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PARTIAL DIFFERENTIATION ON TIME SCALES

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ABSTRACT. In this paper a differential calculus for multivariable functions on time scales is presented. Such a calculus can be used to develop a theory of partial dynamic equations on time scales in order to unify and extend the usual partial differential and partial difference equations.

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1. INTRODUCTION

A time scale is an arbitrary nonempty closed subset of the real numbers. The calculus of time scales was initiated by B. Aulbach and S. Hilger [4, 9] in order to create a theory that can unify discrete and continuous analysis. For a treatment of the single variable calculus of time scales see [5,6,11] and the references given therein.

The present paper deals with the differential calculus for multivariable functions on time scales and intends to prepare an instrument for introducing and investigating partial dynamic equations on time scales. Note that already two papers related to this subject appeared [2, 10]. An integral calculus of multivariable functions on time scales will be given in forthcoming papers of the authors.

There are a number of differences between the calculus of one and of two variables. The calculus of functions of three or more variables differs only slightly from that of two variables. The study in this paper will be therefore limited largely to functions of two variables. Also we mainly consider partial delta derivatives. Partial nabla derivatives and combinations of partial delta and nabla derivatives can be investigated in a similar manner.

The paper is organized as follows. In Section 2 we introduce partial delta and nabla derivatives for multivariable functions on time scales and offer several new concepts related to differentiability. Section 3 deals with a geometric interpretation of delta differentiability. Section 4 contains several useful mean value theorems for derivatives. In Section 5 sufficient conditions to ensure differentiability of functions

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are provided. In Section 6 we present sufficient conditions for equality of mixed partial delta derivatives. Section 7 is devoted to the chain rule for multivariable functions on time scales, while Section 8 treats the concept of the directional derivative. Finally, in Section 9, we study implicit functions on time scales.

2. PARTIAL DERIVATIVES AND DIFFERENTIABILITY

Let $n \in \mathbb{N}$ be fixed. Further, for each $i \in \{1, \ldots, n\}$ let $\mathbb{T}_i$ denote a time scale, that is, $\mathbb{T}_i$ is a nonempty closed subset of the real numbers $\mathbb{R}$. Let us set

$$\Lambda^n = \mathbb{T}_1 \times \ldots \times \mathbb{T}_n = \{t = (t_1, \ldots, t_n) : \ t_i \in \mathbb{T}_i \text{ for all } i \in \{1, \ldots, n\}\}.$$

We call Λ^n an *n-dimensional time scale.* The set Λ^n is a complete metric space with the metric d defined by

$$d(t, s) = \left(\sum_{i=1}^{n} |t_i - s_i|^2 \right)^{1/2} \quad \text{for} \quad t, s \in \Lambda^n.$$

Consequently, according to the well-known theory of general metric spaces, we have for Λ^n the fundamental concepts such as open balls, neighbourhoods of points, open sets, closed sets, compact sets, and so on. In particular, for a given number $\delta > 0$, the *δ-neigbourhood* $U_\delta(t^0)$ of a given point $t^0 \in \Lambda^n$ is the set of all points $t \in \Lambda^n$ such that $d(t^0, t) < \delta$. By a *neigbourhood* of a point $t^0 \in \Lambda^n$ is meant an arbitrary set in Λ^n containing a δ-neighbourhood of the point t^0. Also we have for functions $f : \Lambda^n \to \mathbb{R}$ the concepts of the limit, continuity, and properties of continuous functions on general complete metric spaces.

Our main task in this section is to introduce and investigate partial derivatives for functions $f : \Lambda^n \to \mathbb{R}$. This proves to be possible due to the special structure of the metric space Λ^n.

Let σ_i and ρ_i denote, respectively, the forward and backward jump operators in $\mathbb{T}_i$. Remember that for $u \in \mathbb{T}_i$ the *forward jump operator* $\sigma_i : \mathbb{T}_i \to \mathbb{T}_i$ is defined by

$$\sigma_i(u) = \inf \{v \in \mathbb{T}_i : \ v > u\}$$

and the *backward jump operator* $\rho_i : \mathbb{T}_i \to \mathbb{T}_i$ is defined by

$$\rho_i(u) = \sup \{v \in \mathbb{T}_i : \ v < u\}.$$

In this definition we put $\sigma_i(\max \mathbb{T}_i) = \max \mathbb{T}_i$ if $\mathbb{T}_i$ has a finite maximum, and $\rho_i(\min \mathbb{T}_i) = \min \mathbb{T}_i$ if $\mathbb{T}_i$ has a finite minimum. If $\sigma_i(u) > u$, then we say that u is *right-scattered* (in $\mathbb{T}_i$), while any u with $\rho_i(u) < u$ is called *left-scattered* (in $\mathbb{T}_i$). Also, if $u < \max \mathbb{T}_i$ and $\sigma_i(u) = u$, then u is called *right-dense* (in $\mathbb{T}_i$), and if $u > \min \mathbb{T}_i$ and $\rho_i(u) = u$, then u is called *left-dense* (in $\mathbb{T}_i$). If $\mathbb{T}_i$ has a left-scattered maximum M, then we define $\mathbb{T}_i^\kappa = \mathbb{T}_i \setminus \{M\}$, otherwise $\mathbb{T}_i^\kappa = \mathbb{T}_i$. If $\mathbb{T}_i$ has a right-scattered minimum m, then we define $(\mathbb{T}_i)_\kappa = \mathbb{T}_i \setminus \{m\}$, otherwise $(\mathbb{T}_i)_\kappa = \mathbb{T}_i$.

Let $f : \Lambda^n \to \mathbb{R}$ be a function. The *partial delta derivative* of f with respect to $t_i \in \mathbb{T}_i^\kappa$ is defined as the limit

$$\lim_{\substack{s_i \to t_i \\ s_i \neq \sigma_i(t_i)}} \frac{f(t_1, \ldots, t_{i-1}, \sigma_i(t_i), t_{i+1}, \ldots, t_n) - f(t_1, \ldots, t_{i-1}, s_i, t_{i+1}, \ldots, t_n)}{\sigma_i(t_i) - s_i}$$

provided that this limit exists as a finite number, and is denoted by any of the following symbols:

$$\frac{\partial f(t_1, \ldots, t_n)}{\Delta_i t_i}, \quad \frac{\partial f(t)}{\Delta_i t_i}, \quad \frac{\partial f}{\Delta_i t_i}(t), \quad f_{t_i}^{\Delta_i}(t).$$

If f has partial derivatives $\frac{\partial f(t)}{\Delta_1 t_1}, \ldots, \frac{\partial f(t)}{\Delta_n t_n}$, then we can also consider their partial delta derivatives. These are called *second order* partial delta derivatives. We write

$$\frac{\partial^2 f(t)}{\Delta_i t_i^2} \quad \text{and} \quad \frac{\partial^2 f(t)}{\Delta_j t_j \Delta_i t_i} \quad \left[\text{or } f_{t_i t_i}^{\Delta_i \Delta_i}(t) \quad \text{and} \quad f_{t_i t_j}^{\Delta_i \Delta_j}(t)\right]$$

for the partial delta derivatives of $\frac{\partial f(t)}{\Delta_i t_i}$ with respect to t_i and with respect to t_j, respectively. Thus

$$\frac{\partial^2 f(t)}{\Delta_i t_i^2} = \frac{\partial}{\Delta_i t_i}\left(\frac{\partial f(t)}{\Delta_i t_i}\right) \quad \text{and} \quad \frac{\partial^2 f(t)}{\Delta_j t_j \Delta_i t_i} = \frac{\partial}{\Delta_j t_j}\left(\frac{\partial f(t)}{\Delta_i t_i}\right).$$

Higher order partial delta derivatives are similarly defined. The *partial nabla derivative* of f with respect to $t_i \in (\mathbb{T}_i)_\kappa$ is defined as the limit

$$\lim_{\substack{s_i \to t_i \\ s_i \neq \rho_i(t_i)}} \frac{f(t_1, \ldots, t_{i-1}, \rho_i(t_i), t_{i+1}, \ldots, t_n) - f(t_1, \ldots, t_{i-1}, s_i, t_{i+1}, \ldots, t_n)}{\rho_i(t_i) - s_i}$$

and denoted by $\frac{\partial f(t)}{\nabla_i t_i}$, provided that this limit exists as a finite number. In an obvious way we can define higher order partial nabla derivatives and also mixed derivatives obtained by combining both delta and nabla differentiations such as, for instance,

$$\frac{\partial^2 f(t)}{\Delta_i t_i \nabla_j t_j} \quad \text{or} \quad \frac{\partial^3 f(t)}{\Delta_i t_i^2 \nabla_j t_j}.$$

Definition 2.1. We say that a function $f : \Lambda^n \to \mathbb{R}$ is *completely delta differentiable* at a point $t^0 = (t_1^0, \ldots, t_n^0) \in \mathbb{T}_1^\kappa \times \ldots \times \mathbb{T}_n^\kappa$ if there exist numbers $A_1, \ldots, A_n$ independent of $t = (t_1, \ldots, t_n) \in \Lambda^n$ (but, in general, dependent on t^0) such that for all $t \in U_\delta(t^0)$,

$$f(t_1^0, \ldots, t_n^0) - f(t_1, \ldots, t_n) = \sum_{i=1}^n A_i(t_i^0 - t_i) + \sum_{i=1}^n \alpha_i(t_i^0 - t_i) \tag{2.1}$$

and, for each $j \in \{1, \ldots, n\}$ and all $t \in U_\delta(t^0)$,

$$\begin{aligned} &f(t_1^0, \ldots, t_{j-1}^0, \sigma_j(t_j^0), t_{j+1}^0, \ldots, t_n^0) - f(t_1, \ldots, t_{j-1}, t_j, t_{j+1}, \ldots, t_n) \\ &\quad = A_j\left[\sigma_j(t_j^0) - t_j\right] + \sum_{\substack{i=1 \\ i \neq j}}^n A_i(t_i^0 - t_i) + \beta_{jj}\left[\sigma_j(t_j^0) - t_j\right] + \sum_{\substack{i=1 \\ i \neq j}}^n \beta_{ij}(t_i^0 - t_i), \end{aligned} \tag{2.2}$$

where δ is a sufficiently small positive number, $U_\delta(t^0)$ is the δ-neighbourhood of t^0 in Λ^n, $\alpha_i = \alpha_i(t^0, t)$ and $\beta_{ij} = \beta_{ij}(t^0, t)$ are defined on $U_\delta(t^0)$ such that they are equal to zero at $t = t^0$ and such that

$$\lim_{t\to t^0} \alpha_i(t^0, t) = 0 \quad \text{and} \quad \lim_{t\to t^0} \beta_{ij}(t^0, t) = 0 \quad \text{for all} \quad i, j \in \{1, \ldots, n\}.$$

In the case $\mathbb{T}_1 = \ldots = \mathbb{T}_n = \mathbb{R}$, this definition coincides with the classical (total) differentiability of functions of n real variables (see, for example, [3, 12]).

In the one-variable case, Definition 2.1 becomes the following: A function $f : \mathbb{T} \to \mathbb{R}$ is called completely delta differentiable at a point $t^0 \in \mathbb{T}^\kappa$ if there exists a number A such that

$$f(t^0) - f(t) = A(t^0 - t) + \alpha(t^0 - t) \quad \text{for all} \quad t \in U_\delta(t^0) \tag{2.3}$$

and

$$f(\sigma(t^0)) - f(t) = A\left[\sigma(t^0) - t\right] + \beta\left[\sigma(t^0) - t\right] \quad \text{for all} \quad t \in U_\delta(t^0), \tag{2.4}$$

where $\alpha = \alpha(t^0, t)$ and $\beta = \beta(t^0, t)$ are equal to zero at $t = t^0$ and

$$\lim_{t\to t^0} \alpha(t^0, t) = 0 \quad \text{and} \quad \lim_{t\to t^0} \beta(t^0, t) = 0.$$

If t^0 is right-dense, then the conditions (2.3) and (2.4) coincide and are equivalent to the existence of a usual derivative of f at t^0, being equal to A. If t^0 is right-scattered and left-dense, then (2.3) and (2.4) mean, respectively, that the function f has at t^0 a usual left-sided derivative and a delta derivative and that these derivatives *coincide* and are equal to A. In this place we see a difference between the completely delta differentiability and delta differentiability, where the latter means, simply, the existence of a delta derivative. This is why we use the term "completely delta differentiable" rather than "delta differentiable". Finally, if t^0 is right-scattered and left-scattered at the same time (i.e., an "isolated" point), then the condition (2.3) disappears because both of its sides are zero independent of A and α (for sufficiently small δ, the neighbourhood $U_\delta(t^0)$ consists of the single point t^0), and (2.4) means the existence (which holds, in this case) of a delta derivative of f at t^0, being equal to A.

In the two-variable case, Definition 2.1 becomes the following: A function $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is completely delta differentiable at a point $(t^0, s^0) \in \mathbb{T}_1^\kappa \times \mathbb{T}_2^\kappa$ if there exist numbers A_1 and A_2 such that

$$f(t^0, s^0) - f(t, s) = A_1(t^0 - t) + A_2(s^0 - s) + \alpha_1(t^0 - t) + \alpha_2(s^0 - s) \tag{2.5}$$

and

$$\begin{aligned} f(\sigma_1(t^0), s^0) - f(t, s) = {}& A_1\left[\sigma_1(t^0) - t\right] + A_2(s^0 - s) \\ &+ \beta_{11}\left[\sigma_1(t^0) - t\right] + \beta_{12}(s^0 - s), \end{aligned} \tag{2.6}$$

$$f(t^0, \sigma_2(s^0)) - f(t,s) = A_1(t^0 - t) + A_2\left[\sigma_2(s^0) - s\right] + \beta_{21}(t^0 - t) + \beta_{22}\left[\sigma_2(s^0) - s\right] \tag{2.7}$$

for all $(t,s) \in U_\delta(t^0, s^0)$, where $\alpha_i = \alpha_i(t^0, s^0; t, s)$ and $\beta_{ij} = \beta_{ij}(t^0, s^0; t, s)$ are equal to zero at $(t,s) = (t^0, s^0)$ and

$$\lim_{(t,s)\to(t^0,s^0)} \alpha_i(t^0, s^0; t, s) = 0 \quad \text{and} \quad \lim_{(t,s)\to(t^0,s^0)} \beta_{ij}(t^0, s^0; t, s) = 0$$

for $i, j \in \{1, 2\}$.

Note that in the case $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$, the neighbourhood $U_\delta(t^0, s^0)$ contains the single point (t^0, s^0) for $\delta < 1$. Therefore in this case the condition (2.5) disappears while the conditions (2.6) and (2.7) hold with $\beta_{ij} = 0$ and with

$$A_1 = f(t^0 + 1, s^0) - f(t^0, s^0) = \frac{\partial f(t^0, s^0)}{\Delta_1 t}$$

and

$$A_2 = f(t^0, s^0 + 1) - f(t^0, s^0) = \frac{\partial f(t^0, s^0)}{\Delta_2 s}.$$

This shows that each function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ is completely delta differentiable at every point.

It follows from Definition 2.1 that if the function $f : \Lambda^n \to \mathbb{R}$ is completely delta differentiable at the point $t^0 \in \mathbb{T}_1^\kappa \times \ldots \times \mathbb{T}_n^\kappa$, then it is continuous at that point and has at t^0 the first order partial delta derivatives equal, respectively, to $A_1, \ldots, A_n$:

$$\frac{\partial f(t^0)}{\Delta_1 t_1} = A_1, \quad \ldots, \quad \frac{\partial f(t^0)}{\Delta_n t_n} = A_n.$$

The continuity of f at t^0 follows, in fact, from any one of (2.1) and (2.2) for some $j \in \{1, \ldots, n\}$. Indeed, (2.1) obviously yields the continuity of f at t^0. Let now (2.2) hold for some $j \in \{1, \ldots, n\}$. In the case $\sigma_j(t_j^0) = t_j^0$, (2.2) immediately gives the continuity of f at t^0. Consider the case $\sigma_j(t_j^0) > t_j^0$. Except of $f(t)$, each term in (2.2) has a limit as $t \to t^0$. Therefore $f(t)$ also has a limit as $t \to t^0$, and passing to the limit we get

$$f(t_1^0, \ldots, t_{j-1}^0, \sigma_j(t_j^0), t_{j+1}^0, \ldots, t_n^0) - \lim_{t\to t^0} f(t) = A_j\left[\sigma_j(t_j^0) - t_j^0\right].$$

Further, letting $t = t^0$ in (2.2), we obtain

$$f(t_1^0, \ldots, t_{j-1}^0, \sigma_j(t_j^0), t_{j+1}^0, \ldots, t_n^0) - f(t^0) = A_j\left[\sigma_j(t_j^0) - t_j^0\right].$$

Comparing the last two relations gives

$$\lim_{t\to t^0} f(t) = f(t^0)$$

so that the continuity of f at t^0 is shown. Next, setting in (2.2) $t_i = t_i^0$ for all $i \neq j$ and then dividing both sides by $\sigma_j(t_j^0) - t_j$ and passing to the limit as $t_j \to t_j^0$, we arrive at $\frac{\partial f(t^0)}{\Delta_j t_j} = A_j$. This also shows the uniqueness of the numbers $A_1, \ldots, A_n$

presented in (2.1), (2.2). Note also that due to the continuity of f at t^0 we get from (2.2) in the case $\sigma_j(t_j^0) > t_j^0$ the formula

$$\frac{\partial f(t^0)}{\Delta_j t_j} = \frac{f(t_1^0, \ldots, t_{j-1}^0, \sigma_j(t_j^0), t_{j+1}^0, \ldots, t_n^0) - f(t^0)}{\sigma_j(t_j^0) - t_j^0}.$$

Remark 2.2. The conditions of the form (2.6) and (2.7) can be considered as natural in order to be able to prove the formulas $A_1 = \frac{\partial f(t^0,s^0)}{\Delta_1 t}$ and $A_2 = \frac{\partial f(t^0,s^0)}{\Delta_2 s}$. If we assume f to be not dependent on t or on s, then we can see from (2.6) and (2.7) that the condition (2.3) for one-variable functions is necessary along with condition (2.4). Similarly we can see the necessity of condition (2.5) for two-variable functions by considering analogues of the conditions (2.6) and (2.7) for three variable functions.

Definition 2.3. We say that a function $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is σ_1-*completely delta differentiable* at a point $(t^0, s^0) \in \mathbb{T}_1^\kappa \times \mathbb{T}_2^\kappa$ if it is completely delta differentiable at that point in the sense of conditions (2.5) – (2.7) and moreover, along with the numbers A_1 and A_2 presented in (2.5) – (2.7) there exists also a number B independent of $(t, s) \in \mathbb{T}_1 \times \mathbb{T}_2$ (but, generally, dependent on (t^0, s^0)) such that

$$\begin{aligned} f(\sigma_1(t^0), \sigma_2(s^0)) - f(t, s) = A_1 \left[\sigma_1(t^0) - t\right] + B \left[\sigma_2(s^0) - s\right] \\ + \gamma_1 \left[\sigma_1(t^0) - t\right] + \gamma_2 \left[\sigma_2(s^0) - s\right] \end{aligned} \tag{2.8}$$

for all $(t, s) \in V^{\sigma_1}(t^0, s^0)$, where $V^{\sigma_1}(t^0, s^0)$ is a union of some neighbourhoods of the points (t^0, s^0) and $(\sigma_1(t^0), s^0)$, and the functions $\gamma_1 = \gamma_1(t^0, s^0; t, s)$ and $\gamma_2 = \gamma_2(t^0, s^0; s)$ are equal to zero for $(t, s) = (t^0, s^0)$ and

$$\lim_{(t,s)\to(t^0,s^0)} \gamma_1(t^0, s^0; t, s) = 0 \quad \text{and} \quad \lim_{s\to s^0} \gamma_2(t^0, s^0; s) = 0.$$

Note that in (2.8) the function γ_2 depends only on the variable s. Setting $t = \sigma_1(t^0)$ in (2.8) yields (here it is essential that γ_2 does not depend on t)

$$B = \frac{\partial f(\sigma_1(t^0), s^0)}{\Delta_2 s}.$$

Definition 2.4. We say that a function $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ is σ_2-*completely delta differentiable* at a point $(t^0, s^0) \in \mathbb{T}_1^\kappa \times \mathbb{T}_2^\kappa$ if it is completely delta differentiable at that point in the sense of conditions (2.5) – (2.7) and moreover, along with the numbers A_1 and A_2 presented in (2.5) – (2.7) there exists also a number D independent of $(t, s) \in \mathbb{T}_1 \times \mathbb{T}_2$ (but, generally, dependent on (t^0, s^0)) such that

$$\begin{aligned} f(\sigma_1(t^0), \sigma_2(s^0)) - f(t, s) = D \left[\sigma_1(t^0) - t\right] + A_2 \left[\sigma_2(s^0) - s\right] \\ + \eta_1 \left[\sigma_1(t^0) - t\right] + \eta_2 \left[\sigma_2(s^0) - s\right] \end{aligned} \tag{2.9}$$

for all $(t, s) \in V^{\sigma_2}(t^0, s^0)$, where $V^{\sigma_2}(t^0, s^0)$ is a union of some neighbourhoods of the points (t^0, s^0) and $(t^0, \sigma_2(s^0))$, and the functions $\eta_1 = \eta_1(t^0, s^0; t)$ and $\eta_2 =$

$\eta_2(t^0, s^0; t, s)$ are equal to zero for $(t, s) = (t^0, s^0)$ and

$$\lim_{t\to t^0} \eta_1(t^0, s^0; t) = 0 \quad \text{and} \quad \lim_{(t,s)\to(t^0,s^0)} \eta_2(t^0, s^0; t, s) = 0.$$

Note that in (2.9) the function η_1 depends only on the variable t. Setting $s = \sigma_2(s^0)$ in (2.9) yields

$$D = \frac{\partial f(t^0, \sigma_2(s^0))}{\Delta_1 t}.$$

Concluding this section, let us give two simple examples.

Example 2.5. Let $\mathbb{T} = [0, 1] \cup \{2\}$, where $[0, 1]$ is the real number interval. Define the function $f : \mathbb{T} \to \mathbb{R}$ by $f(t) = t^2$ for $t \in [0, 1]$ and $f(2) = 2$. Then $f^\Delta(t) = 2t$ for $t \in [0, 1)$ and $f^\Delta(1) = 1$. We see that $f^\Delta(1^-) = 2 \neq 1 = f^\Delta(1)$, and therefore the function f is not completely delta differentiable at the point $t = 1$. However, if we set $f(2) = 3$, then we have $f^\Delta(1) = 2$ and f becomes completely delta differentiable at $t = 1$.

Example 2.6. Let $\mathbb{T}_1 = [0, 1] \cup \{2\}$ and $\mathbb{T}_2 = [0, 1]$, where $[0, 1]$ is the real number interval. Define the function $f : \mathbb{T}_1\times\mathbb{T}_2 \to \mathbb{R}$ by $f(t, s) = t^2+s^2$ for $(t, s) \in [0, 1]\times[0, 1]$ and $f(2, s) = 3 + \sqrt{s}$ for $s \in [0, 1]$. Then we have

$$\frac{\partial f(t, s)}{\Delta_1 t} = 2t \text{ for } (t, s) \in [0, 1) \times [0, 1], \quad \frac{\partial f(1, s)}{\Delta_1 t} = 2 + \sqrt{s} - s^2 \text{ for } s \in [0, 1]$$

and

$$\frac{\partial f(t, s)}{\Delta_2 s} = 2s \text{ for } (t, s) \in [0, 1] \times [0, 1), \quad \frac{\partial f(2, s)}{\Delta_2 s} = \frac{1}{2\sqrt{s}} \text{ for } s \in (0, 1).$$

This function f is completely delta differentiable at the point $(1, 0)$, but it is not σ_1-completely delta differentiable at that point. However, if we set $f(2, s) = 3 + s$ for $s \in [0, 1]$, say, then f becomes σ_1-completely delta differentiable at the point $(1, 0)$.

3. GEOMETRIC SENSE OF DIFFERENTIABILITY

First we consider the geometric sense of complete delta differentiability in the case of single variable functions on time scales (see also [8]). Let $\mathbb{T}$ be a time scale with the forward jump operator σ and the delta differentiation operator Δ. Consider a real-valued continuous function

$$u = f(t) \quad \text{for} \quad t \in \mathbb{T}. \tag{3.1}$$

Let Γ be the "curve" represented by the function (3.1), that is, the set of points $\{(t, f(t)) : t \in \mathbb{T}\}$ in the xy-plane. Let t^0 be a fixed point in $\mathbb{T}^\kappa$. Then $P_0 = (t^0, f(t^0))$ is a point on Γ.

Definition 3.1. A line $\mathcal{L}_0$ passing through the point P_0 is called the *delta tangent line* to the curve Γ at the point P_0 if

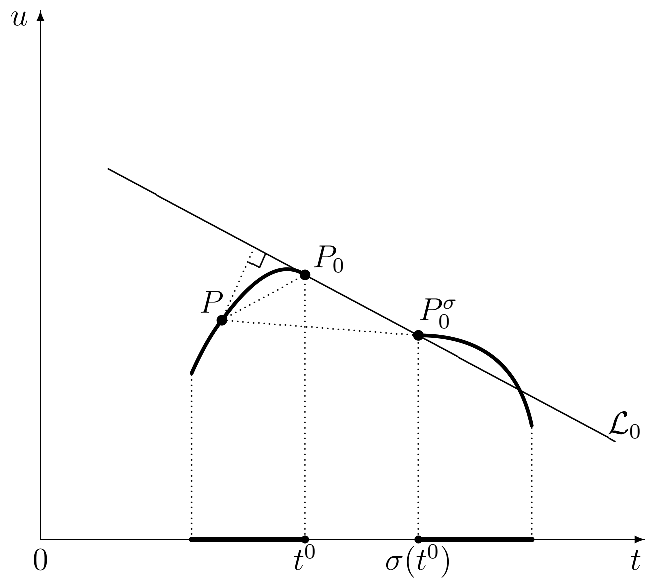
(i) $\mathcal{L}_0$ passes also through the point $P_0^\sigma = (\sigma(t^0), f(\sigma(t^0)))$;

(ii) if P_0 is not an isolated point of the curve Γ, then

$$\lim_{\substack{P\to P_0\\ P\neq P_0}} \frac{d(P,\mathcal{L}_0)}{d(P,P_0)} = 0, \tag{3.2}$$

where P is the moving point of the curve Γ, $d(P,\mathcal{L}_0)$ is the distance from the point P to the line $\mathcal{L}_0$, and $d(P,P_0^\sigma)$ is the distance from the point P to the point P_0^σ.

FIGURE 3.1. $\mathbb{T}$ consists of two separate real number intervals. Accordingly, the (time scale) curve Γ consists of two arcs of usual curves. In order the curve Γ to have a delta tangent line $\mathcal{L}_0$ at the point P_0, there must exist the usual left-sided tangent line to Γ at P_0 and, moreover, that line must pass through the point P_0^σ.



Theorem 3.2. *If the function f is completely delta differentiable at the point t^0, then the curve represented by this function has the uniquely determined delta tangent line at the point $P_0 = (t^0, f(t^0))$ specified by the equation*

$$y - f(t^0) = f^\Delta(t^0)(x - t^0), \tag{3.3}$$

where (x,y) is the current point of the line.

Proof. Let f be a completely delta differentiable function at a point $t^0 \in \mathbb{T}^\kappa$, Γ be the curve represented by this function, and $\mathcal{L}_0$ be the line described by equation (3.3). Let us show that $\mathcal{L}_0$ passes also through the point P_0^σ. Indeed, if $\sigma(t^0) = t^0$, then

$P_0^\sigma = P_0$ and the statement is true. Let now $\sigma(t^0) > t^0$. Substituting the coordinates $(\sigma(t^0), f(\sigma(t^0)))$ of the point P_0^σ into equation (3.3), we get

$$f(\sigma(t^0)) - f(t^0) = f^\Delta(t^0)\left[\sigma(t^0) - t^0\right],$$

which is obviously true by virtue of the continuity of f at t^0. Now we check condition (3.2). Assume that P_0 is not an isolated point of the curve Γ (note that if P_0 is an isolated point of Γ, then from $P \to P_0$ we get $P = P_0$ and the left-hand side of (3.2) becomes meaningless). The variable point $P \in \Gamma$ has the coordinates $(t, f(t))$. As is known from analytic geometry, the distance of the point P from the line $\mathcal{L}_0$ with equation (3.3) is expressed by the formula

$$d(P, \mathcal{L}_0) = \frac{1}{M}\left|f(t) - f(t^0) - f^\Delta(t^0)\left(t - t^0\right)\right|,$$

where

$$M = \sqrt{1 + [f^\Delta(t^0)]^2}.$$

Hence, by the differentiability condition (2.3) in which we have $A = f^\Delta(t^0)$ due to the other differentiability condition (2.4),

$$d(P, \mathcal{L}_0) = \frac{1}{M}\left|\alpha(t - t^0)\right| = \frac{1}{M}\left|\alpha\right|\left|t - t^0\right|.$$

Next,

$$d(P, P_0) = \sqrt{\left(t - t^0\right)^2 + [f(t) - f(t^0)]^2} \geq \left|t - t^0\right|.$$

Therefore

$$\frac{d(P, \mathcal{L}_0)}{d(P, P_0)} \leq \frac{1}{M}|\alpha| \to 0 \quad \text{as} \quad P \to P_0.$$

Thus we have proved that the line specified by equation (3.3) is the delta tangent line to Γ at the point P_0.

Now we must show that there are no other delta tangent lines to Γ at the point P_0 distinct from $\mathcal{L}_0$. If $P_0 \neq P_0^\sigma$, then the delta tangent line (provided it exists) is unique as it passes through the distinct points P_0 and P_0^σ. Let now $P_0 = P_0^\sigma$ so that P_0 is nonisolated. Suppose that there is a delta tangent line $\mathcal{L}$ to Γ at the point P_0 described by an equation

$$a(x - t^0) - b\left[y - f(t^0)\right] = 0 \quad \text{with} \quad a^2 + b^2 = 1. \tag{3.4}$$

Let $P = (t, f(t))$ be a variable point on Γ. Using equation (3.4), we have

$$d(P, \mathcal{L}) = \left|a\left(t - t^0\right) - b\left[f(t) - f(t^0)\right]\right|.$$

Hence, by the differentiability condition (2.3) with $A = f^\Delta(t^0)$, the latter being a result of the condition (2.4),

$$d(P, \mathcal{L}) = \left|a - b\left[f^\Delta(t^0) + \alpha\right]\right|\left|t - t^0\right|.$$

Next, by the same differentiability condition,

$$\begin{aligned} d(P,P_0) &= \sqrt{(t-t^0)^2 + [f(t)-f(t^0)]^2} \\ &= \sqrt{(t-t^0)^2 + [f^\Delta(t^0)+\alpha]^2 (t-t^0)^2} \\ &= \sqrt{1+[f^\Delta(t^0)+\alpha]^2}\,\left|t-t^0\right|. \end{aligned}$$

So we have

$$\frac{d(P,\mathcal{L})}{d(P,P_0)} = \frac{\left|a-b\left[f^\Delta(t^0)+\alpha\right]\right|}{\sqrt{1+[f^\Delta(t^0)+\alpha]^2}}.$$

Passing here to the limit as $t \to t^0$ and taking into account that the left-hand side (by the definition of delta tangent line) and α tend to zero, we obtain

$$a - bf^\Delta(t^0) = 0.$$

We now see that $b \neq 0$ for if otherwise, we would have $a = b = 0$. Hence the line $\mathcal{L}$ is described by equation (3.3). □

Remark 3.3. If P_0 is an isolated point of the curve Γ (hence $P_0 \neq P_0^\sigma$), then there exists a delta tangent line at the point P_0 to the curve Γ that coincides with the unique line through the points P_0 and P_0^σ.

Remark 3.4. If P_0 is not an isolated point of the curve Γ and if Γ has a delta tangent line at the point P_0, then the line PP_0, where $P \in \Gamma$ (and $P \neq P_0$), approaches this tangent line as $P \to P_0$. Conversely, if the line PP_0 approaches as $P \to P_0$ some line $\mathcal{L}_0$ passing through the point P_0^σ, then this limiting line is a delta tangent line at P_0. For the proof it is sufficient to note that if φ is the angle between the lines $\mathcal{L}_0$ and PP_0, then (see Figure 3.1)

$$\frac{d(P,\mathcal{L}_0)}{d(P,P_0)} = \sin\varphi.$$

Passing now to the two-variable case, let us consider the "surface" $\mathcal{S}$ represented by a real-valued continuous function $u = f(t,s)$ defined on $\mathbb{T}_1 \times \mathbb{T}_2$, that is, the set of points $\{(t,s,f(t,s)) : (t,s) \in \mathbb{T}_1 \times \mathbb{T}_2\}$ in the xyz-space. Let (t^0,s^0) be a fixed point in $\mathbb{T}_1^\kappa \times \mathbb{T}_2^\kappa$.

Definition 3.5. A plane Ω_0 passing through the point $P_0 = (t^0,s^0,f(t^0,s^0))$ is called the *delta tangent plane* to the surface $\mathcal{S}$ at the point P_0 if

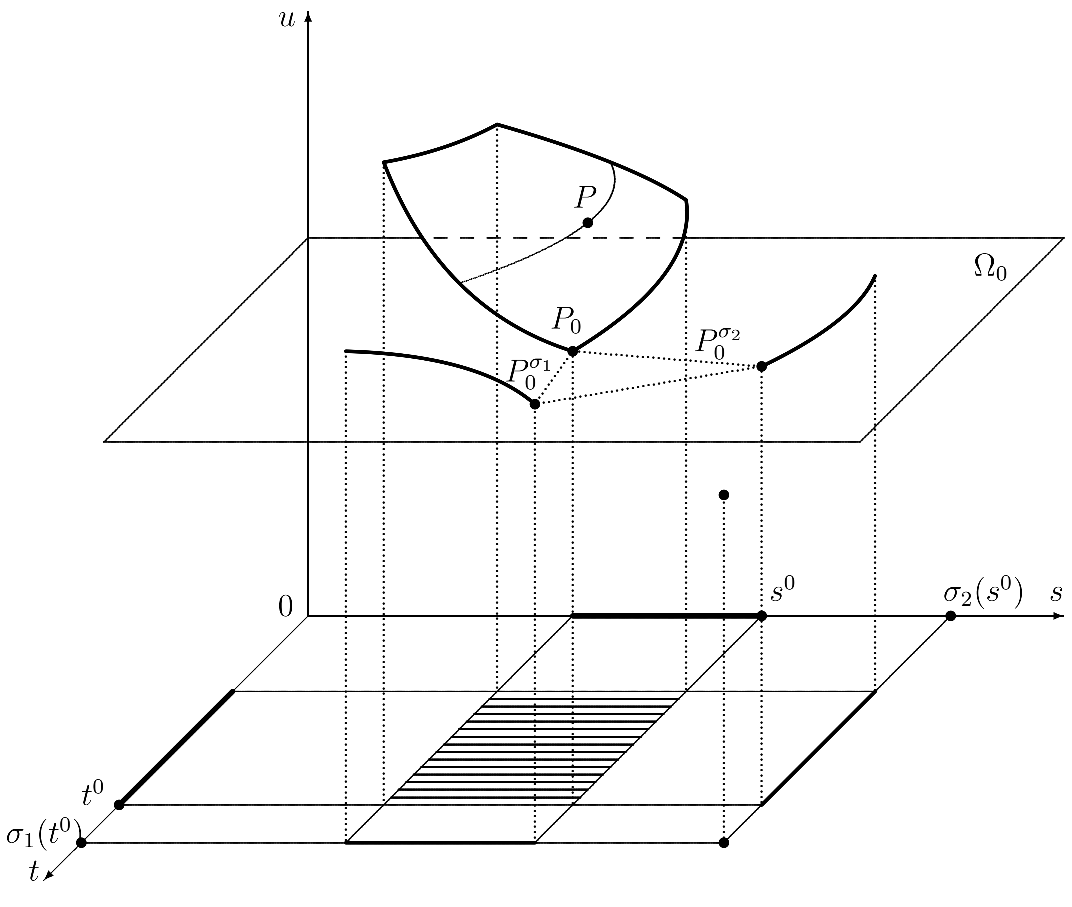
(i) Ω_0 passes also through the points $P_0^{\sigma_1} = (\sigma_1(t^0),s^0,f(\sigma_1(t^0),s^0))$ and $P_0^{\sigma_2} = (t_0,\sigma_2(s^0),f(t^0,\sigma_2(s^0)))$;

(ii) if P_0 is not an isolated point of the surface $\mathcal{S}$, then

$$\lim_{\substack{P\to P_0\\ P\neq P_0}} \frac{d(P,\Omega_0)}{d(P,P_0)} = 0, \tag{3.5}$$

where P is the moving point of the surface $\mathcal{S}$, $d(P, \Omega_0)$ is the distance from the point P to the plane Ω_0, and $d(P, P_0)$ is the distance of the point P from the point P_0.

FIGURE 3.2. Each of $\mathbb{T}_1$ and $\mathbb{T}_2$ consists of a real number interval and a separate point. Accordingly, the (time scale) surface $\mathcal{S}$ consists of one piece of a usual surface, two arcs of usual curves and one separate point. In order the surface $\mathcal{S}$ to have a delta tangent plane Ω_0 at the point P_0, there must exist the usual "left-sided" tangent plane to $\mathcal{S}$ at P_0 and, moreover, that plane must pass through both points $P_0^{\sigma_1}$ and $P_0^{\sigma_2}$.



Theorem 3.6. *If the function f is completely delta differentiable at the point (t^0, s^0), then the surface represented by this function has the uniquely determined delta tangent plane at the point $P_0 = (t^0, s^0, f(t^0, s^0))$ specified by the equation*

$$z - f(t^0, s^0) = \frac{\partial f(t^0, s^0)}{\Delta_1 t}(x - t^0) + \frac{\partial f(t^0, s^0)}{\Delta_2 s}(y - s^0), \tag{3.6}$$

where (x, y, z) is the current point of the plane.

Proof. Let f be a completely delta differentiable function at a point $(t^0, s^0) \in \mathbb{T}_1^\kappa \times \mathbb{T}_2^\kappa$, $\mathcal{S}$ be the surface represented by this function, and Ω_0 be the plane described by equation (3.6). Let us show that Ω_0 passes also through the point $P_0^{\sigma_1}$. Indeed, if $\sigma_1(t^0) = t^0$, then $P_0^{\sigma_1} = P_0$ and the statement is true. Let now $\sigma_1(t^0) > t^0$. Substituting the coordinates $(\sigma_1(t^0), s^0, f(\sigma_1(t^0), s^0))$ of the point $P_0^{\sigma_1}$ into equation (3.6), we get

$$f(\sigma_1(t^0), s^0) - f(t^0, s^0) = \frac{\partial f(t^0, s^0)}{\Delta_1 t}\left[\sigma_1(t^0) - t^0\right],$$

which is obviously true due to the continuity of f at (t^0, s^0). Similarly we can see that Ω_0 passes also through the point $P_0^{\sigma_2}$. Now we check (3.5). Assume that P_0 is not an isolated point of the surface $\mathcal{S}$. The variable point $P \in \mathcal{S}$ has the coordinates $(t, s, f(t, s))$. As is known from analytic geometry, the distance between P and Ω_0 with equation (3.6) is expressed by the formula

$$d(P, \Omega_0) = \frac{1}{N}\left| f(t, s) - f(t^0, s^0) - \frac{\partial f(t^0, s^0)}{\Delta_1 t}(t - t^0) - \frac{\partial f(t^0, s^0)}{\Delta_2 s}(s - s^0)\right|,$$

where

$$N = \sqrt{1 + \left[\frac{\partial f(t^0, s^0)}{\Delta_1 t}\right]^2 + \left[\frac{\partial f(t^0, s^0)}{\Delta_2 s}\right]^2}.$$

Hence, by the differentiability condition (2.5) in which $A_1 = \frac{\partial f(t^0,s^0)}{\Delta_1 t}$ and $A_2 = \frac{\partial f(t^0,s^0)}{\Delta_2 s}$ due to the other differentiability conditions (2.6) and (2.7),

$$\begin{aligned} d(P, \Omega_0) &= \frac{1}{N}\left|\alpha_1(t - t^0) + \alpha_2(s - s^0)\right| \\ &\le \frac{1}{N}\sqrt{\alpha_1^2 + \alpha_2^2}\sqrt{(t - t^0)^2 + (s - s^0)^2}. \end{aligned}$$

Next,

$$\begin{aligned} d(P, P_0) &= \sqrt{(t - t^0)^2 + (s - s^0)^2 + [f(t, s) - f(t^0, s^0)]^2} \\ &\ge \sqrt{(t - t^0)^2 + (s - s^0)^2}. \end{aligned}$$

Therefore

$$\frac{d(P, \Omega_0)}{d(P, P_0)} \le \frac{1}{N}\sqrt{\alpha_1^2 + \alpha_2^2} \to 0 \quad \text{as} \quad P \to P_0.$$

Thus we have proved that the plane specified by equation (3.6) is the delta tangent plane to $\mathcal{S}$ at the point P_0.

Now we must show that there are no other delta tangent planes to $\mathcal{S}$ at the point P_0 distinct from Ω_0. If $\sigma_1(t^0) > t^0$ and $\sigma_2(s^0) > s^0$ at the same time, the points P_0, $P_0^{\sigma_1}$, and $P_0^{\sigma_2}$ are pairwise distinct. In this case the delta tangent plane (provided it exists) is unique as it passes through the three distinct points P_0, $P_0^{\sigma_1}$, and $P_0^{\sigma_2}$. Further we have to consider the remaining possible cases. Suppose that there is a delta tangent plane Ω to $\mathcal{S}$ at the point P_0 described by an equation

$$a(x - t^0) + b(y - s^0) - c\left[z - f(t^0, s^0)\right] = 0 \quad \text{with} \quad a^2 + b^2 + c^2 = 1. \tag{3.7}$$

Let $P = (t, s, f(t,s))$ be a variable point on $\mathcal{S}$. Using equation (3.7), we have

$$d(P,\Omega) = \left|a\left(t - t^0\right) + b(s - s^0) - c\left[f(t,s) - f(t^0,s^0)\right]\right|.$$

Hence, by the differentiability condition (2.6) in which $A_1 = \frac{\partial f(t^0,s^0)}{\Delta_1 t}$ and $A_2 = \frac{\partial f(t^0,s^0)}{\Delta_2 s}$, the latter being a result of the conditions (2.6) and (2.7),

$$d(P,\Omega) = \left|\left[a - c(A_1+\alpha_1)\right](t-t^0) + \left[b - c(A_2+\alpha_2)\right](s-s^0)\right|.$$

Next, by the same differentiability condition,

$$\begin{aligned} d(P,P_0) &= \sqrt{(t-t^0)^2 + (s-s^0)^2 + \left[f(t,s) - f(t^0,s^0)\right]^2} \\ &= \sqrt{(t-t^0)^2 + (s-s^0)^2 + \left[(A_1+\alpha_1)(t-t^0) + (A_2+\alpha_2)(s-s^0)\right]^2}. \end{aligned}$$

So we have

$$\frac{d(P,\Omega)}{d(P,P_0)} = \frac{\left|\left[a - c(A_1+\alpha_1)\right](t-t^0) + \left[b - c(A_2+\alpha_2)\right](s-s^0)\right|}{\sqrt{(t-t^0)^2 + (s-s^0)^2 + \left[(A_1+\alpha_1)(t-t^0) + (A_2+\alpha_2)(s-s^0)\right]^2}}. \tag{3.8}$$

Now we discuss the remaining possible cases.

(a) Let $\sigma_1(t^0) = t^0$ and $\sigma_2(s^0) = s^0$. By our assumption, the left-hand side of (3.8) tends to zero as $(t,s) \to (t^0,s^0)$, that is, as $P \to P_0$. On putting $s = s^0$ in (3.8), dividing in the right-hand side the numerator and denominator by $|t - t^0|$, and passing then to the limit as $t \to t^0$, we get

$$a - cA_1 = 0, \quad \text{that is,} \quad a - c\frac{\partial f(t^0,s^0)}{\Delta_1 t} = 0. \tag{3.9}$$

Similarly, putting $t = t^0$ in (3.8), cancelling $|s - s^0|$, and passing then to the limit as $s \to s^0$, we obtain

$$b - cA_2 = 0, \quad \text{that is,} \quad b - c\frac{\partial f(t^0,s^0)}{\Delta_2 s} = 0. \tag{3.10}$$

We see that $c \neq 0$ because, if otherwise, we would have $a = b = c = 0$. Hence the plane Ω is described by equation (3.6).

(b) Let now $\sigma_1(t^0) = t^0$ and $\sigma_2(s^0) > s^0$. In this case we obtain (3.9) from (3.8) as in case (a). However, now we can, in general, not get (3.10) from (3.8) as in case (a) because the point s^0 may be isolated in $\mathbb{T}_2$, and therefore for all points (t,s) in a sufficiently small neighbourhood of (t^0,s^0) we may have $s = s^0$ (and hence we cannot divide by $|s - s^0|$ in order to pass then to the limit as $s \to s^0$). We proceed as follows. Since by definition of the delta tangent plane, the plane Ω must also pass through the point $P_0^{\sigma_2}$, we also have the equation

$$a(x - t^0) + b\left[y - \sigma_2(s^0)\right] - c\left[z - f(t^0, \sigma_2(s^0))\right] = 0$$

for the same plane Ω. Using this equation, we obtain

$$d(P,\Omega) = \left|a(t - t^0) + b\left[s - \sigma_2(s^0)\right] - c\left[f(t,s) - f(t^0,\sigma_2(s^0))\right]\right|.$$

Hence, by the differentiability condition (2.7), we get

$$d(P,\Omega) = \left|[a - c(A_1 + \beta_{21})]\,(t - t^0) + [b - c(A_2 + \beta_{22})]\left[\sigma_2(s^0) - s\right]\right|.$$

Therefore

$$\frac{d(P,\Omega)}{d(P,P_0)} = \frac{|[a - c(A_1 + \beta_{21})]\,(t - t^0) + [b - c(A_2 + \beta_{22})]\,[\sigma_2(s^0) - s]|}{\sqrt{(t - t^0)^2 + (s - s^0)^2 + [(A_1 + \alpha_1)(t - t^0) + (A_2 + \alpha_2)(s - s^0)]^2}}.$$

Setting here $s = s^0$ and taking into account (3.9) proved in the considered case, we obtain

$$\left.\frac{d(P,\Omega)}{d(P,P_0)}\right|_{s=s^0} = \frac{|-c\beta_{21}(t - t^0) + [b - c(A_2 + \beta_{22})]\,[\sigma_2(s^0) - s^0]|}{|t - t^0|\sqrt{1 + (A_1 + \alpha_1)^2}}.$$

Passing here to the limit as $t \to t^0$, we see that

$$b - cA_2 = 0, \quad \text{that is,} \quad b - c\frac{\partial f(t^0, s^0)}{\Delta_2 s} = 0$$

because, if otherwise, the right-hand side would tend to infinity. Further, the proof is completed as in case (a).

(c) Finally, suppose $\sigma_1(t^0) > t^0$ and $\sigma_2(s^0) = s^0$. In this case the proof is analogous to that in case (b) and uses the equation (Ω also passes through $P_0^{\sigma_1}$)

$$a\left[x - \sigma_1(t^0)\right] + b(y - s^0) - c\left[z - f(\sigma_1(t^0), s^0)\right] = 0$$

and the differentiability condition (2.6). $\square$

Remark 3.7. If $P_0^{\sigma_1} \neq P_0$ and $P_0^{\sigma_2} \neq P_0$ (hence also $P_0^{\sigma_1} \neq P_0^{\sigma_2}$) at the same time and if there is a delta tangent plane at the point P_0 to the surface $\mathcal{S}$, then it coincides with the unique plane through the three points P_0, $P_0^{\sigma_1}$, and $P_0^{\sigma_2}$.

4. MEAN VALUE THEOREMS

First we present mean value results in the single variable case (see [7] or [6, Chapter 1]) that will be used to obtain mean value results in the multivariable case. They will also be used below in Sections 5, 6, and 9.

Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$ with $a < b$. Further, let $f : \mathbb{T} \to \mathbb{R}$ be a function. Below all intervals are time scales intervals.

Theorem 4.1. *Suppose that f is continuous on $[a,b]$ and has a delta derivative at each point of $[a,b)$. If $f(a) = f(b)$, then there exist points $\xi, \xi' \in [a,b)$ such that $f^{\Delta}(\xi') \leq 0 \leq f^{\Delta}(\xi)$.*

Proof. Since the function f is continuous on the compact set $[a,b]$, f assumes its minimum m and its maximum M. Therefore there exist $\xi, \xi' \in [a,b]$ such that $m = f(\xi)$ and $M = f(\xi')$. Since $f(a) = f(b)$, we may assume that $\xi, \xi' \in [a,b)$. It is not difficult to see that $f^{\Delta}(\xi') \leq 0$ and $f^{\Delta}(\xi) \geq 0$. $\square$

Theorem 4.2 (Mean Value Theorem)**.** *Suppose that f is a continuous function on $[a, b]$ and has a delta derivative at each point of $[a, b)$. Then there exist $\xi, \xi' \in [a, b)$ such that*

$$f^{\Delta}(\xi')(b-a) \leq f(b) - f(a) \leq f^{\Delta}(\xi)(b-a). \tag{4.1}$$

Proof. Consider the function φ defined on $[a, b]$ by

$$\varphi(t) = f(t) - f(a) - \frac{f(b) - f(a)}{b-a}(t-a).$$

Clearly φ is continuous on $[a, b]$ and has a delta derivative at each point of $[a, b)$. Also $\varphi(a) = \varphi(b) = 0$. Therefore by Theorem 4.1 there exist $\xi, \xi' \in [a, b)$ such that

$$\varphi^{\Delta}(\xi') \leq 0 \leq \varphi^{\Delta}(\xi).$$

Hence, taking into account that

$$\varphi^{\Delta}(t) = f^{\Delta}(t) - \frac{f(b) - f(a)}{b-a},$$

we arrive at the statement of the theorem. □

Corollary 4.3. *Let f be a continuous function on $[a, b]$ that has a delta derivative at each point of $[a, b)$. If $f^{\Delta}(t) = 0$ for all $t \in [a, b)$, then f is a constant function on $[a, b]$.*

Corollary 4.4. *Let f be a continuous function on $[a, b]$ that has a delta derivative at each point of $[a, b)$. Then f is increasing, decreasing, nondecreasing, and nonincreasing on $[a, b]$ if $f^{\Delta}(t) > 0$, $f^{\Delta}(t) < 0$, $f^{\Delta}(t) \geq 0$, and $f^{\Delta}(t) \leq 0$ for all $t \in [a, b)$, respectively.*

Above in Theorem 4.2 we assumed that $a < b$. We can remove that restriction as follows. If $a = b$, then we may assume that (4.1) is true as all three parts of the inequalities (4.1) become zero in this case independent of ξ and ξ'. Further, we always can rewrite (4.1), multiplying it by -1, in the form

$$f^{\Delta}(\xi)(a-b) \leq f(a) - f(b) \leq f^{\Delta}(\xi')(a-b).$$

These considerations allow us to state the following result.

Theorem 4.5. *Let a and b be two arbitrary points in $\mathbb{T}$ and let us set $\alpha = \min\{a, b\}$ and $\beta = \max\{a, b\}$. Let, further, f be a continuous function on $[\alpha, \beta]$ that has a delta derivative at each point of $[\alpha, \beta)$. Then there exist $\xi, \xi' \in [\alpha, \beta)$ such that*

$$f^{\Delta}(\xi')(a-b) \leq f(a) - f(b) \leq f^{\Delta}(\xi)(a-b).$$

Passing now to the two-variable case, we consider functions $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ of the variables $(t, s) \in \mathbb{T}_1 \times \mathbb{T}_2$.

Theorem 4.6 (Mean Value Theorem)**.** *Let* (a_1, a_2) *and* (b_1, b_2) *be any two points in* $\mathbb{T}_1 \times \mathbb{T}_2$ *and let us set*

$$\alpha_i = \min\{a_i, b_i\} \quad \textit{and} \quad \beta_i = \max\{a_i, b_i\} \quad \textit{for} \quad i \in \{1, 2\}.$$

Let, further, f *be a continuous function on* $[\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \subset \mathbb{T}_1 \times \mathbb{T}_2$ *that has first order partial delta derivatives* $\frac{\partial f(t,a_2)}{\Delta_1 t}$ *for each* $t \in [\alpha_1, \beta_1)$ *and* $\frac{\partial f(b_1,s)}{\Delta_2 s}$ *for each* $s \in [\alpha_2, \beta_2)$*. Then there exist* $\xi, \xi' \in [\alpha_1, \beta_1)$ *and* $\eta, \eta' \in [\alpha_2, \beta_2)$ *such that*

$$\begin{aligned} \frac{\partial f(\xi', a_2)}{\Delta_1 t}(a_1 - b_1) + \frac{\partial f(b_1, \eta')}{\Delta_2 s}(a_2 - b_2) &\le f(a_1, a_2) - f(b_1, b_2) \\ &\le \frac{\partial f(\xi, a_2)}{\Delta_1 t}(a_1 - b_1) + \frac{\partial f(b_1, \eta)}{\Delta_2 s}(a_2 - b_2). \end{aligned} \tag{4.2}$$

Also, if f *has first order partial derivatives* $\frac{\partial f(t,b_2)}{\Delta_1 t}$ *for each* $t \in [\alpha_1, \beta_1)$ *and* $\frac{\partial f(a_1,s)}{\Delta_2 s}$ *for each* $s \in [\alpha_2, \beta_2)$*, then there exist* $\tau, \tau' \in [\alpha_1, \beta_1)$ *and* $\theta, \theta' \in [\alpha_2, \beta_2)$ *such that*

$$\begin{aligned} \frac{\partial f(\tau', b_2)}{\Delta_1 t}(a_1 - b_1) + \frac{\partial f(b_1, \theta')}{\Delta_2 s}(a_2 - b_2) &\le f(a_1, a_2) - f(b_1, b_2) \\ &\le \frac{\partial f(\tau, b_2)}{\Delta_1 t}(a_1 - b_1) + \frac{\partial f(a_1, \theta)}{\Delta_2 s}(a_2 - b_2). \end{aligned} \tag{4.3}$$

Proof. To prove (4.2) we consider the difference

$$f(a_1, a_2) - f(b_1, b_2) = [f(a_1, a_2) - f(b_1, a_2)] + [f(b_1, a_2) - f(b_1, b_2)]\,. \tag{4.4}$$

By Theorem 4.5 there exist $\xi, \xi' \in [\alpha_1, \beta_1)$ and $\eta, \eta' \in [\alpha_2, \beta_2)$ such that

$$\frac{\partial f(\xi', a_2)}{\Delta_1 t}(a_1 - b_1) \le f(a_1, a_2) - f(b_1, a_2) \le \frac{\partial f(\xi, a_2)}{\Delta_1 t}(a_1 - b_1)$$

and

$$\frac{\partial f(b_1, \eta')}{\Delta_2 s}(a_2 - b_2) \le f(b_1, a_2) - f(b_1, b_2) \le \frac{\partial f(b_1, \eta)}{\Delta_2 s}(a_2 - b_2).$$

Adding these inequalities side by side and taking into account (4.4), we obtain (4.2). Using the relation

$$f(a_1, a_2) - f(b_1, b_2) = [f(a_1, b_2) - f(b_1, b_2)] + [f(a_1, a_2) - f(a_1, b_2)]\,,$$

the inequalities (4.3) can be proved similarly. ☐

From Theorem 4.6 we get the following corollary.

Corollary 4.7. *Let* f *be a continuous function on* $\mathbb{T}_1 \times \mathbb{T}_2$ *that has first order partial derivatives* $\frac{\partial f(t,s)}{\Delta_1 t}$ *and* $\frac{\partial f(t,s)}{\Delta_2 s}$ *for* $(t, s) \in \mathbb{T}_1^\kappa \times \mathbb{T}_2$ *and* $(t, s) \in \mathbb{T}_1 \times \mathbb{T}_2^\kappa$*, respectively. If these derivatives are identically zero, then* f *is a constant function on* $\mathbb{T}_1 \times \mathbb{T}_2$*.*

Note that the statement of Corollary 4.7 can be obtained also from Corollary 4.3.

5. SUFFICIENT CONDITIONS FOR DIFFERENTIABILITY

The existence of all partial delta derivatives (even in the single variable time scales case) is in general not sufficient for the completely delta differentiability. The next theorem gives sufficient conditions for the completely delta differentiability.

Theorem 5.1. *Let a function $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be continuous and have first order partial delta derivatives $\frac{\partial f(t,s)}{\Delta_1 t}$ and $\frac{\partial f(t,s)}{\Delta_2 s}$ in some δ-neighbourhood $U_\delta(t^0, s^0)$ of the point $(t^0, s^0) \in \mathbb{T}_1^\kappa \times \mathbb{T}_2^\kappa$. If these derivatives are continuous at the point (t^0, s^0), then f is completely delta differentiable at (t^0, s^0).*

Proof. For better clearness of the proof we first consider the single variable case. So, let $f : \mathbb{T} \to \mathbb{R}$ be a function that has a delta derivative $f^\Delta(t)$ in some δ-neighbourhood $U_\delta(t^0)$ of the point $t^0 \in \mathbb{T}^\kappa$ (note that, in contrast to the multivariable case, in the single variable case existence of the derivative at a point implies continuity of the function at that point). The relation (2.4) with $A = f^\Delta(t^0)$ follows immediately from the definition of the delta derivative

$$f(\sigma(t^0)) - f(t) = f^\Delta(t^0)\left[\sigma(t^0) - t\right] + \beta\left[\sigma(t^0) - t\right], \tag{5.1}$$

where $\beta = \beta(t^0, t)$ and $\beta \to 0$ as $t \to t^0$. In order to prove (2.3), we consider all possible cases separately.

(i) If the point t^0 is isolated in $\mathbb{T}$, then (2.3) is satisfied independent of A and α, since in this case $U_\delta(t^0)$ consists of the single point t^0 for sufficiently small $\delta > 0$.

(ii) Let t^0 be right-dense. Regardless whether t^0 is left-scattered or left-dense, we have in this case $\sigma(t^0) = t^0$ and (5.1) coincides with (2.3).

(iii) Finally, let t^0 be left-dense and right-scattered. Then for sufficiently small $\delta > 0$, any point $t \in U_\delta(t^0) \setminus \{t^0\}$ must satisfy $t < t^0$. Applying Theorem 4.5, we obtain

$$f^\Delta(\xi')(t^0 - t) \le f(t^0) - f(t) \le f^\Delta(\xi)(t^0 - t),$$

where $\xi, \xi' \in [t, t^0)$. Since $\xi \to t^0$ and $\xi' \to t^0$ as $t \to t^0$, we get from the latter inequalities by the assumed continuity of the delta derivative

$$\lim_{t \to t^0} \frac{f(t^0) - f(t)}{t^0 - t} = f^\Delta(t^0).$$

Therefore

$$\frac{f(t^0) - f(t)}{t^0 - t} = f^\Delta(t^0) + \alpha,$$

where $\alpha = \alpha(t^0, t)$ and $\alpha \to 0$ as $t \to t^0$. Consequently, in the considered case we obtain (2.3) with $A = f^\Delta(t^0)$ as well.

Now we consider the two-variable case as it is stated in the theorem. To prove (2.5), we take the difference

$$f(t^0,s^0)-f(t,s)=\left[f(t^0,s^0)-f(t,s^0)\right]+\left[f(t,s^0)-f(t,s)\right]. \tag{5.2}$$

By the one-variable case considered above, we have

$$f(t^0,s^0)-f(t,s^0)=\frac{\partial f(t^0,s^0)}{\Delta_1 t}(t^0-t)+\alpha_1(t^0-t)\quad\text{for}\quad (t,s^0)\in U_\delta(t^0,s^0), \tag{5.3}$$

where $\alpha_1=\alpha_1(t^0,s^0;t)$ and $\alpha_1\to 0$ as $t\to t^0$. Further, applying the one-variable mean value result, Theorem 4.5, for fixed t and variable s, we have

$$\frac{\partial f(t,\xi')}{\Delta_2 s}(s^0-s)\le f(t,s^0)-f(t,s)\le\frac{\partial f(t,\xi)}{\Delta_2 s}(s^0-s), \tag{5.4}$$

where $\xi,\xi'\in[\alpha,\beta)$ and $\alpha=\min\{s^0,s\}$, $\beta=\max\{s^0,s\}$. Since $\xi\to s^0$ and $\xi'\to s^0$ as $s\to s^0$, by the assumed continuity of the partial delta derivatives at (t^0,s^0) we have

$$\lim_{(t,s)\to(t^0,s^0)}\frac{\partial f(t,\xi')}{\Delta_2 s}=\lim_{(t,s)\to(t^0,s^0)}\frac{\partial f(t,\xi)}{\Delta_2 s}=\frac{\partial f(t^0,s^0)}{\Delta_2 s}.$$

Therefore from (5.4) we obtain

$$f(t,s^0)-f(t,s)=\frac{\partial f(t^0,s^0)}{\Delta_2 s}(s^0-s)+\alpha_2(s^0-s), \tag{5.5}$$

where $\alpha_2=\alpha_2(t^0,s^0;t,s)$ and $\alpha_2\to 0$ as $(t,s)\to(t^0,s^0)$. Substituting (5.3) and (5.5) in (5.2), we get a relation of the form (2.5) with $A_1=\frac{\partial f(t^0,s^0)}{\Delta_1 t}$ and $A_2=\frac{\partial f(t^0,s^0)}{\Delta_2 s}$. To prove (2.6) we take the difference

$$f(\sigma_1(t^0),s^0)-f(t,s)=\left[f(\sigma_1(t^0),s^0)-f(t,s^0)\right]+\left[f(t,s^0)-f(t,s)\right]. \tag{5.6}$$

By the definition of the partial delta derivative we have

$$f(\sigma_1(t^0),s^0)-f(t,s^0)=\frac{\partial f(t^0,s^0)}{\Delta_1 t}\left[\sigma_1(t^0)-t\right]+\beta_{11}\left[\sigma_1(t^0)-t\right], \tag{5.7}$$

where $\beta_{11}=\beta_{11}(t^0,s^0;t)$ and $\beta_{11}\to 0$ as $t\to t^0$. Now substituting (5.7) and (5.5) into (5.6), we obtain a relation of the form (2.6) with $A_1=\frac{\partial f(t^0,s^0)}{\Delta_1 t}$ and $A_2=\frac{\partial f(t^0,s^0)}{\Delta_2 s}$. The equality (2.7) can be proved similarly by considering the difference

$$f(t^0,\sigma_2(s^0))-f(t,s)=\left[f(t^0,s)-f(t,s)\right]+\left[f(t^0,\sigma_2(s^0))-f(t^0,s)\right].$$

The proof is complete. □

The next theorem presents sufficient conditions for σ_1-completely delta differentiability.

Theorem 5.2. *Let $f:\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{R}$ be a continuous function that has the partial derivatives $\frac{\partial f(t,s)}{\Delta_1 t}$ and $\frac{\partial f(t,s)}{\Delta_2 s}$ in a union of some neighbourhoods of the points $(t^0,s^0)\in\mathbb{T}_1^\kappa\times\mathbb{T}_2^\kappa$ and $(\sigma_1(t^0),s^0)$. If these derivatives are continuous at the point (t^0,s^0) and, moreover, $\frac{\partial f(\sigma_1(t^0),s)}{\Delta_2 s}$ is continuous at $s=s^0$, then f is σ_1-completely delta differentiable at (t^0,s^0).*

Proof. It follows from Theorem 5.1 that f is completely delta differentiable at (t^0, s^0). It remains to prove (2.8). To this end consider the difference

$$f(\sigma_1(t^0), \sigma_2(s^0)) - f(t, s) = \left[f(\sigma_1(t^0), s) - f(t, s)\right] + \left[f(\sigma_1(t^0), \sigma_2(s^0)) - f(\sigma_1(t^0), s)\right]. \tag{5.8}$$

Applying the one-variable mean value theorem, Theorem 4.5, for fixed s and variable t, we have

$$\frac{\partial f(\xi', s)}{\Delta_1 t}\left[\sigma_1(t^0) - t\right] \leq f(\sigma_1(t^0), s) - f(t, s) \leq \frac{\partial f(\xi, s)}{\Delta_1 t}\left[\sigma_1(t^0) - t\right],$$

where $\xi, \xi' \in [\alpha, \beta)$ and $\alpha = \min\{\sigma_1(t^0), t\}$, $\beta = \max\{\sigma_1(t^0), t\}$. Hence, since $\xi \to t^0$ and $\xi' \to t^0$ as $t \to t^0$, by the assumed continuity of the partial delta derivative $\frac{\partial f(t,s)}{\Delta_1 t}$ at (t^0, s^0), we obtain

$$f(\sigma_1(t^0), s) - f(t, s) = \frac{\partial f(t^0, s^0)}{\Delta_1 t}\left[\sigma_1(t^0) - t\right] + \gamma_1\left[\sigma_1(t^0) - t\right], \tag{5.9}$$

where $\gamma_1 = \gamma_1(t^0, s^0; t, s)$ and $\gamma_1 \to 0$ as $(t, s) \to (t^0, s^0)$. Further, by the definition of the partial delta derivative, we have

$$f(\sigma_1(t^0), \sigma_2(s^0)) - f(\sigma_1(t^0), s) = \frac{\partial f(\sigma_1(t^0), s)}{\Delta_2 s}\left[\sigma_2(s^0) - s\right] + \gamma_2\left[\sigma_2(s^0) - s\right], \tag{5.10}$$

where $\gamma_2 = \gamma_2(t^0, s^0; s)$ and $\gamma_2 \to 0$ as $s \to s^0$. Now substituting (5.9) and (5.10) into (5.8) and taking into account the continuity of $\frac{\partial f(\sigma_1(t^0),s)}{\Delta_2 s}$ at $s = s^0$, we get a relation of the form (2.8) with $A_1 = \frac{\partial f(t^0,s^0)}{\Delta_1 t}$ and $B = \frac{\partial f(\sigma_1(t^0),s^0)}{\Delta_2 s}$. $\square$

The next theorem can be proved similarly as Theorem 5.2 by considering the difference

$$f(\sigma_1(t^0), \sigma_2(s^0)) - f(t, s) = \left[f(\sigma_1(t^0), \sigma_2(s^0)) - f(t, \sigma_2(s^0))\right] + \left[f(t, \sigma_2(s^0)) - f(t, s)\right].$$

Theorem 5.3. *Let $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be a continuous function that has the partial derivatives $\frac{\partial f(t,s)}{\Delta_1 t}$ and $\frac{\partial f(t,s)}{\Delta_2 s}$ in a union of some neighbourhoods of the points $(t^0, s^0) \in \mathbb{T}_1^\kappa \times \mathbb{T}_2^\kappa$ and $(t^0, \sigma_2(s^0))$. If these derivatives are continuous at the point (t^0, s^0), and, moreover, $\frac{\partial f(t,\sigma_2(s^0))}{\Delta_1 t}$ is continuous at $t = t^0$, then f is σ_2-completely delta differentiable at (t^0, s^0).*

6. EQUALITY OF MIXED PARTIAL DERIVATIVES

The next theorem gives us a sufficient condition for the independence of mixed partial delta derivatives of the order of differentiation.

Theorem 6.1. *Let a function $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ have the mixed partial delta derivatives $\frac{\partial^2 f(t,s)}{\Delta_1 t \Delta_2 s}$ and $\frac{\partial^2 f(t,s)}{\Delta_2 s \Delta_1 t}$ in some neighbourhood of the point $(t^0, s^0) \in \mathbb{T}_1^\kappa \times \mathbb{T}_2^\kappa$. If these derivatives are continuous at the point (t^0, s^0), then*

$$\frac{\partial^2 f(t^0, s^0)}{\Delta_1 t \Delta_2 s} = \frac{\partial^2 f(t^0, s^0)}{\Delta_2 s \Delta_1 t}.$$

Proof. Consider the function

$$\Phi(t,s) = f(\sigma_1(t^0), \sigma_2(s^0)) - f(\sigma_1(t^0), s) - f(t, \sigma_2(s^0)) + f(t,s),$$

where $(t,s) \in \mathbb{T}_1 \times \mathbb{T}_2$. Setting

$$\varphi(t) = f(t, \sigma_2(s^0)) - f(t,s),$$

we can write

$$\Phi(t,s) = \varphi(\sigma_1(t^0)) - \varphi(t).$$

Therefore, applying the mean value theorem, Theorem 4.5, we have

$$\varphi^{\Delta_1}(\xi') \left[\sigma_1(t^0) - t\right] \le \Phi(t,s) \le \varphi^{\Delta_1}(\xi) \left[\sigma_1(t^0) - t\right],$$

where $\xi, \xi' \in [\alpha, \beta)$ and $\alpha = \min\{t, \sigma_1(t^0)\}$, $\beta = \max\{t, \sigma_1(t^0)\}$. That is,

$$\begin{aligned}\left[\frac{\partial f(\xi', \sigma_2(s^0))}{\Delta_1 t} - \frac{\partial f(\xi', s)}{\Delta_1 t}\right] \left[\sigma_1(t^0) - t\right] &\le \Phi(t,s) \\ &\le \left[\frac{\partial f(\xi, \sigma_2(s^0))}{\Delta_1 t} - \frac{\partial f(\xi, s)}{\Delta_1 t}\right] \left[\sigma_1(t^0) - t\right].\end{aligned}$$

Hence

$$\frac{\partial f(\xi', \sigma_2(s^0))}{\Delta_1 t} - \frac{\partial f(\xi', s)}{\Delta_1 t} \le \frac{\Phi(t,s)}{\sigma_1(t^0) - t} \le \frac{\partial f(\xi, \sigma_2(s^0))}{\Delta_1 t} - \frac{\partial f(\xi, s)}{\Delta_1 t} \quad \text{if} \quad t < \sigma_1(t^0)$$

and

$$\frac{\partial f(\xi, \sigma_2(s^0))}{\Delta_1 t} - \frac{\partial f(\xi, s)}{\Delta_1 t} \le \frac{\Phi(t,s)}{\sigma_1(t^0) - t} \le \frac{\partial f(\xi', \sigma_2(s^0))}{\Delta_1 t} - \frac{\partial f(\xi', s)}{\Delta_1 t} \quad \text{if} \quad t > \sigma_1(t^0).$$

Further, again by the same mean value result, Theorem 4.5, there exist η, η' and θ, θ' in the interval $[\gamma, \delta)$, where $\gamma = \min\{s, \sigma_2(s^0)\}$, $\delta = \max\{s, \sigma_2(s^0)\}$, such that

$$\frac{\partial^2 f(\xi, \eta')}{\Delta_2 s \Delta_1 t} \left[\sigma_2(s^0) - s\right] \le \frac{\partial f(\xi, \sigma_2(s^0))}{\Delta_1 t} - \frac{\partial f(\xi, s)}{\Delta_1 t} \le \frac{\partial^2 f(\xi, \eta)}{\Delta_2 s \Delta_1 t} \left[\sigma_2(s^0) - s\right]$$

and

$$\frac{\partial^2 f(\xi', \theta')}{\Delta_2 s \Delta_1 t} \left[\sigma_2(s^0) - s\right] \le \frac{\partial f(\xi', \sigma_2(s^0))}{\Delta_1 t} - \frac{\partial f(\xi', s)}{\Delta_1 t} \le \frac{\partial^2 f(\xi', \theta)}{\Delta_2 s \Delta_1 t} \left[\sigma_2(s^0) - s\right].$$

From the above four inequalities, by the assumed continuity of $\frac{\partial^2 f(t,s)}{\Delta_2 s \Delta_1 t}$, we conclude that

$$\lim_{\substack{(t,s) \to (t^0, s^0) \\ t \neq \sigma_1(t^0),\, s \neq \sigma_2(s^0)}} \frac{\Phi(t,s)}{\left[\sigma_1(t^0) - t\right]\left[\sigma_2(s^0) - s\right]} = \frac{\partial^2 f(t^0, s^0)}{\Delta_2 s \Delta_1 t}. \tag{6.1}$$

Similarly, using the expression of $\Phi(t,s)$ in the form

$$\Phi(t,s) = \psi(\sigma_2(s^0)) - \psi(s), \quad \text{where} \quad \psi(s) = f(\sigma_1(t^0), s) - f(t,s),$$

and the continuity of $\frac{\partial^2 f(t,s)}{\Delta_1 t \Delta_2 s}$, we can prove the equality

$$\lim_{\substack{(t,s)\to(t^0,s^0) \\ t\neq\sigma_1(t^0),\, s\neq\sigma_2(s^0)}} \frac{\Phi(t,s)}{[\sigma_1(t^0)-t]\,[\sigma_2(s^0)-s]} = \frac{\partial^2 f(t^0,s^0)}{\Delta_1 t \Delta_2 s}. \tag{6.2}$$

Now, the left-hand sides of (6.1) and (6.2) are the same. Consequently, their right-hand sides must be equal to each other. This completes the proof. □

7. THE CHAIN RULE

The chain rule for one-variable functions on time scales has been presented in [1] and [5, Chapter 1]. To get an extension to two-variable functions on time scales we start with a time scale $\mathbb{T}$. Denote its forward jump operator by σ and its delta differentiation operator by Δ. Let, further, two functions

$$\varphi : \mathbb{T} \to \mathbb{R} \quad \text{and} \quad \psi : \mathbb{T} \to \mathbb{R}$$

be given. Let us set

$$\varphi(\mathbb{T}) = \mathbb{T}_1 \quad \text{and} \quad \psi(\mathbb{T}) = \mathbb{T}_2.$$

We will assume that $\mathbb{T}_1$ and $\mathbb{T}_2$ are time scales. Denote by σ_1, Δ_1 and σ_2, Δ_2 the forward jump operators and delta operators for $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively. Take a point $\xi^0 \in \mathbb{T}^\kappa$ and put

$$t^0 = \varphi(\xi^0) \quad \text{and} \quad s^0 = \psi(\xi^0).$$

We will also assume that

$$\varphi(\sigma(\xi^0)) = \sigma_1(\varphi(\xi^0)) \quad \text{and} \quad \psi(\sigma(\xi^0)) = \sigma_2(\psi(\xi^0)). \tag{7.1}$$

Under the above assumptions let a function $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be given.

Theorem 7.1. *Let the function f be σ_1-completely delta differentiable at the point (t^0, s^0). If the functions φ and ψ have delta derivatives at the point ξ^0, then the composite function*

$$F(\xi) = f(\varphi(\xi), \psi(\xi)) \quad \text{for} \quad \xi \in \mathbb{T} \tag{7.2}$$

has a delta derivative at that point which is expressed by the formula

$$F^\Delta(\xi^0) = \frac{\partial f(t^0,s^0)}{\Delta_1 t}\varphi^\Delta(\xi^0) + \frac{\partial f(\sigma_1(t^0),s^0)}{\Delta_2 s}\psi^\Delta(\xi^0). \tag{7.3}$$

Proof. Using (7.1) and (2.8) with $A_1 = \frac{\partial f(t^0,s^0)}{\Delta_1 t}$ and $B = \frac{\partial f(\sigma_1(t^0),s^0)}{\Delta_2 s}$, we have

$$\begin{aligned}
F(\sigma(\xi^0)) - F(\xi) &= f(\varphi(\sigma(\xi^0)),\psi(\sigma(\xi^0))) - f(\varphi(\xi),\psi(\xi))\\
&= f(\sigma_1(\varphi(\xi^0)),\sigma_2(\psi(\xi^0))) - f(\varphi(\xi),\psi(\xi))\\
&= \frac{\partial f(\varphi(\xi^0),\psi(\xi^0))}{\Delta_1 t}\left[\sigma_1(\varphi(\xi^0)) - \varphi(\xi)\right] + \frac{\partial f(\sigma_1(\varphi(\xi^0)),\psi(\xi^0))}{\Delta_2 s}\left[\sigma_2(\psi(\xi^0)) - \psi(\xi)\right]\\
&\quad +\gamma_1\left[\sigma_1(\varphi(\xi^0)) - \varphi(\xi)\right] + \gamma_2\left[\sigma_2(\psi(\xi^0)) - \psi(\xi)\right]\\
&= \frac{\partial f(t^0,s^0)}{\Delta_1 t}\left[\varphi(\sigma(\xi^0)) - \varphi(\xi)\right] + \frac{\partial f(\sigma_1(t^0),s^0)}{\Delta_2 s}\left[\psi(\sigma(\xi^0)) - \psi(\xi)\right]\\
&\quad +\gamma_1\left[\varphi(\sigma(\xi^0)) - \varphi(\xi)\right] + \gamma_2\left[\psi(\sigma(\xi^0)) - \psi(\xi)\right].
\end{aligned}$$

Dividing both sides of this equality by $\sigma(\xi^0) - \xi$ and passing then to the limit as $\xi \to \xi^0$, we get the formula (7.3) because $\xi \to \xi^0$ implies $\gamma_1 \to 0$ and $\gamma_2 \to 0$. □

The next theorem can be proved similarly to Theorem 7.1 by using (2.9).

Theorem 7.2. *Let the function f be σ_2-completely delta differentiable at the point (t^0,s^0). If the functions φ and ψ have delta derivatives at the point ξ^0, then the composite function F defined by* (7.2) *has the delta derivative $F^\Delta(\xi^0)$ which is expressed by the formula*

$$F^\Delta(\xi^0) = \frac{\partial f(t^0,\sigma_2(s^0))}{\Delta_1 t}\varphi^\Delta(\xi^0) + \frac{\partial f(t^0,s^0)}{\Delta_2 s}\psi^\Delta(\xi^0).$$

Remark 7.3. One or both of the functions φ and ψ may be constant. In that case one or both of $\mathbb{T}_1$ and $\mathbb{T}_2$ will be a single point time scale. For a single point time scale $\mathbb{T}_1 = \{t_1\}$ we assume that $\sigma_1(t_1) = t_1$ and for each function $g : \mathbb{T}_1 \to \mathbb{R}$ we assume that $g^{\Delta_1}(t_1) = 0$.

Let now two time scales $\mathbb{T}_{(1)}$ and $\mathbb{T}_{(2)}$ be given. Denote their forward jump operators and delta differentiation operators by $\sigma_{(1)}$, $\Delta_{(1)}$ and $\sigma_{(2)}$, $\Delta_{(2)}$, respectively. Let, further, two functions

$$\varphi : \mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \to \mathbb{R} \quad \text{and} \quad \psi : \mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \to \mathbb{R}$$

of two variables $(\xi,\eta) \in \mathbb{T}_{(1)} \times \mathbb{T}_{(2)}$, and a fixed point $(\xi^0,\eta^0) \in \mathbb{T}^\kappa_{(1)} \times \mathbb{T}^\kappa_{(2)}$ be given. Let us set

$$\mathbb{T}_1 = \mathbb{T}_1(\eta^0) = \varphi(\mathbb{T}_{(1)},\eta^0) \quad \text{and} \quad \mathbb{T}_2 = \mathbb{T}_2(\xi^0) = \psi(\xi^0,\mathbb{T}_{(2)})$$

and

$$t^0 = \varphi(\xi^0,\eta^0) \quad \text{and} \quad s^0 = \psi(\xi^0,\eta^0).$$

We will assume that $\mathbb{T}_1$ and $\mathbb{T}_2$ are time scales. Denote their forward jump operators and delta differentiation operators by σ_1, Δ_1 and σ_2, Δ_2, respectively. We will also assume

$$\varphi(\sigma_{(1)}(\xi^0),\eta^0) = \sigma_1(\varphi(\xi^0,\eta^0)), \quad \psi(\sigma_{(1)}(\xi^0),\eta^0) = \sigma_2(\psi(\xi^0,\eta^0)) \tag{7.4}$$

and

$$\varphi(\xi^0, \sigma_{(2)}(\eta^0)) = \sigma_1(\varphi(\xi^0, \eta^0)), \quad \psi(\xi^0, \sigma_{(2)}(\eta^0)) = \sigma_2(\psi(\xi^0, \eta^0)). \tag{7.5}$$

Under the above conditions let a function $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ of two variables $(t, s) \in \mathbb{T}_1 \times \mathbb{T}_2$ be given.

Theorem 7.4. *Let the function f be σ_1-completely delta differentiable at the point (t^0, s^0). If the functions φ and ψ have first order partial delta derivatives at the point (ξ^0, η^0), then the composite function*

$$F(\xi, \eta) = f(\varphi(\xi, \eta), \psi(\xi, \eta)) \quad \text{for} \quad (\xi, \eta) \in \mathbb{T}_{(1)} \times \mathbb{T}_{(2)} \tag{7.6}$$

has first order partial delta derivatives at (ξ^0, η^0) which are expressed by the formulas

$$\frac{\partial F(\xi^0, \eta^0)}{\Delta_{(1)}\xi} = \frac{\partial f(t^0, s^0)}{\Delta_1 t} \frac{\partial \varphi(\xi^0, \eta^0)}{\Delta_{(1)}\xi} + \frac{\partial f(\sigma_1(t^0), s^0)}{\Delta_2 s} \frac{\partial \psi(\xi^0, \eta^0)}{\Delta_{(1)}\xi} \tag{7.7}$$

and

$$\frac{\partial F(\xi^0, \eta^0)}{\Delta_{(2)}\eta} = \frac{\partial f(t^0, s^0)}{\Delta_1 t} \frac{\partial \varphi(\xi^0, \eta^0)}{\Delta_{(2)}\eta} + \frac{\partial f(\sigma_1(t^0), s^0)}{\Delta_2 s} \frac{\partial \psi(\xi^0, \eta^0)}{\Delta_{(2)}\eta}. \tag{7.8}$$

Proof. Using (7.4) and (2.8) with $A_1 = \frac{\partial f(t^0, s^0)}{\Delta_1 t}$ and $B = \frac{\partial f(\sigma_1(t^0), s^0)}{\Delta_2 s}$, we have

$$\begin{aligned}
F(\sigma_{(1)}(\xi^0), \eta^0) - F(\xi, \eta^0) &= f(\varphi(\sigma_{(1)}(\xi^0), \eta^0), \psi(\sigma_{(1)}(\xi^0), \eta^0)) - f(\varphi(\xi, \eta^0), \psi(\xi, \eta^0)) \\
&= f(\sigma_1(\varphi(\xi^0, \eta^0)), \sigma_2(\psi(\xi^0, \eta^0))) - f(\varphi(\xi, \eta^0), \psi(\xi, \eta^0)) \\
&= \frac{\partial f(\varphi(\xi^0, \eta^0), \psi(\xi^0, \eta^0))}{\Delta_1 t} \left[\sigma_1(\varphi(\xi^0, \eta^0)) - \varphi(\xi, \eta^0)\right] \\
&\quad + \frac{\partial f(\sigma_1(\varphi(\xi^0, \eta^0)), \psi(\xi^0, \eta^0))}{\Delta_2 s} \left[\sigma_2(\psi(\xi^0, \eta^0)) - \psi(\xi, \eta^0)\right] \\
&\quad + \gamma_1 \left[\sigma_1(\varphi(\xi^0, \eta^0)) - \varphi(\xi, \eta^0)\right] + \gamma_2 \left[\sigma_2(\psi(\xi^0, \eta^0)) - \psi(\xi, \eta^0)\right] \\
&= \frac{\partial f(t^0, s^0)}{\Delta_1 t} \left[\varphi(\sigma_{(1)}(\xi^0), \eta^0) - \varphi(\xi, \eta^0)\right] \\
&\quad + \frac{\partial f(\sigma_1(t^0), s^0)}{\Delta_2 s} \left[\psi(\sigma_{(1)}(\xi^0), \eta^0) - \psi(\xi, \eta^0)\right] \\
&\quad + \gamma_1 \left[\varphi(\sigma_{(1)}(\xi^0), \eta^0) - \varphi(\xi, \eta^0)\right] + \gamma_2 \left[\psi(\sigma_{(1)}(\xi^0), \eta^0) - \psi(\xi, \eta^0)\right].
\end{aligned}$$

On dividing both sides of this equality by $\sigma_{(1)}(\xi^0) - \xi$ and passing to the limit as $\xi \to \xi^0$, we get the formula (7.7) because $\xi \to \xi^0$ implies $\gamma_1 \to 0$ and $\gamma_2 \to 0$.

Similarly, using (7.5) and (2.8), we obtain

$$\begin{aligned}
&F(\xi^0,\sigma_{(2)}(\eta^0))-F(\xi^0,\eta)\\
&= f(\varphi(\xi^0,\sigma_{(2)}(\eta^0)),\psi(\xi^0,\sigma_{(2)}(\eta^0)))-f(\varphi(\xi^0,\eta),\psi(\xi^0,\eta))\\
&= f(\sigma_1(\varphi(\xi^0,\eta^0)),\sigma_2(\psi(\xi^0,\eta^0)))-f(\varphi(\xi^0,\eta),\psi(\xi^0,\eta))\\
&= \frac{\partial f(\varphi(\xi^0,\eta^0),\psi(\xi^0,\eta^0))}{\Delta_1 t}\left[\sigma_1(\varphi(\xi^0,\eta^0))-\varphi(\xi^0,\eta)\right]\\
&\quad+\frac{\partial f(\sigma_1(\varphi(\xi^0,\eta^0)),\psi(\xi^0,\eta^0))}{\Delta_2 s}\left[\sigma_2(\psi(\xi^0,\eta^0))-\psi(\xi^0,\eta)\right]\\
&\quad+\gamma_1\left[\sigma_1(\varphi(\xi^0,\eta^0))-\varphi(\xi^0,\eta)\right]+\gamma_2\left[\sigma_2(\psi(\xi^0,\eta^0))-\psi(\xi^0,\eta)\right]\\
&= \frac{\partial f(t^0,s^0)}{\Delta_1 t}\left[\varphi(\xi^0,\sigma_{(2)}(\eta^0))-\varphi(\xi^0,\eta)\right]\\
&\quad+\frac{\partial f(\sigma_1(t^0),s^0)}{\Delta_2 s}\left[\psi(\xi^0,\sigma_{(2)}(\eta^0))-\psi(\xi^0,\eta)\right]\\
&\quad+\gamma_1\left[\varphi(\xi^0,\sigma_{(2)}(\eta^0))-\varphi(\xi^0,\eta)\right]+\gamma_2\left[\psi(\xi^0,\sigma_{(2)}(\eta^0))-\psi(\xi^0,\eta)\right].
\end{aligned}$$

On dividing both sides of this equality by $\sigma_{(2)}(\eta^0)-\eta$ and passing to the limit as $\eta\to\eta^0$, we get the formula (7.8) because $\eta\to\eta^0$ implies $\gamma_1\to 0$ and $\gamma_2\to 0$. The theorem is proved. □

The next theorem can be proved in a similar way by using (7.4), (7.5), and (2.9).

Theorem 7.5. *Let the function f be σ_2-completely delta differentiable at the point (t^0,s^0). If the functions φ and ψ have first order partial delta derivatives at the point (ξ^0,η^0), then the composite function $F(\xi,\eta)$ defined by (7.6) has first order partial delta derivatives at (ξ^0,η^0) which are expressed by the formulas*

$$\frac{\partial F(\xi^0,\eta^0)}{\Delta_{(1)}\xi}=\frac{\partial f(t^0,\sigma_2(s^0))}{\Delta_1 t}\frac{\partial\varphi(\xi^0,\eta^0)}{\Delta_{(1)}\xi}+\frac{\partial f(t^0,s^0)}{\Delta_2 s}\frac{\partial\psi(\xi^0,\eta^0)}{\Delta_{(1)}\xi}$$

and

$$\frac{\partial F(\xi^0,\eta^0)}{\Delta_{(2)}\eta}=\frac{\partial f(t^0,\sigma_2(s^0))}{\Delta_1 t}\frac{\partial\varphi(\xi^0,\eta^0)}{\Delta_{(2)}\eta}+\frac{\partial f(t^0,s^0)}{\Delta_2 s}\frac{\partial\psi(\xi^0,\eta^0)}{\Delta_{(2)}\eta}.$$

8. THE DIRECTIONAL DERIVATIVE

Let $\mathbb{T}$ be a time scale with the forward jump operator σ and the delta operator Δ. We will assume that $0\in\mathbb{T}$. Further, let $\omega=\binom{\omega_1}{\omega_2}\in\mathbb{R}^2$ be a unit vector and let (t^0,s^0) be a fixed point in $\mathbb{R}^2$. Let us set

$$\mathbb{T}_1=\left\{t=t^0+\xi\omega_1:\ \xi\in\mathbb{T}\right\}\quad\text{and}\quad\mathbb{T}_2=\left\{s=s^0+\xi\omega_2:\ \xi\in\mathbb{T}\right\}.$$

Then $\mathbb{T}_1$ and $\mathbb{T}_2$ are time scales and $t^0\in\mathbb{T}_1$, $s^0\in\mathbb{T}_2$. Denote the forward jump operators of $\mathbb{T}_1$ and $\mathbb{T}_2$ by σ_1 and σ_2, the delta operators by Δ_1 and Δ_2, respectively.

Definition 8.1. Let a function $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ be given. The directional delta derivative of the function f at the point (t^0, s^0) in the direction of the vector ω (along ω) is defined as the number

$$\frac{\partial f(t^0, s^0)}{\Delta \omega} = F^{\Delta}(0), \tag{8.1}$$

provided it exists, where

$$F(\xi) = f(t^0 + \xi\omega_1, s^0 + \xi\omega_2) \quad \text{for} \quad \xi \in \mathbb{T}. \tag{8.2}$$

Theorem 8.2. *Suppose that the function f is σ_1-completely delta differentiable at the point (t^0, s^0). Then the directional delta derivative of f at (t^0, s^0) in the direction of the vector ω exists and is expressed by the formula*

$$\frac{\partial f(t^0, s^0)}{\Delta \omega} = \frac{\partial f(t^0, s^0)}{\Delta_1 t}\omega_1 + \frac{\partial f(\sigma_1(t^0), s^0)}{\Delta_2 s}\omega_2. \tag{8.3}$$

Proof. The proof is obtained from the definitions (8.1) and (8.2) by applying Theorem 7.1. □

The next theorem follows similarly by applying Theorem 7.2.

Theorem 8.3. *Suppose that the function f is σ_2-completely delta differentiable at the point (t^0, s^0). Then the directional delta derivative of f at (t^0, s^0) in the direction of the vector ω exists and is expressed by the formula*

$$\frac{\partial f(t^0, s^0)}{\Delta \omega} = \frac{\partial f(t^0, \sigma_2(s^0))}{\Delta_1 t}\omega_1 + \frac{\partial f(t^0, s^0)}{\Delta_2 s}\omega_2.$$

Remark 8.4. For $\omega_1 = 1$ and $\omega_2 = 0$, (8.3) coincides with $\frac{\partial f(t^0,s^0)}{\Delta_1 t}$, while for $\omega_1 = 0$ and $\omega_2 = 1$ it coincides with $\frac{\partial f(t^0,s^0)}{\Delta_2 s}$ because then $\mathbb{T}_1 = \{t^0\}$ and hence $\sigma_1(t^0) = t^0$ (see Remark 7.3).

9. IMPLICIT FUNCTIONS

Let $\mathbb{T}$ be a time scale with the forward jump operator σ and the delta differentiation Δ. Take an arbitrary real-valued function f defined on $\mathbb{T} \times \mathbb{R}$ and consider the equation

$$f(t, y) = 0 \quad \text{for} \quad (t, y) \in \mathbb{T} \times \mathbb{R}. \tag{9.1}$$

Let $\mathcal{M}$ denote the set of all points $(t, y) \in \mathbb{T} \times \mathbb{R}$ for which equation (9.1) is fulfilled and let (t^0, y^0) be a point belonging to this set, that is, $f(t^0, y^0) = 0$.

If no additional conditions are imposed on the function f, then the set $\mathcal{M}$ can be of an arbitrary structure. We also often encounter the case when, at least in a sufficiently small neighbourhood of (t^0, y^0), the set $\mathcal{M}$ is a "curve" described by a continuous (single-valued) function

$$y = \psi(t). \tag{9.2}$$

The problem is to decide whether the equation (9.1) determines y as a function of t. If so, we have (9.2) for some function ψ. We say that ψ is defined "implicitly" by (9.1). The implicit function theorems give a description of conditions under which there exists the function ψ as well as some conclusions about this function.

Theorem 9.1. *Let an equation* (9.1) *satisfy the following conditions:*

(i) *The function f is defined in a neigbourhood U of the point $(t^0, y^0) \in \mathbb{T}^\kappa \times \mathbb{R}$ and is continuous in U together with its partial derivatives $\frac{\partial f(t,y)}{\Delta t}$ and $\frac{\partial f(t,y)}{\partial y}$;*

(ii) $f(t^0, y^0) = 0$;

(iii) $\frac{\partial f(t^0,y^0)}{\partial y} \neq 0$.

Then the following statements are true:

(a) *There is a "rectangle" (a neighbourhood of the point (t^0, y^0) in $\mathbb{T} \times \mathbb{R}$)*

$$\mathcal{N} = \left\{(t, y) \in \mathbb{T} \times \mathbb{R} : \ |t - t^0| < \delta, \ |y - y^0| < \delta'\right\} \tag{9.3}$$

belonging to U such that the set $\mathcal{M} \cap \mathcal{N}$ is described by a (uniquely determined) single-valued function

$$y = \psi(t) \quad \text{for} \quad t \in \mathcal{N}^0, \tag{9.4}$$

where

$$\mathcal{M} = \{(t, y) \in \mathbb{T} \times \mathbb{R} : \ f(t, y) = 0\} \quad \text{and} \quad \mathcal{N}^0 = \left\{t \in \mathbb{T} : \ |t - t^0| < \delta\right\};$$

(b) $y^0 = \psi(t^0)$;

(c) *the function $\psi(t)$ is continuous in $\mathcal{N}^0$;*

(d) *the function $\psi(t)$ has a delta derivative $\psi^\Delta(t)$ on $\mathcal{N}^0$.*

Proof. Without loss of generality we can suppose that U is an open rectangle of the form

$$U = \left\{(t, y) \in \mathbb{T} \times \mathbb{R} : \ |t - t^0| < a, \ |y - y^0| < b\right\},$$

and also for the definiteness we can suppose that $\frac{\partial f(t^0,y^0)}{\partial y} > 0$. From the continuity of $\frac{\partial f(t,y)}{\partial y}$ on U we then also have

$$\frac{\partial f(t, y)}{\partial y} > 0 \tag{9.5}$$

in a small neighbourhood $U_1 \subset U$ of the point (t^0, y^0) of the form

$$U_1 = \left\{(t, y) \in \mathbb{T} \times \mathbb{R} : \ |t - t^0| < a_1, \ |y - y^0| < b_1\right\}.$$

Then the function $f(t^0, y)$ of the single variable y is continuous on the closed interval $[y^0 - b_1, y^0 + b_1]$, is strictly increasing on that interval, and turns into zero at the point $y = y^0$ $(f(t^0, y^0) = 0)$. It follows that

$$f(t^0, y^0 - b_1) < 0 \quad \text{and} \quad f(t^0, y^0 + b_1) > 0$$

and, by the continuity of f, there is a sufficiently small number $\delta > 0$ with $\delta < a_1$ such that

$$f(t, y^0 - b_1) < 0 \quad \text{and} \quad f(t, y^0 + b_1) > 0 \quad \text{for all} \quad t \in \mathcal{N}^0 = \left\{t \in \mathbb{T} : \ |t - t^0| < \delta\right\}.$$

Now we choose an arbitrary point $t \in \mathcal{N}^0$, fix it temporarily, and consider the function $f(t, y)$ of one real variable y on the interval $(y^0 - b_1, y^0 + b_1) \subset \mathbb{R}$. By the properties of f, this function is continuous, strictly increasing (by (9.5)), and assumes values of opposite signs at the end points of the interval. Therefore there exists a single value $y \in (y^0 - b_1, y^0 + b_1)$ which we denote by $y = \psi(t)$ for which $f(t, \psi(t)) = 0$. Thus, letting $\delta' = b_1$, we see that in the neigbhourhood $\mathcal{N}$ of the point (t^0, y^0), defined by (9.3), the equation (9.1) really determines y as a unique function of t: $y = \psi(t)$. This completes the proof of (a).

From condition (ii) and by the uniqueness of the function ψ, we obtain $y^0 = \psi(t^0)$, i.e., (b) is true.

Let us prove (c), i.e., that the function ψ is continuous (in the time scale topology). To this end, it is sufficient to show that it is continuous at the point $t = t^0$. Indeed, the same proof can then be extended to any other point t' of the interval $\mathcal{N}^0$ because the point $(t', \psi(t'))$ can be enclosed in a neighbourhood $\mathcal{N}' \subset \mathcal{N}$ such that all the conditions of the theorem are fulfilled for it if $\mathcal{N}$, t^0, and y^0 are replaced by $\mathcal{N}'$, t', and $y' = \psi(t')$, respectively. Let us now take an arbitrary positive number $\varepsilon' \in (0, \delta')$. By what has been already proved, there is a positive number $\varepsilon \in (0, \delta)$ such that if we define the rectangle

$$\mathcal{N}_* = \left\{(t, y) \in \mathbb{T} \times \mathbb{R} : \ |t - t^0| < \varepsilon, \ |y - y^0| < \varepsilon'\right\},$$

then there is a function $y = \psi_*(t)$ for $t \in \mathcal{N}_*^0 = \{t \in \mathbb{T} : \ |t - t^0| < \varepsilon\}$ which describes the set $\mathcal{M} \cap \mathcal{N}_*$. We have $\mathcal{N}_* \subset \mathcal{N}$ and therefore, obviously, $\psi(t) = \psi_*(t)$ for $t \in \mathcal{N}_*^0$. This shows that for any sufficiently small $\varepsilon' > 0$ there exists $\varepsilon > 0$ such that $|\psi(t) - \psi(t^0)| < \varepsilon'$ provided that $|t - t^0| < \varepsilon$, which means that ψ is continuous at the point $t = t^0$.

Finally we show (d), i.e., that the function $\psi(t)$ possesses the delta derivative $\psi^\Delta(t)$ with respect to t. We take a fixed point $t_1 \in \mathcal{N}^0$ and show that the delta derivative $\psi^\Delta(t_1)$ exists. We consider the two possible cases separately.

First suppose that t_1 is right-scattered, i.e., $\sigma(t_1) > t_1$. In this case it follows from the continuity of ψ at t_1 that it has the delta derivative $\psi^\Delta(t_1)$ and

$$\psi^\Delta(t_1) = \frac{\psi(\sigma(t_1)) - \psi(t_1)}{\sigma(t_1) - t_1}.$$

Now suppose that t_1 is right-dense. Let us take any variable point $t_2 \in \mathcal{N}^0$ such that $t_2 \neq t_1$. It is sufficient to show that there exists the finite limit

$$\lim_{t_2 \to t_1} \frac{\psi(t_1) - \psi(t_2)}{t_1 - t_2},$$

which then is obviously equal to $\psi^\Delta(t_1)$ in the considered case. By the equations $f(t_1, \psi(t_1)) = 0$ and $f(t_2, \psi(t_2)) = 0$ we can write

$$f(t_2, \psi(t_1)) - f(t_1, \psi(t_1)) = f(t_2, \psi(t_1)) - f(t_2, \psi(t_2)). \tag{9.6}$$

On the other hand, by the usual mean value theorem we have

$$f(t_2, \psi(t_1)) - f(t_2, \psi(t_2)) = \frac{\partial f(t_2, \theta)}{\partial y} \left[\psi(t_1) - \psi(t_2)\right],$$

where θ is a real number between $\psi(t_1)$ and $\psi(t_2)$ (hence $\theta \to \psi(t_1)$ as $t_2 \to t_1$ by the continuity of ψ). Further, by the mean value theorem for delta derivatives, Theorem 4.5, we have

$$\frac{\partial f(\xi', \psi(t_1))}{\Delta t}(t_2 - t_1) \le f(t_2, \psi(t_1)) - f(t_1, \psi(t_1)) \le \frac{\partial f(\xi, \psi(t_1))}{\Delta t}(t_2 - t_1),$$

where $\xi, \xi' \in [\alpha, \beta)$ and $\alpha = \min\{t_1, t_2\}$, $\beta = \max\{t_1, t_2\}$ (hence $\xi \to t_1$ and $\xi' \to t_1$ as $t_2 \to t_1$). Consequently, we obtain from (9.6)

$$\frac{\partial f(\xi', \psi(t_1))}{\Delta t}(t_2 - t_1) \le \frac{\partial f(t_2, \theta)}{\partial y} \left[\psi(t_1) - \psi(t_2)\right] \le \frac{\partial f(\xi, \psi(t_1))}{\Delta t}(t_2 - t_1). \tag{9.7}$$

Dividing (9.7) by $\frac{\partial f(t_2,\theta)}{\partial y}(t_1 - t_2)$ and taking into account (9.5) and the continuity of the partial derivatives $\frac{\partial f(t,y)}{\Delta t}$ and $\frac{\partial f(t,y)}{\partial y}$, we see that there exists the finite limit

$$\psi^\Delta(t_1) = \lim_{t_2 \to t_1} \frac{\psi(t_1) - \psi(t_2)}{t_1 - t_2} = -\frac{\frac{\partial f(t_1, \psi(t_1))}{\Delta t}}{\frac{\partial f(t_1, \psi(t_1))}{\partial y}}.$$

The theorem is proved. □

Similarly to the equation (9.1) we can consider an equation with more variables. Let us, for instance, consider the equation

$$f(t, s, y) = 0 \quad \text{for} \quad (t, s, y) \in \mathbb{T}_1 \times \mathbb{T}_2 \times \mathbb{R}, \tag{9.8}$$

where $\mathbb{T}_1$ and $\mathbb{T}_2$ are given time scales. The following theorem can be proved analogously to Theorem 9.1.

Theorem 9.2. *Let an equation* (9.8) *satisfy the following conditions:*

(i) *The function f is defined in a neigbourhood U of the point $(t^0, s^0, y^0) \in \mathbb{T}_1^\kappa \times \mathbb{T}_2^\kappa \times \mathbb{R}$ and is continuous in U together with its partial derivatives $\frac{\partial f(t,s,y)}{\Delta_1 t}$, $\frac{\partial f(t,s,y)}{\Delta_2 s}$, and $\frac{\partial f(t,s,y)}{\partial y}$;*

(ii) $f(t^0, s^0, y^0) = 0$;

(iii) $\frac{\partial f(t^0, s^0, y^0)}{\partial y} \neq 0$.

Then the following statements are true:

(a) *There is a neighbourhood of the point* (t^0, s^0, y^0) *in* $\mathbb{T}_1 \times \mathbb{T}_2 \times \mathbb{R}$

$$\mathcal{N} = \left\{(t, s, y) \in \mathbb{T}_1 \times \mathbb{T}_2 \times \mathbb{R} : \ |t - t^0| < \delta, \ |s - s^0| < \delta', \ |y - y^0| < \delta''\right\}$$

belonging to U *such that the set* $\mathcal{M} \cap \mathcal{N}$ *is described by a (uniquely determined) single-valued function*

$$y = \psi(t, s) \quad \textit{for} \quad (t, s) \in \mathcal{N}^0,$$

where

$$\mathcal{M} = \{(t, s, y) \in \mathbb{T}_1 \times \mathbb{T}_2 \times \mathbb{R} : \ f(t, s, y) = 0\}$$

and

$$\mathcal{N}^0 = \left\{(t, s) \in \mathbb{T}_1 \times \mathbb{T}_2 : \ |t - t^0| < \delta, \ |s - s^0| < \delta'\right\};$$

(b) $y^0 = \psi(t^0, s^0)$*;*

(c) *the function* $\psi(t, s)$ *is continuous in* $\mathcal{N}^0$*;*

(d) *the function* $\psi(t, s)$ *has partial delta derivatives* $\frac{\partial\psi(t,s)}{\Delta_1 t}$ *and* $\frac{\partial\psi(t,s)}{\Delta_2 s}$ *on* $\mathcal{N}^0$.

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