq u \geq t_1 \), we have

\[
x(u) > x(u) - x(v) = -\int_{u}^{v} x'(s) \, ds = -\int_{u}^{v} \frac{r_2(s) x'(s)}{r_2(s)} \, ds
\]

\[
\geq -\int_{u}^{v} \frac{r_2(v) x'(v)}{r_2(s)} \, ds = -R_2(v, u) r_2(v) x'(v).
\]  

(2.8)

Integrating (2.5) from \( h(t) \geq t_1 \) to \( t \), we obtain

\[
-r_2(h(t)) x'(h(t)) > r_2(t) x'(t) - r_2(h(t)) x'(h(t)) = \int_{h(t)}^{t} Q(s) x(h(s)) \, ds
\]

\[
\geq -\left[ \int_{h(t)}^{t} Q(s) R_2(h(t), h(s)) \, ds \right] r_2(h(t)) x'(h(t)),
\]

i.e.,

\[
1 > \int_{h(t)}^{t} Q(s) R_2(h(t), h(s)) \, ds.
\]  

(2.9)

(2.10)

Taking \( \limsup \) as \( t \to \infty \) on both sides of (2.10) yields a contradiction to (2.6) and completes the proof. □

Lemma 2.5. If

\[
\limsup_{t \to \infty} \int_{h(t)}^{t} \left( \frac{1}{R_2(u)} \int_{u}^{t} Q(s) \, ds \right) \, du > 1,
\]  

then all bounded solutions of (2.5) are oscillatory.

Proof. Let \( x \) be a bounded nonoscillatory solution of (2.5), say \( x(t) > 0 \) and \( x(h(t)) > 0 \) for \( t \geq t_1 \) for some \( t_1 \geq t_0 \). As in Lemma 2.4, we obtain (2.7). Integrating (2.5) from \( u \) to \( t \), we get

\[
-r_2(u) x'(u) > r_2(t) x'(t) - r_2(u) x'(u) = \int_{u}^{t} Q(s) x(h(s)) \, ds
\]

\[
\geq \left[ \int_{u}^{t} Q(s) \, ds \right] x(h(t)),
\]

i.e.,

\[
x'(u) > \left( \frac{1}{r_2(u)} \int_{u}^{t} Q(s) \, ds \right) x(h(t)).
\]  

(2.11)

Integrating (2.12) from \( h(t) \) to \( t \), we get

\[
x(h(t)) > x(h(t)) - x(t)
\]

\[
\geq \left[ \int_{h(t)}^{t} \left( \frac{1}{r_2(u)} \int_{u}^{t} Q(s) \, ds \right) \, du \right] x(h(t)),
\]

i.e.,

\[
1 > \int_{h(t)}^{t} \left( \frac{1}{r_2(u)} \int_{u}^{t} Q(s) \, ds \right) \, du.
\]  

(2.13)

Taking \( \limsup \) as \( t \to \infty \) on both sides of (2.13) yields a contradiction to (2.11) and completes the proof. □
3. Oscillation—Riccati method

Now, we are ready to establish the main results of this paper.

**Theorem 3.1.** Assume (2.1), (2.2), and $\alpha \geq \beta$. Suppose (1.2) is nonoscillatory. If there exist two functions $\rho, h \in C^1(I, \mathbb{R})$ such that

$$g(t) \leq h(t) \leq t, \quad h'(t) \geq 0, \quad \text{and} \quad \rho(t) > 0 \quad \text{for all} \quad t \geq t_0,$$

satisfying

$$\limsup_{t \to \infty} \int_{t_1}^{t} \left[ k \rho(s) q(s) - \frac{A^2(s)}{4B(s)} \right] ds = \infty \quad \text{for any} \quad t_1 \in I,$$

(3.1)

where, for $t \geq t_1$,

$$A(t) = \rho'(t) - \frac{\rho(t)}{t_1(t)} R_2(t, t_1),$$

$$B(t) = c^* \rho^{-1}(t) g'(t)(R^*(g(t), t_1))^{\beta^{-1}} \left( \frac{R_2(g(t), t_1)}{r_1(g(t))} \right)^{1/\alpha},$$

(3.2)

and (2.6) or (2.11) holds with

$$Q(t) = \left[ c k q(t)(R_1(h(t), g(t)))^{\beta} - \frac{\rho(t)}{r_1(t)} \right] \geq 0 \quad \text{for all} \quad t \geq t_1,$$

(3.3)

with $c, c^* > 0$, then every solution $y$ of (1.1), or $L_2 y$, is oscillatory.

**Proof.** Let $y$ be a nonoscillatory solution of (1.1) on $[t_1, \infty)$, $t_1 \geq t_0$. Without loss of generality, we may assume that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq t_1$. From Lemma 2.2, it follows that $L_1 y(t) < 0$ or $L_1 y(t) > 0$ for $t \geq t_1$. First, we assume $L_1 y > 0$ on $[t_1, \infty)$. By (1.1), $L_2 y$ is strictly decreasing. Hence, for any $t_2 \geq t_1$, we have

$$L_1 y(t) = L_1 y(t_2) + \int_{t_2}^{t} (L_1 y)'(s) ds = L_1 y(t_2) + \int_{t_2}^{t} \frac{L_2 y(s)}{r_2(s)} ds$$

$$\leq L_1 y(t_2) + \int_{t_2}^{t} \frac{L_2 y(t_2)}{r_2(s)} ds = L_1 y(t_2) + L_2 y(t_2) R_2(t, t_2),$$

(3.4)

so $L_2 y(t_2) > 0$ as otherwise (2.2) would imply $L_1 y(t) \to -\infty$ as $t \to \infty$, a contradiction to the positivity of $L_1 y$. Altogether, $L_2 y > 0$ on $[t_1, \infty)$. Define

$$w = \rho \frac{L_2 y}{(y \circ g)^\beta} \quad \text{on} \quad [t_1, \infty).$$

(3.5)

Differentiating $w$ and using (1.1), (2.3), and the assumption (iii) on $f$, we obtain

$$w' = \frac{\rho'}{\rho} L_2 y + \frac{(L_2 y)'(y \circ g)^\beta - \beta (y \circ g)^{\beta-1} (y' \circ g) g L_2 y}{(y \circ g)^{2\beta}}$$

$$\leq \frac{\rho'}{\rho} w + \frac{L_2 y}{L_2 y} w - \beta g' \frac{y' \circ g}{y \circ g} w$$

$$\geq \frac{\rho'}{\rho} w - \frac{L_2 y}{L_2 y} w - \beta g' \frac{y' \circ g}{y \circ g} w$$

$$\leq -k \rho q + Aw - \beta g \left( \frac{y' \circ g}{y \circ g} \right) w.$$
Thus, 
\[
\frac{y'(g(t))}{y(g(t))} = \left( \frac{R_2(g(t), t_1)}{\rho(t)r_1(g(t))} \right)^{1/\alpha} \frac{\rho^{1/\alpha}(t)(L_2y(t))^{1/\alpha}}{y^{\beta/\alpha}(g(t))} y^{\beta/\alpha-1}(g(t))
\]
\[
\overset{(3.4)}{=} \left( \frac{R_2(g(t), t_1)}{\rho(t)r_1(g(t))} \right)^{1/\alpha} \omega^{1/\alpha}(t)y^{\beta/\alpha-1}(g(t))
\]
and (3.5) implies
\[
w'(t) \leq -k\rho(t)q(t) + A(t)w(t) - \beta g'(t)w^{1+1/\alpha}(t)y^{\beta/\alpha-1}(g(t)) \left( \frac{R_2(g(t), t_1)}{\rho(t)r_1(g(t))} \right)^{1/\alpha}.
\]
(3.6)
It follows from \(L_2y(t) < 0\) that \(0 \leq L_2y(t) \leq L_2y(t_1) = c_1\) for \(t \geq t_1\). Hence
\[
r_2(t)(L_1y)'(t) = (L_2y(t)) \leq c_1 \text{ for all } t \geq t_1
\]
and thus we have for all \(t \geq t_2 := t_1 + 1\) that
\[
r_1(t)(y')^\alpha(t) = (L_1y)(t) = (L_1y)(t_1) + \int_{t_1}^{t} (L_1y)'(s)ds
\]
\[
\leq (L_1y)(t_1) + c_1 \int_{t_1}^{t} \frac{ds}{r_2(s)} = (L_1y)(t_1) + c_1R_2(t, t_1)
\]
\[
= \left[ \frac{(L_1y)(t_1)}{R_2(t, t_1)} + c_1 \right] R_2(t, t_1)
\]
\[
\leq \left[ \frac{(L_1y)(t_1)}{R_2(t_2, t_1)} + c_1 \right] R_2(t, t_1) = \tilde{c}_1R_2(t, t_1),
\]
holds (note \((L_1y)(t_1) > 0\), where
\[
\tilde{c}_1 = c_1 + \frac{(L_1y)(t_1)}{R_2(t_2, t_1)}.
\]
Therefore, we have for all \(t \geq t_2\) that
\[
y(t) = y(t_2) + \int_{t_2}^{t} y'(s)ds \leq y(t_2) + \int_{t_2}^{t} \left( \frac{\tilde{c}_1R_2(s, t_1)}{r_1(s)} \right)^{1/\alpha} ds
\]
\[
\leq y(t_2) + \int_{t_2}^{t} \left( \frac{\tilde{c}_1R_2(s, t_1)}{r_1(s)} \right)^{1/\alpha} ds = y(t_2) + \tilde{c}_1^{1/\alpha} R^*(t, t_1)
\]
\[
= \left[ \frac{y(t_2)}{R^*(t, t_1)} + \tilde{c}_1^{1/\alpha} \right] R^*(t, t_1)
\]
\[
\leq \left[ \frac{y(t_2)}{R^*(t_2, t_1)} + \tilde{c}_1^{1/\alpha} \right] R^*(t, t_1) = c_2R^*(t, t_1)
\]
holds (note \(y(t_2) > 0\), where
\[
c_2 = \tilde{c}_1^{1/\alpha} + \frac{y(t_2)}{R^*(t_2, t_1)}.
\]
Thus we have
\[
y^{\beta/\alpha-1}(g(t)) \geq c_2^{\beta/\alpha-1}(R^*(g(t), t_1))^{\beta/\alpha-1} \text{ for } t \geq t_2.
\]
(3.7)
From (3.4) and (2.4), we get
\[
w(t) = \rho(t) \frac{L_2y(t)}{y^{\beta}(g(t))} \leq \rho(t) \frac{L_2y(g(t))}{y^{\beta}(g(t))} \leq \rho(t)(R^*(g(t), t_1))^{-\alpha} y^{\beta-\alpha}(g(t))
\]
(3.8)
for \(t \geq t_1\). Using (3.7) in (3.8), we obtain
\[
w(t) \leq c_2^{\beta-\alpha} \rho(t)(R^*(g(t), t_1))^{-\beta},
\]
and hence
\[
w^{1/\alpha-1}(t) \geq c_2^{(\alpha-\beta)/(1/\alpha-1)} \rho^{1/\alpha-1}(t)(R^*(g(t), t_1))^{-\beta(1/\alpha-1)}
\]
(3.9)
for $t \geq t_2$. Using (3.7) and (3.9) in (3.6), we obtain

$$\begin{align*}
w'(t) &\leq -k\rho(t)q(t) + A(t)w(t) - \beta c_2^\beta t^{\alpha - \beta} \rho^{-1}(t)g(t)\left(R^*(g(t), t_1)\right)^{\beta - 1} \left(\frac{R_2(g(t), t_1)}{r_1(g(t))}\right)^{1/\nu} w^2(t) \\
&\leq -k\rho(t)q(t) + A(t)w(t) - B(t)w^2(t) \\
&= -k\rho(t)q(t) - \left(\frac{\sqrt{B(t)}}{2}\right)^2 + A^2(t) \\
&\leq -k\rho(t)q(t) + A^2(t) 4B(t)
\end{align*}
$$

for $t \geq t_2$, where $A$ and $B$ are as in (3.2) with $c^\star = \beta c_2^\beta$. Integrating (3.10) from $t_2$ to $t$, we see that

$$\int_{t_2}^t \left[ k\rho(s)q(s) - \frac{A^2(s)}{4B(s)} \right] ds \leq w(t_2) - w(t) \leq w(t_2),$$

which contradicts (3.1). Next, we assume $L_1y < 0$ on $[t_1, \infty)$. We consider the function $L_2y$. The case $L_2y(t) \leq 0$ cannot hold for all large $t$, since by a double integration of

$$y'(t) = \left(\frac{L_1y(t)}{r_1(t)}\right)^{1/\alpha} \leq \left(\frac{L_1y(t_2)}{r_1(t)}\right)^{1/\alpha}, \quad t \geq t_2,$$

we obtain from (2.1) that $y(t) < 0$ for all large $t$, which is a contradiction. Thus, assume $y(t) > 0$, $L_1y(t) < 0$, and $L_2y(t) \geq 0$ for all large $t$, say $t \geq t_3 \geq t_2$. Now, for $u \geq v \geq t_3$, we have

$$y(u) - y(v) = -\int_v^u r_1^{-1/\alpha}(\tau) \left(\frac{r_1(\tau) y'(\tau)^{\alpha}}{r_1(\tau)}\right)^{1/\alpha} d\tau \\
\geq \left(\int_v^u r_1^{-1/\alpha}(\tau) d\tau\right) (-L_1y(v))^{1/\alpha}$$

Setting $u = g(t)$ and $v = h(t)$, we get

$$y(g(t)) \geq R_1(v, u)(-L_1y(h(t)))^{1/\alpha} = R_1(h(t), g(t))x(h(t))$$

for $t \geq t_3$, where $x(t) = (-L_1y(t))^{1/\alpha} > 0$ for $t \geq t_3$. From (1.1), the fact that $x$ is decreasing, and $g(t) \leq h(t) \leq t$, we obtain

$$(r_2z')'(t) \geq \left(\frac{p(t)}{r_1(t)}\right)z(h(t))$$

$$\geq kq(t)\left(R_1(h(t), g(t))\right)^{\beta} z(h(t)) \left(z(h(t))\right)^{\beta - 1},$$

where $z = x^\alpha$. Since $z$ is decreasing and $\alpha \geq \beta$, there exists a constant $c_4 > 0$ such that $z^{\beta - 1}(t) \geq c_4$ for $t \geq t_2$. Thus,

$$(r_2z')'(t) \geq \left(c_4kq(t)\left(R_1(h(t), g(t))\right)^{\beta} \left(p(t)\right)\right)z(h(t)).$$

Proceeding exactly as in the proofs of Lemmas 2.4 and 2.5, we arrive at the desired conclusion, thus completing the proof. \(\Box\)

The following corollary is immediate.

**Corollary 3.2.** Assume (2.1), (2.2), and $\alpha \geq \beta$. Suppose (1.2) is nonoscillatory and $A \leq 0$, where $A$ is defined as in (3.2). If there exist two functions $\rho, h \in C^1(I, \mathbb{R})$ such that

$$g(t) \leq h(t) \leq t, \quad h'(t) \geq 0, \quad \text{and} \quad \rho(t) > 0 \quad \text{for all} \quad t \geq t_0,$$

and

$$\limsup_{t \to \infty} \int_{t_1}^t \rho(s)q(s)ds = \infty \quad \text{for any} \quad t_1 \in I,$$

and (2.6) or (2.11) holds with $Q$ as in Theorem 3.1, then every solution $y$ of (1.1), or $L_2y$, is oscillatory.

The following examples are illustrative.

**Example 3.3.** Consider the equation

$$((y')^3)' + 9(y')^3 + 6y\left(t - \frac{3\pi}{2}\right) = 0.$$  \hfill (3.11)
It is easy to check that all conditions of Corollary 3.2 with $\rho(t) = 1$ are satisfied, and hence (3.11) is oscillatory. One oscillatory solution is $y(t) = \sin t$.

**Example 3.4.** Consider the equation

$$((y')^3)'(t) + (y'(t))^3 + \frac{10}{e^2}y^3(t - 1) = 0. \tag{3.12}$$

Here we take $k = 1$, $\rho(t) = 1$ and $h(t) = t - 1/2$. Now, it is easy to check that all conditions of Theorem 3.1 are fulfilled except that $Q(t)$ is negative. We note that (3.12) admits the nonoscillatory solution $y(t) = e^{-t}$.

For $t \geq t_1 \geq t_0$, we let

$$P(t) = \frac{p(t)}{r_1(t)}R_2(t, t_1), \quad Q_1(t) = kq(t)(R^*(g(t), t_1))^\beta,$$

and

$$\mu(t) = \exp \left( \int_{t_1}^{t} P(s)ds \right).$$

Now, we present the following comparison result.

**Theorem 3.5.** Assume (2.1), (2.2), and $\alpha \geq \beta$. Suppose (1.2) is nonoscillatory. Assume there exists a function $h \in C^1(I, \mathbb{R})$ such that

$$g(t) \leq h(t) \leq t \quad \text{and} \quad h'(t) \geq 0 \quad \text{for all} \quad t \geq t_0,$$

and (2.6) or (2.11) holds with $Q$ as in Theorem 3.1. If every solution of the first-order delay equation

$$z'(t) + (\mu(g(t)))^{1-\beta/\alpha}Q_1(t)z^{\beta/\alpha}(g(t)) = 0 \tag{3.13}$$

is oscillatory, then every solution $y$ of (1.1) on $[t_1, \infty)$ is oscillatory.

**Proof.** Let $y$ be a nonoscillatory solution of (1.1) on $[t_1, \infty)$. We may assume that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq t_1$. From Lemma 2.2, it follows that $L_1y(t) < 0$ or $L_1y(t) > 0$ for $t \geq t_1$. If $L_1y > 0$ on $[t_1, \infty)$, then as in the proof of Theorem 3.1, we get $L_2y > 0$ on $[t_1, \infty)$. Since $\lim_{t \to \infty} g(t) = \infty$, we can choose $t_2 \geq t_1$ such that $g(t) \geq t_1$ for all $t \geq t_2$, and so (2.4) gives

$$y(g(t)) \geq R^*(g(t), t_1)L_2^1y(g(t)) \quad \text{for all} \quad t \geq t_2. \tag{3.14}$$

Using (2.3) and (3.14) in (1.1), we obtain

$$(L_2y)'(t) + \left( \frac{p(t)}{r_1(t)} \right)R_2(t, t_1)L_2y(t) + kq(t)(R^*(g(t), t_1))^\beta (L_2y(g(t)))^{\beta/\alpha} \leq 0$$

for $t \geq t_2$, which can be written as

$$w'(t) + P(t)w(t) + Q_1(t)w^{\beta/\alpha}(g(t)) \leq 0 \quad \text{for all} \quad t \geq t_2,$$

where $w(t) = L_2y(t)$, i.e.,

$$(\mu w)'(t) + \mu(t)Q_1(t)w^{\beta/\alpha}(g(t)) \leq 0 \quad \text{for all} \quad t \geq t_2.$$

Setting $z = \mu w > 0$ in the above inequality and noting that $\mu(g(t)) \leq \mu(t)$, we obtain

$$z'(t) + (\mu(g(t)))^{1-\beta/\alpha}Q_1(t)z^{\beta/\alpha}(g(t)) \leq 0.$$

This inequality has a positive solution, and by [5, Corollary 2.3.5], we see that (3.13) has a positive solution, which is a contradiction. The case when $L_1y < 0$ on $[t_1, \infty)$ is similar to that of Theorem 3.1 and hence is omitted. This completes the proof. □

The following corollary is immediate.

**Corollary 3.6.** Assume (2.1), (2.2), and $\alpha \geq \beta$. Suppose (1.2) is nonoscillatory. Assume there exists a function $h \in C^1(I, \mathbb{R})$ such that

$$g(t) \leq h(t) \leq t \quad \text{and} \quad h'(t) \geq 0 \quad \text{for all} \quad t \geq t_0,$$

and (2.6) or (2.11) holds with $Q$ as in Theorem 3.1. If

$$\lim_{s \to \infty} \frac{1}{g(s)} \int_{g(s)}^{t} Q_1(s)ds > \frac{1}{e} \quad \text{when} \quad \alpha = \beta,$$

$$\int_{s}^{\infty} \mu^{1-\beta/\alpha}(g(s))Q_1(s)ds = \infty \quad \text{when} \quad \alpha > \beta,$$

then every solution $y$ of (1.1) on $[t_1, \infty)$ is oscillatory.
Next, if (1.2) is oscillatory, we give the following result.

**Theorem 3.7.** Assume (2.1), (2.2), and \( \alpha \geq \beta \). Suppose (1.2) is oscillatory. If there exists a function \( h \in C^1(I, \mathbb{R}) \) such that
\[
g(t) \leq h(t) \leq t \quad \text{and} \quad h'(t) \geq 0 \quad \text{for all} \quad t \geq t_0,
\]
and (2.6) or (2.11) holds with \( Q \) as in Theorem 3.1, then every solution \( y \) of (1.1) is oscillatory or \( y' \) is oscillatory.

**Proof.** Let \( y \) be a nonoscillatory solution of (1.1) on \([t_1, \infty)\), \( t_1 \geq t_0 \). Without loss of generality, we may assume that \( y(t) > 0 \) and \( y'(t) > 0 \) for \( t \geq t_1 \). If \( L_1y > 0 \) holds on \([t_1, \infty)\), then (1.1) becomes
\[
(r_2x')''(t) + \left( \frac{p(t)}{r_1(t)} \right) x(t) \leq 0 \quad \text{for all} \quad t \geq t_2 \geq t_1,
\]
where \( x = L_1y > 0 \). By [10, Lemma 2.6], (1.2) has a positive solution, which is a contradiction. The proof of the case when \( L_1y < 0 \) on \([t_1, \infty)\) is similar to that of Theorem 3.1 and hence is omitted. This completes the proof. \( \square \)

**Example 3.8.** As an illustrative example, we consider the equation
\[
y''(t) + \frac{1}{2} y'(t) + \frac{1}{2} y(t - \frac{3\pi}{2}) = 0. \quad (3.15)
\]
Here, \( \alpha = \beta = 1 \). Let \( h(t) = t - \pi \). It is easy to check that all hypotheses of Theorem 3.7 are satisfied, and hence every solution \( y \) of (3.15) is oscillatory or \( y' \) is oscillatory. One such solution is \( y(t) = \sin t \). We note that none of the results in [7,12,15–17,20–22] is applicable to (3.15).

Finally, we can easily extend Theorem 3.7 to the equation
\[
(r_2(y')''(t) + p(t)(y'(h(t)))^\alpha + q(t)f(y(g(t))) = 0, \quad (3.16)
\]
where \( h \in C^1(I, \mathbb{R}) \) is such that
\[
g(t) \leq h(t) \leq t \quad \text{and} \quad h'(t) \geq 0 \quad \text{for all} \quad t \geq t_0.
\]

**Theorem 3.9.** Assume (2.1), (2.2), and \( \alpha \geq \beta \). Suppose the equation
\[
(r_2x')'(t) + \left( \frac{p(t)}{r_1(h(t))} \right) x(h(t)) = 0 \quad (3.17)
\]
is oscillatory. If (2.6) or (2.11) holds with
\[
Q(t) = ckq(t)(R_1(h(t), g(t)))^\beta - \left( \frac{p(t)}{r_1(h(t))} \right) \geq 0 \quad \text{for all} \quad t \geq t_1,
\]
where \( c \) is any positive constant, then every solution \( y \) of (3.16) oscillatory or \( y' \) is oscillatory.

**Proof.** Let \( y \) be a nonoscillatory solution of (3.16) on \([t_1, \infty)\), \( t_1 \geq t_0 \). Without loss of generality, we may assume that \( y(t) > 0 \) and \( y'(t) > 0 \) for \( t \geq t_1 \). As in the proof of Theorem 3.7, we obtain either \( L_1y < 0 \) or \( L_1y > 0 \) on \([t_1, \infty)\). If \( L_1y > 0 \) holds on \([t_1, \infty)\), then (3.16) becomes
\[
(r_2x')''(t) + \left( \frac{p(t)}{r_1(h(t))} \right) x(h(t)) \leq 0 \quad \text{for all} \quad t \geq t_2 \geq t_1,
\]
where \( x = L_1y > 0 \). By [10, Lemma 2.6], (3.17) has a positive solution, which is a contradiction. The proof of the case when \( L_1y < 0 \) on \([t_1, \infty)\) is similar to that of Theorem 3.1 and hence is omitted. This completes the proof. \( \square \)

We note that there are many criteria in the literature for the oscillation of second-order dynamic equations, and so, by applying these results to (1.1) and (3.16), we can obtain many oscillation results, more, for example, than those presented in [6,21,22].

The following examples are illustrative.

**Example 3.10.** Consider the equation
\[
y''(t) + y'(t - \pi) + 2y(t - \frac{3\pi}{2}) = 0. \quad (3.18)
\]
It is easy to check that all hypotheses of Theorem 3.9 are satisfied with \( \alpha = \beta = 1 \), and hence every solution \( y \) of (3.18) is oscillatory or \( y' \) is oscillatory. One oscillatory solution is \( y(t) = \sin t \). We note that none of the known results that appeared in the literature is applicable to (3.18) because of the delay that appears in the damping term. Now, the same equation without delays, namely
\[
y''(t) + y'(t) + 2y(t) = 0, \quad (3.19)
\]
In order to apply our results to (1.3), we can rewrite (1.3) in the form

\[ A \text{ and } B \text{ defined as in Theorem 3.1.} \]

If (2.6) or (2.11) holds with \( Q \) as in Theorem 3.1, then every solution \( y \) of (1.1) and a function \( H \) has a continuous and nonpositive partial derivative on

\[ \text{Function 4. Oscillation—integral averaging method} \]

The formulation of the results as a special case of the results obtained in this paper is left to the reader.

4. Oscillation—integral averaging method

In this section, we establish new oscillation results for (1.1) by using the integral averaging technique due to Philos [18]. Following Philos [18], let us introduce now the class of functions \( \mathcal{P} \) which will be used in this section. Let

\[ D_0 = \{(t, s) : t > s > t_0 \} \quad \text{and} \quad D = \{(t, s) : t \geq s > t_0 \}. \]

A function \( H \in C(D, \mathbb{R}) \) is said to belong to the class \( \mathcal{P} \) if

\[ H(t, s) > 0 \quad \text{for all} \quad (t, s) \in D_0, \quad H(t, t) = 0. \]

\( H \) has a continuous and nonpositive partial derivative on \( D_0 \) with respect to the second variable, and for a positive continuous function \( \bar{h} \),

\[ \frac{-\partial H(t, s)}{\partial s} = \bar{h}(t, s)\sqrt{H(t, s)} \quad \text{for all} \quad (t, s) \in D_0. \]

For the choice \( H(t, s) = (t - s)^n, \quad n \in \mathbb{N} \), the Philos-type conditions reduce to the Kamenev type ones.

\textbf{Theorem 4.1.} Assume (2.1), (2.2), and \( \alpha \geq \beta \). Suppose (1.2) is nonoscillatory. Assume that there exist two functions \( \rho, h \in C^1(t, \mathbb{R}) \) such that

\[ g(t) \leq h(t) \leq t, \quad h'(t) \geq 0, \quad \text{and} \quad \rho(t) > 0 \quad \text{for all} \quad t \geq t_0, \]

and a function \( H \in \mathcal{P} \) satisfying

\[ \limsup_{t \to \infty} \frac{1}{H(t, t_1)} \int_{t_1}^{t} \left[ k\rho(s)q(s)H(t, s) - \frac{p^2(t, s)}{4B(s)} \right] ds = \infty \quad \text{for all large} \quad t \geq t_1, \quad \text{where} \]

\[ P(t, s) = \bar{h}(t, s) - A(s)\sqrt{H(t, s)}, \]

with \( A \) and \( B \) defined as in Theorem 3.1. If (2.6) or (2.11) holds with \( Q \) as in Theorem 3.1, then every solution \( y \) of (1.1), or \( L_2y \), is oscillatory.

\textbf{Proof.} Let \( y \) be a nonoscillatory solution of (1.1) on \([t_1, \infty), \quad t_1 \geq t_0\). Without loss of generality, we may assume \( y(t) > 0 \) and \( y'(t) > 0 \) for \( t \geq t_1 \). Proceeding as in the proof of Theorem 3.1, we obtain (3.10), i.e.,

\[ w'(t) \leq -k\rho(t)q(t) + A(t)w(t) - B(t)w^2(t), \]

and so

\[ \int_{t_1}^{t} k\rho(s)q(s)H(t, s) ds \leq \int_{t_1}^{t} H(t, s) \left[ -w'(s) + A(s)w(s) - B(s)w^2(s) \right] ds \]

\[ = -H(t, s)w(s) \Bigg|_{s=t_1}^{s=t} + \int_{t_1}^{t} \left[ \frac{\partial H(t, s)}{\partial s}w(s) + H(t, s)(A(s)w(s) - B(s)w^2(s)) \right] ds \]
Theorem 4.2. Let the hypotheses, except (4.1), of Theorem 4.1 hold. Moreover, suppose that for every t

Proof. Let

The remainder of the proof is similar to that of [18, Theorem 2] and hence is omitted. The rest of the proof is similar to that of Theorem 3.1 and hence is omitted. □

Theorem 4.2. Let the hypotheses, except (4.1), of Theorem 4.1 hold. Moreover, suppose that for every \( t_1 > t_0 \),

\[
0 < \inf_{s \leq t_1} \left( \liminf_{t \to \infty} \frac{H(t, s)}{H(t, t_1)} \right) < \infty,
\]

\[
\limsup_{t \to \infty} \frac{1}{H(t, t_1)} \int_{t_1}^{t} \frac{c_3P^2(t, s)}{B(s)} \, ds < \infty,
\]

and there exists \( \Psi \in C(I) \) such that

\[
\int_{t_1}^{\infty} \frac{1}{c_3} \Psi^2(s)B(s) \, ds = \infty, \quad \Psi_+(s) = \max \{ \Psi(s), 0 \},
\]

and

\[
\limsup_{t \to \infty} \frac{1}{H(t, t_1)} \int_{t_1}^{t} \left[ k\rho(s)q(s)H(t, s) - \frac{P^2(t, s)}{4B(s)} \right] \, ds \geq \Psi(t_1). \tag{4.2}
\]

Then every solution \( y \) of (1.1), or \( L_2y \), is oscillatory.

Proof. Let \( y \) be a nonoscillatory solution of (1.1) on \([t_1, \infty)\) Without loss of generality, we may assume \( y(t) > 0 \) and \( y(g(t)) > 0 \) for \( t \geq t_1 \). Proceeding as in the proof of Theorem 4.1, we obtain

\[
\int_{t_1}^{t} k\rho(s)q(s)H(t, s) \, ds \leq H(t, t_1)w(t_1) + \int_{t_1}^{t} \frac{P^2(t, s)}{4B(s)} \, ds - \int_{t_1}^{t} \left[ \sqrt{H(t, s)B(s)w(s) + \frac{P(t, s)}{2\sqrt{B(s)}}} \right]^2 \, ds.
\]

Using (4.2), we obtain

\[
\Psi(t_1) \leq \limsup_{t \to \infty} \frac{1}{H(t, t_1)} \int_{t_1}^{t} \left[ k\rho(s)q(s)H(t, s) - \frac{P^2(t, s)}{4B(s)} \right] \, ds
\]

\[
\leq w(t_1) - \liminf_{t \to \infty} \frac{1}{H(t, t_1)} \int_{t_1}^{t} \left[ \sqrt{H(t, s)B(s)w(s) + \frac{P(t, s)}{2\sqrt{B(s)}}} \right]^2 \, ds,
\]

and hence

\[
\liminf_{t \to \infty} \frac{1}{H(t, t_1)} \int_{t_1}^{t} \left[ \sqrt{H(t, s)B(s)w(s) + \frac{P(t, s)}{2\sqrt{B(s)}}} \right]^2 \, ds < \infty. \tag{4.3}
\]

Define

\[
c_1(t) = \frac{1}{H(t, t_1)} \int_{t_1}^{t} H(t, s)B(s)w^2(s) \, ds
\]

and

\[
c_2(t) = \frac{1}{H(t, t_1)} \int_{t_1}^{t} \sqrt{H(t, s)}p(t, s)w(s) \, ds.
\]

It follows from (4.3) that

\[
\liminf_{t \to \infty} [c_1(t) + c_2(t)] < \infty.
\]

The remainder of the proof is similar to that of [18, Theorem 2] and hence is omitted. The rest of the proof of the case if \( y(t) > 0 \) and \( L_1y(t) < 0 \) is similar to that of the proof of Theorem 3.1 and hence is omitted. □
5. General remarks

1. The results of this paper are presented in a form that is essentially new and of a high degree of generality.
2. It would be of interest to consider Eqs. (1.1) and (3.16) and try to obtain some oscillation criteria if \( p(t) < 0 \) and \( q(t) < 0 \).
3. We note that the results in [21] are applicable to (1.1) if \( g(t) \leq t \), while our oscillation results are applicable to (1.1) if \( g(t) < t \). Thus, as is well known, it is the delay in (1.1) that can generate oscillation.
4. The results of this paper could be extended to dynamic equations on time scales of the form

\[
(r_2(r_1(y^\alpha)))^\beta(t) + p(t)(y^\alpha(y(t)))^\alpha + q(t)f(y(g(t))) = 0,
\]

where \( r_1, r_2, p, q, g \) are rd-continuous functions defined on any time scale \( T \) with \( \sup T = \infty \). The function \( f \) and the constant \( \alpha \) are as in (1.1), and when \( T = \mathbb{Z} \), i.e., in the discrete case, similar results can be obtained. The details are left to the reader.

References