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Periodicity of scalar dynamic equations and applications to population models ☆

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Abstract

Easily verifiable sufficient criteria are established for the existence of periodic solutions of a class of nonautonomous scalar dynamic equations on time scales, which incorporate as special cases many single species models governed by ordinary differential and difference equations when the time scale is the set of all real and all integer numbers, respectively.

Keywords: Time scales; Periodic solution; Coincidence degree

1. Introduction

The theory of calculus on time scales (see [2,3] and references cited therein) was initiated by Stefan Hilger in his PhD thesis in 1988 [11] in order to unify continuous and discrete analysis, and it has a tremendous potential for applications and has recently received much attention since the foundational work. It has been created in order to unify the study of differential and difference equations.

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In this paper, we will explore the existence of periodic solutions of a class of scalar nonautonomous dynamic equations on time scales, which incorporate as special cases many single species models governed by ordinary differential (difference) equations when the time scale $\mathbb{T}$ is set to be $\mathbb{R}$ ($\mathbb{Z}$). The approach is based on a continuation theorem in coincidence degree, which has been extensively applied to explore the existence problem in ordinary differential (difference) equations but is only once [1] applied to dynamic equations on general time scales.

First, we give some elements of the times scales calculus and introduce the continuation theorem. Then, we explore the existence of positive periodic solutions of a class of nonautonomous scalar equations with deviating argument on time scales.

2. Preliminaries

In this section, we briefly give some elements of the time scales calculus, recall the continuation theorem from coincidence degree theory, and state an auxiliary result that is needed in the paper.

First, let us present some foundational definitions and results from the calculus on time scales so that the paper is self-contained. For more details, one can see [2,3,11].

Notation 2.1. Throughout this paper, the symbol $\mathbb{T}$ denotes a *time scale*, i.e., an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Let $\omega > 0$. Throughout, the time scale $\mathbb{T}$ is assumed to be *ω-periodic*, i.e., $t \in \mathbb{T}$ implies $t + \omega \in \mathbb{T}$. In particular, the time scale $\mathbb{T}$ under consideration is unbounded above and below. Some examples of such time scales are

$$\mathbb{R}, \qquad \mathbb{Z}, \qquad \bigcup_{k\in\mathbb{Z}} [2k, 2k+1], \qquad \bigcup_{k\in\mathbb{Z}} \bigcup_{n\in\mathbb{N}} \left\{ k + \frac{1}{n} \right\},$$

whose periods are any real number, any integer, any even integer, and any integer, respectively.

Definition 2.1. We define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$, the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+ = [0, \infty)$ by

$$\sigma(t) := \inf\{s \in \mathbb{T}:\ s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T}:\ s < t\}, \quad \text{and}$$
$$\mu(t) = \sigma(t) - t \quad \text{for } t \in \mathbb{T},$$

respectively. If $\sigma(t) = t$, then t is called right-dense (otherwise: right-scattered), and if $\rho(t) = t$, then t is called left-dense (otherwise: left-scattered).

Definition 2.2. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\big|\big[f(\sigma(t)) - f(s)\big] - f^\Delta(t)\big[\sigma(t) - s\big]\big| \leqslant \varepsilon \big|\sigma(t) - s\big| \quad \text{for all } s \in U.$$

In this case, $f^\Delta(t)$ is called the delta (or Hilger) derivative of f at t. Moreover, f is said to be delta or Hilger differentiable on $\mathbb{T}$ if $f^\Delta(t)$ exists for all $t \in \mathbb{T}$. A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}$. Then we define

$$\int_r^s f(t)\Delta t = F(s) - F(r) \quad \text{for } r, s \in \mathbb{T}.$$

Definition 2.3. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exists (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathrm{C_{rd}}(\mathbb{T})$.

Lemma 2.1. *Every rd-continuous function has an antiderivative.*

Lemma 2.2. *If $a, b \in \mathbb{T}$, $\alpha, \beta \in \mathbb{R}$ and $f, g \in \mathrm{C_{rd}}(\mathbb{T})$, then*

(a) $\int_a^b [\alpha f(t) + \beta g(t)]\Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t$;
(b) *if $f(t) \geqslant 0$ for all $a \leqslant t < b$, then $\int_a^b f(t)\Delta t \geqslant 0$;*
(c) *if $|f(t)| \leqslant g(t)$ for all $a \leqslant t < b$, then $|\int_a^b f(t)\Delta t| \leqslant \int_a^b g(t)\Delta t$.*

Definition 2.4. If $a \in \mathbb{T}$, $\inf \mathbb{T} = -\infty$, and f is rd-continuous on $(-\infty, a]$, then we define the *improper integral* by

$$\int_{-\infty}^{a} f(t)\Delta t := \lim_{T \to -\infty} \int_{T}^{a} f(t)\Delta t$$

provided this limit exists, and we say that the improper integral converges in this case.

Notation 2.2. To facilitate the discussion below, we now introduce some notation to be used throughout this paper. Let

$$\kappa = \min\{[0, \infty) \cap \mathbb{T}\}, \qquad I_\omega = [\kappa, \kappa + \omega] \cap \mathbb{T}, \qquad \bar{g} = \frac{1}{\omega} \int_{I_\omega} g(s)\Delta s = \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} g(s)\Delta s,$$

where $g \in \mathrm{C_{rd}}(\mathbb{T})$ is an ω-periodic real function, i.e., $g(t + \omega) = g(t)$ for all $t \in \mathbb{T}$.

Next, let us recall the continuation theorem in coincidence degree theory, borrowing notations and terminology from [7], which will come into play later on.

Notation 2.3. Let X, Z be normed vector spaces, $L : \operatorname{Dom} L \subset X \to Z$ be a linear mapping, $N : X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero and there exist continuous projections $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$, then it follows that $L|\operatorname{Dom} L \cap \operatorname{Ker} P : (I - P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Lemma 2.3 *(Continuation theorem). Let L be a Fredholm mapping of index zero and N be L-compact on $\overline{\Omega}$. Suppose*

(a) *for each $\lambda \in (0, 1)$, every solution z of $Lz = \lambda Nz$ is such that $z \notin \partial\Omega$;*
(b) *$QNz \neq 0$ for each $z \in \partial\Omega \cap \operatorname{Ker} L$ and the Brouwer degree* $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.

Then the operator equation $Lz = Nz$ has at least one solution lying in $\operatorname{Dom} L \cap \overline{\Omega}$.

In order to achieve the priori estimation in the case of dynamic equations on a time scale $\mathbb{T}$, we now give the following inequality which is proved in [1, Lemma 2.4].

Lemma 2.4. *Let $t_1, t_2 \in I_\omega$ and $t \in \mathbb{T}$. If $g : \mathbb{T} \to \mathbb{R}$ is ω-periodic, then*

$$g(t) \leqslant g(t_1) + \int_\kappa^{\kappa+\omega} \left|g^\Delta(s)\right| \Delta s \quad \text{and} \quad g(t) \geqslant g(t_2) - \int_\kappa^{\kappa+\omega} \left|g^\Delta(s)\right| \Delta s.$$

3. Main result

Consider the scalar dynamic equation of first order on a time scale

$$x^\Delta(t) = G\Bigg(t, \exp\{x(g_1(t))\}, \exp\{x(g_2(t))\}, \ldots, \exp\{x(g_n(t))\}, \int_{-\infty}^{t} c(t,s) \exp\{x(s)\} \Delta s\Bigg), \tag{3.1}$$

where:

- (H$_1$) $G : \mathbb{T} \times \mathbb{R}^{n+1} \to \mathbb{R}$, $G(t, \cdot)$ is continuous on $\mathbb{R}^{n+1}$ for all $t \in \mathbb{T}$ and is ω-periodic in t, i.e., $G(t+\omega, u) = G(t, u)$ for all $u \in \mathbb{R}^{n+1}$ and all $t \in \mathbb{T}$, where $\omega > 0$ is called the period of (3.1).
- (H$_2$) $g_i : \mathbb{T} \to \mathbb{T}$ for $1 \leqslant i \leqslant n$ and $c : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ satisfy $g_i(t+\omega) = g_i(t)$ and $c(t+\omega, s+\omega) = c(t,s)$, and $\int_{-\infty}^t c(t,s)\Delta s$ is rd-continuous in $t \in \mathbb{T}$.

Remark 3.1. Let $\tilde{x}(t) = \exp\{x(t)\}$. If $\mathbb{T} = \mathbb{R}$, then (3.1) reduces to the continuous nonautonomous scalar equation with deviating arguments

$$\tilde{x}'(t) = \tilde{x}(t) G\Bigg(t, \tilde{x}(g_1(t)), \tilde{x}(g_2(t)), \ldots, \tilde{x}(g_n(t)), \int_{-\infty}^{t} c(t,s)\tilde{x}(s)\, \mathrm{d}s\Bigg),$$

while if $\mathbb{T} = \mathbb{Z}$, then (3.1) is reformulated as the discrete equation

$$\tilde{x}(t+1) = \tilde{x}(t) \exp\Bigg\{G\Bigg(t, \tilde{x}(g_1(t)), \tilde{x}(g_2(t)), \ldots, \tilde{x}(g_n(t)), \sum_{s=-\infty}^{t-1} c(t,s)\tilde{x}(s)\Bigg)\Bigg\}.$$

Both of these nonautonomous scalar equations with deviating arguments have been studied in [5].

In order to explore the existence of periodic solutions of (3.1), first we should embed our problem in the frame of coincidence degree theory. Define

$$\mathcal{L}^\omega = \{u \in C(\mathbb{T}, \mathbb{R}):\ u(t+\omega) = u(t) \text{ for all } t \in \mathbb{T}\}, \qquad \|u\| = \max_{t \in I_\omega} \left|u(t)\right| \quad \text{for } u \in \mathcal{L}^\omega.$$

It is not difficult to show that $(\mathcal{L}^\omega, \|\cdot\|)$ is a Banach space. Let

$$\mathcal{L}_0^\omega = \{u \in \mathcal{L}^\omega:\ \bar{u} = 0\}, \qquad \mathcal{L}_c^\omega = \{u \in \mathcal{L}^\omega:\ u(t) \equiv h \in \mathbb{R} \text{ for } t \in \mathbb{T}\}.$$

Then it is easy to show that $\mathcal{L}_0^\omega$ and $\mathcal{L}_c^\omega$ are both closed linear subspaces of $\mathcal{L}^\omega$, $\mathcal{L}^\omega = \mathcal{L}_0^\omega \oplus \mathcal{L}_c^\omega$, and $\dim \mathcal{L}_c^\omega = 1$.

Theorem 3.1. *Let* (H_1) *and* (H_2) *hold. Moreover, assume*:

(H_3) *there exists a constant $M > 0$ such that for any ω-periodic function $x : \mathbb{T} \to \mathbb{R}$, if*

$$\int_{\kappa}^{\kappa+\omega} G\left(t, \exp\{x(g_1(t))\}, \ldots, \exp\{x(g_n(t))\}, \int_{-\infty}^{t} c(t,s)\exp\{x(s)\}\Delta s\right)\Delta t = 0,$$

then

$$\int_{\kappa}^{\kappa+\omega} \left|G\left(t, \exp\{x(g_1(t))\}, \ldots, \exp\{x(g_n(t))\}, \int_{-\infty}^{t} c(t,s)\exp\{x(s)\}\Delta s\right)\right|\Delta t \leqslant M;$$

(H_4) *there exist constants $A_2 > A_1 > 0$ such that if $u_i \geqslant A_2$ for all $1 \leqslant i \leqslant n+1$, then*

$$\int_{\kappa}^{\kappa+\omega} G\left(t, u_1, \ldots, u_n, \int_{-\infty}^{t} c(t,s)u_{n+1}\Delta s\right)\Delta t < 0;$$

and if $0 < u_i \leqslant A_1$ for all $1 \leqslant i \leqslant n+1$, then

$$\int_{\kappa}^{\kappa+\omega} G\left(t, u_1, \ldots, u_n, \int_{-\infty}^{t} c(t,s)u_{n+1}\Delta s\right)\Delta t > 0.$$

Then Eq. (3.1) *has at least one ω-periodic solution.*

Proof. Let $X = Z = \mathcal{L}^{\omega}$ and define

$$Nx = G\left(t, \exp\{x(g_1(t))\}, \ldots, \exp\{x(g_n(t))\}, \int_{-\infty}^{t} c(t,s)\exp\{x(s)\}\Delta s\right),$$

$$Lx = x^{\Delta}, \qquad Px = Qx = \bar{x}.$$

Then $\operatorname{Ker} L = \mathcal{L}_c^{\omega}$, $\operatorname{Im} L = \mathcal{L}_0^{\omega}$, and $\dim \operatorname{Ker} L = 1 = \operatorname{codim} \operatorname{Im} L$. Since $\mathcal{L}_0^{\omega}$ is closed in $\mathcal{L}^{\omega}$, it follows that L is a Fredholm mapping of index zero. It is not difficult to show that P and Q are continuous projections such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$. Furthermore, the generalized inverse (to L) $K_P : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ exists and is given by

$$K_P x = \hat{x} - \bar{\hat{x}}, \quad \text{where } \hat{x}(t) = \int_{\kappa}^{t} x(s)\Delta s.$$

Thus

$$QNx = \frac{1}{\omega}\int_{\kappa}^{\kappa+\omega} G\left(t, \exp\{x(g_1(t))\}, \ldots, \exp\{x(g_n(t))\}, \int_{-\infty}^{t} c(t,s)\exp\{x(s)\}\Delta s\right)\Delta t$$

and

$$K_P(I-Q)Nx = \int_\kappa^t (Nx)(s)\Delta s - \frac{1}{\omega}\int_\kappa^{\kappa+\omega}\int_\kappa^t (Nx)(s)\Delta s\Delta t - \left(t-\kappa-\frac{1}{\omega}\int_\kappa^{\kappa+\omega}(t-\kappa)\Delta t\right)\overline{Nx}.$$

Obviously, QN and $K_P(I-Q)N$ are continuous. Since X is a Banach space, using the Arzelà–Ascoli theorem, it is easy to show that $\overline{K_P(I-Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\overline{\Omega})$ is bounded. Thus, N is L-compact on $\overline{\Omega}$ with any open bounded set $\Omega \subset X$.

Now we are in the position to search for an appropriate open, bounded subset Ω for the application of the continuation theorem, Lemma 2.3. For the operator equation $Lx = \lambda Nx$, $\lambda \in (0,1)$, we have

$$x^\Delta(t) = \lambda G\left(t, \exp\{x(g_1(t))\}, \ldots, \exp\{x(g_n(t))\}, \int_{-\infty}^t c(t,s)\exp\{x(s)\}\Delta s\right). \tag{3.2}$$

Assume that $x \in X$ is an arbitrary solution of Eq. (3.2) for a certain $\lambda \in (0,1)$. Integrating both sides of (3.2) over the interval $[\kappa, \kappa+\omega]$, we obtain

$$\int_\kappa^{\kappa+\omega} G\left(t, \exp\{x(g_1(t))\}, \ldots, \exp\{x(g_n(t))\}, \int_{-\infty}^t c(t,s)\exp\{x(s)\}\Delta s\right)\Delta t = 0. \tag{3.3}$$

From (3.2) and (H_3), we have

$$\begin{aligned}\int_\kappa^{\kappa+\omega}\left|x^\Delta(t)\right|\Delta t &= \lambda\int_\kappa^{\kappa+\omega}\Bigg|G\Bigg(t, \exp\{x(g_1(t))\}, \ldots, \exp\{x(g_n(t))\}, \\ &\qquad \int_{-\infty}^t c(t,s)\exp\{x(s)\}\Delta s\Bigg)\Bigg|\Delta t \\ &\leqslant M.\end{aligned} \tag{3.4}$$

Note that since $x \in X$, there exist $\xi, \eta \in I_\omega$, such that

$$x(\xi) = \min_{t\in I_\omega} x(t) \quad \text{and} \quad x(\eta) = \max_{t\in I_\omega} x(t). \tag{3.5}$$

Next, we assume that $x(\xi) \geqslant \ln(A_2)$. Then by (3.5), we have $x(t) \geqslant \ln(A_2)$ for any $t \in I_\omega$. From (H_4), we have

$$\int_\kappa^{\kappa+\omega} G\left(t, \exp\{x(g_1(t))\}, \ldots, \exp\{x(g_n(t))\}, \int_{-\infty}^t c(t,s)\exp\{x(s)\}\Delta s\right)\Delta t < 0,$$

which contradicts (3.3). Thus we arrive at

$$x(\xi) < \ln(A_2) \quad \text{and} \quad x(\eta) > \ln(A_1), \tag{3.6}$$

where we used a similar argument for the second inequality. From (3.4), (3.6), and Lemma 2.4, we have

$$x(t) \leqslant x(\xi) + \int_{\kappa}^{\kappa+\omega} \left|x^{\Delta}(t)\right| \Delta t < \ln(A_2) + M,$$

$$x(t) \geqslant x(\eta) - \int_{\kappa}^{\kappa+\omega} \left|x^{\Delta}(t)\right| \Delta t > \ln(A_1) - M,$$

and therefore

$$\max_{t\in\mathbb{R}} \left|x(t)\right| = \max_{t\in I_\omega} \left|x(t)\right| < \max\left\{\left|\ln(A_2) + M\right|, \left|\ln(A_1) - M\right|\right\} =: A_3.$$

Now we define $\Omega := \{x \in X\colon \|x\| < B\}$, where $B := \max\{A_3, |\ln(A_1)|, |\ln(A_2)|\}$. It is clear that Ω satisfies the requirement (a) in Lemma 2.3. If $x \in \partial\Omega \cap \operatorname{Ker} L$, then by (H_4)

$$QNx = \frac{1}{\omega} \int_{\kappa}^{\kappa+\omega} G\left(t, \exp\left\{x\left(g_1(t)\right)\right\}, \ldots, \exp\left\{x\left(g_n(t)\right)\right\}, \int_{-\infty}^{t} c(t,s) \exp\left\{x(s)\right\} \Delta s\right) \Delta t \neq 0.$$

Moreover, note that $J = I$ since $\operatorname{Im} Q = \operatorname{Ker} L$. In order to compute the Brouwer degree, let us consider the homotopy $H(\nu, x) = \nu x - (1-\nu)QNx$, $\nu \in [0,1]$. For any $x \in \partial\Omega \cap \operatorname{Ker} L$, $\nu \in [0,1]$, we have $xH(\nu, x) > 0$, so $H(\nu, x) \neq 0$. By the homotopy invariance of topological degree, we have $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} = \deg\{QNx, \Omega \cap \operatorname{Ker} L, 0\} = \deg\{x, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, where $\deg(\cdot, \cdot, \cdot)$ is the Brouwer degree. Now we have proved that Ω satisfies all requirements in Lemma 2.3. Thus $Lx = Nx$ has at least one solution in $\operatorname{Dom} L \cap \overline{\Omega}$, that is, (3.1) has at least one ω-periodic solution in $\operatorname{Dom} L \cap \overline{\Omega}$. The proof is complete. □

Similarly, we can prove the following two results.

Theorem 3.2. *Let* (H_1)–(H_3) *hold. In addition, assume*

(H_5) *there exist constants* $A_2 > A_1 > 0$ *such that if* $u_i \geqslant A_2$ *for all* $1 \leqslant i \leqslant n+1$, *then*

$$\int_{\kappa}^{\kappa+\omega} G\left(t, u_1, \ldots, u_n, \int_{-\infty}^{t} c(t,s) u_{n+1} \Delta s\right) \Delta t > 0;$$

and if $0 < u_i \leqslant A_1$ *for all* $1 \leqslant i \leqslant n+1$, *then*

$$\int_{\kappa}^{\kappa+\omega} G\left(t, u_1, \ldots, u_n, \int_{-\infty}^{t} c(t,s) u_{n+1} \Delta s\right) \Delta t < 0.$$

Then Eq. (3.1) *has at least one ω-periodic solution.*

Corollary 3.1. *Let* (H_1)–(H_3) *hold. Moreover, assume that there exists a constant* $A > 0$ *such that if* $u_i \leqslant A$ *for all* $1 \leqslant i \leqslant n+1$, *then for any* $t \in I_\omega$, *we always have*

$$G\left(t, e^{u_1}, \ldots, e^{u_n}, \int_{-\infty}^{t} c(t,s) e^{u_{n+1}} \Delta s\right) > 0,$$

$$G\left(t, e^{-u_1}, \ldots, e^{-u_n}, \int_{-\infty}^{t} c(t,s)e^{-u_{n+1}}\,\Delta s\right) < 0$$

or

$$G\left(t, e^{u_1}, \ldots, e^{u_n}, \int_{-\infty}^{t} c(t,s)e^{u_{n+1}}\,\Delta s\right) < 0,$$

$$G\left(t, e^{-u_1}, \ldots, e^{-u_n}, \int_{-\infty}^{t} c(t,s)e^{-u_{n+1}}\,\Delta s\right) > 0.$$

Then Eq. (3.1) *has at least one ω-periodic solution.*

4. Some specific cases

Next, we consider the scalar equation on a time scale

$$x^{\Delta}(t) = G\bigl(t, \exp\bigl\{x\bigl(g(t)\bigr)\bigr\}\bigr), \tag{4.1}$$

where $G : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{T} \to \mathbb{T}$ are ω-periodic.

Following a similar strategy as in [5], one can reach the following conclusion.

Theorem 4.1. *Assume that there exist constants $B, \alpha, \beta > 0$ such that*

(H_6) *if $|x| \leqslant B$, then $\int_{\kappa}^{\kappa+\omega} |G(t, e^x)|\Delta t < \beta$; and if $|x| > B$, then $xG(t, e^x) > 0$;*
(H_7) *if $x < -B$, then $\int_{\kappa}^{\kappa+\omega} G(t, e^x)\Delta t > -\alpha$; or if $x > B$, then $\int_{\kappa}^{\kappa+\omega} G(t, e^x)\Delta t \leqslant \alpha$.*

Then Eq. (4.1) *has at least one ω-periodic solution.*

Finally, in order to illustrate some features of our main theorems, let us consider the specific dynamic equations

$$N^{\Delta}(t) = a(t) - \sum_{i=1}^{n} b_i(t)\exp\bigl\{N\bigl(g_i(t)\bigr)\bigr\} - \int_{-\infty}^{t} c(t,s)\exp\bigl\{N(s)\bigr\}\Delta s, \tag{4.2}$$

$$N^{\Delta}(t) = a(t) - \prod_{i=1}^{n} b_i(t)\exp\bigl\{N\bigl(g_i(t)\bigr)\bigr\}, \tag{4.3}$$

$$N^{\Delta}(t) = r(t)\frac{K(t) - \exp\{N(g(t))\}}{K(t) + c(t)\exp\{N(g(t))\}}, \tag{4.4}$$

$$N^{\Delta}(t) = r(t) - \sum_{i=1}^{n} \frac{a_i(t)\exp\{N(g_i(t))\}}{1 + c_i(t)\exp\{N(g_i(t))\}}, \tag{4.5}$$

$$N^{\Delta}(t) = a(t) + b(t)\exp\bigl\{pN\bigl(g(t)\bigr)\bigr\} - c(t)\exp\bigl\{qN\bigl(g(t)\bigr)\bigr\}, \tag{4.6}$$

$$N^{\Delta}(t) = r(t) - \frac{\exp\{\theta N(g(t))\}}{K(t)^{\theta}}, \tag{4.7}$$

where $a, a_i, b, b_i, c, c_i, r, K : \mathbb{T} \to \mathbb{R}$ are rd-continuous ω-periodic functions such that $\bar{a} > 0$, $c(t) > 0$, $a_i(t) \geqslant 0$, $b_i(t) \geqslant 0$, $c_i(t) \geqslant 0$, $K(t) > 0$, $\bar{r} > 0$ in (4.5) and (4.7), $r(t) > 0$ in (4.4), $g : \mathbb{T} \to \mathbb{T}$ and $g_i : \mathbb{T} \to \mathbb{T}$ are ω-periodic, and p, q, θ are positive constants with $q > p$, moreover, $c : \mathbb{T} \times \mathbb{T} \to \mathbb{R}^+$ satisfies $c(t + \omega, s + \omega) = c(t, s)$, and $\int_{-\infty}^{t} c(t, s)\Delta s$ is rd-continuous in t.

By Theorems 3.1, 3.2 and 4.1, one can easily reach the following claims.

Theorem 4.2. *Each of* (4.2)–(4.7) *has at least one ω-periodic solution.*

Remark 4.1. Let $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$ and $\tilde{x}(t) = \exp\{x(t)\}$. Then the dynamic equations (4.2)–(4.7) reduce to the well-known continuous or discrete time nonautonomous logistic equation with several deviating arguments [5,12,16], multiplicative logistic type equation with several deviating arguments [9,10,15–17], generalized food-limited population model with deviating argument [4–6], Michaelis–Menton type single species growth model with several deviating arguments [5,12,13], Lotka–Volterra type single species growth model with deviating arguments [5,14,16], and nonautonomous Gilpin–Ayala single species model [5,8], respectively, which have been studied extensively in the literature.

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