Proceedings of the Edinburgh Mathematical Society (2003) **46**, 627–636 ©
DOI:10.1017/S0013091502001086 Printed in the United Kingdom

POSITIVE SEMIDEFINITENESS OF DISCRETE QUADRATIC FUNCTIONALS

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(Received 25 November 2002)

Abstract We consider symplectic difference systems, which contain as special cases linear Hamiltonian difference systems and Sturm–Liouville difference equations of any even order. An associated discrete quadratic functional is important in discrete variational analysis, and while its positive definiteness has been characterized and is well understood, a characterization of its positive semidefiniteness remained an open problem. In this paper we present the solution to this problem and offer necessary and sufficient conditions for such discrete quadratic functionals to be non-negative definite.

Keywords: Hamiltonian systems; symplectic systems; discrete quadratic functionals; non-negativity

2000 *Mathematics subject classification:* Primary 39A12; 39A13
Secondary 34B24; 49K99

1. Introduction and main results

The so-called Reid Roundabout Theorem for Hamiltonian differential systems has been well known for a long time (cf. [**9**] or [**7**, Theorem 2.4.1]). It characterizes the positivity of a corresponding (continuous) quadratic functional by the disconjugacy (non-oscillation) of the differential system. This result was carried over to discrete quadratic functionals and corresponding Hamiltonian difference systems in [**2**, Theorem 2], and it was proven for more general symplectic difference systems in [**3**]. In contrast to the continuous case, the discrete results do not require us to assume controllability, which was shown in [**4**]. The characterization of non-negativity rather than positivity of quadratic functionals remained an open problem, and its solution is the content of this paper. There exists a recent approach to this problem in [**5**] (see also the references given there), but along different lines.

Here we consider *symplectic difference systems*

$$x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \quad u_{k+1} = \mathcal{C}_k x_k + \mathcal{D}_k u_k, \quad k \in [0, N] \cap \mathbb{Z}, \tag{1.1}$$

where $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$, $\mathcal{D}_k$ are real $n \times n$ matrices, $x_k, u_k \in \mathbb{R}^n$, and $N \in \mathbb{N}$. We will assume throughout that the $2n \times 2n$ matrices

$$\mathcal{S}_k := \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix}$$

are *symplectic*, i.e.

$$\mathcal{S}_k^{\mathrm{T}} \mathcal{J} \mathcal{S}_k = \mathcal{J} \quad \text{with } \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{1.2}$$

where I denotes the $n \times n$ identity matrix. Using this notation with the 'big' matrices $\mathcal{S}_k$ and putting $z_k = \binom{x_k}{u_k} \in \mathbb{R}^{2n}$, the difference system (1.1) is the same as

$$z_{k+1} = \mathcal{S}_k z_k, \quad k \in [0, N] \cap \mathbb{Z}. \tag{1.3}$$

The symplecticity, i.e. (1.2), implies that $\mathcal{S}_k$ is invertible with

$$\mathcal{S}_k^{-1} = \begin{pmatrix} \mathcal{D}_k^{\mathrm{T}} & -\mathcal{B}_k^{\mathrm{T}} \\ -\mathcal{C}_k^{\mathrm{T}} & \mathcal{A}_k^{\mathrm{T}} \end{pmatrix}.$$

This fact and (1.2) are equivalent to the following formulae:

$$\begin{gathered} \mathcal{A}_k^{\mathrm{T}} \mathcal{C}_k,\ \mathcal{D}_k^{\mathrm{T}} \mathcal{B}_k,\ \mathcal{A}_k \mathcal{B}_k^{\mathrm{T}},\ \mathcal{C}_k \mathcal{D}_k^{\mathrm{T}} \text{ are symmetric,} \\ \mathcal{A}_k^{\mathrm{T}} \mathcal{D}_k - \mathcal{C}_k^{\mathrm{T}} \mathcal{B}_k = \mathcal{A}_k \mathcal{D}_k^{\mathrm{T}} - \mathcal{B}_k \mathcal{C}_k^{\mathrm{T}} = I. \end{gathered}$$

By [**3**, Lemma 1] the symplectic system (1.1) is a Hamiltonian difference system if and only if the matrices $\mathcal{A}_k$ are invertible.

Moreover, we deal with the corresponding *discrete quadratic functional*

$$\mathcal{F}(z) = \sum_{k=0}^{N} \{x_k^{\mathrm{T}} \mathcal{A}_k^{\mathrm{T}} \mathcal{C}_k x_k + u_k^{\mathrm{T}} \mathcal{D}_k^{\mathrm{T}} \mathcal{B}_k u_k + 2 x_k^{\mathrm{T}} \mathcal{C}_k^{\mathrm{T}} \mathcal{B}_k u_k\} \tag{1.4}$$

for *admissible sequences*

$$z = (z_k)_{k=0}^{N+1} = \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} x_k \\ u_k \end{pmatrix}_{k=0}^{N+1},$$

i.e. z satisfies the first equation of (1.1), the so-called *equation of motion*, and the Dirichlet boundary conditions, more precisely:

$$x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k \quad \text{for } k \in [0, N] \cap \mathbb{Z}, \quad x_0 = x_{N+1} = 0. \tag{1.5}$$

Note also that

$$\mathcal{F}(z) = \sum_{k=0}^{N} z_k^{\mathrm{T}} \{\mathcal{S}_k^{\mathrm{T}} \mathcal{K} \mathcal{S}_k - \mathcal{K}\} z_k \quad \text{with } \mathcal{K} = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}.$$

We need some further *notation*. By $M^\dagger$ we denote the *Moore–Penrose inverse* of a matrix M (cf. [**1**]). For a real and symmetric matrix P we write $P \geqslant 0$ if P is non-negative definite. By $\operatorname{Ker} M$, $\operatorname{image} M$, $\operatorname{rank} M$, M^{T} and M^{-1} we denote the kernel, image, rank, transpose and inverse of a matrix M, respectively. We shall deal only with the so-called *principal solution*

$$\begin{pmatrix} X \\ U \end{pmatrix} = \begin{pmatrix} X_k \\ U_k \end{pmatrix}_{k=0}^{N+1}$$

of (1.1) at 0, i.e. X_k and U_k are real $n \times n$ matrices which satisfy

$$X_{k+1} = \mathcal{A}_k X_k + \mathcal{B}_k U_k, \quad U_{k+1} = \mathcal{C}_k X_k + \mathcal{D}_k U_k, \quad k \in [0, N] \cap \mathbb{Z}, \tag{1.6}$$

and the initial conditions

$$X_0 = 0 \quad \text{and} \quad U_0 = I. \tag{1.7}$$

Then $\binom{X}{U}$ is a *conjoined basis* of (1.1) (cf. [**2**, Definition 1]), i.e. it satisfies $\operatorname{rank}(X_k^{\mathrm{T}}, U_k^{\mathrm{T}}) = n$ and $X_k^{\mathrm{T}} U_k = U_k^{\mathrm{T}} X_k$ for $0 \leqslant k \leqslant N+1$. It follows from (1.2) that

$$X_k = \mathcal{D}_k^{\mathrm{T}} X_{k+1} - \mathcal{B}_k^{\mathrm{T}} U_{k+1}, \quad U_k = -\mathcal{C}_k^{\mathrm{T}} X_{k+1} + \mathcal{A}_k^{\mathrm{T}} U_{k+1} \quad \text{for } k \in [0, N] \cap \mathbb{Z}. \tag{1.8}$$

The following related matrices were introduced in [**8**], but note that we here use P instead of D:

$$\left.\begin{aligned} M_k &= (I - X_{k+1} X_{k+1}^\dagger) \mathcal{B}_k, & T_k &= I - M_k^\dagger M_k, \\ P_k &= T_k^{\mathrm{T}} X_k X_{k+1}^\dagger \mathcal{B}_k T_k, & \tilde{P}_k &= \mathcal{B}_k^{\mathrm{T}} \mathcal{D}_k - \mathcal{B}_k^{\mathrm{T}} Q_{k+1} \mathcal{B}_k, \end{aligned}\right\} \tag{1.9}$$

for $0 \leqslant k \leqslant N$, where Q_k denotes a symmetric matrix with

$$Q_k X_k = U_k X_k^\dagger X_k, \tag{1.10}$$

and, by [**2**, Lemma 2], we may choose

$$Q_k = U_k X_k^\dagger + (U_k X_k^\dagger \tilde{X}_k - \tilde{U}_k)(I - X_k^\dagger X_k) U_k^{\mathrm{T}},$$

where $\binom{X}{U}$ and $\binom{\tilde{X}}{\tilde{U}}$ are normalized conjoined bases of (1.1). If the matrices X_k are invertible, then the matrices Q_k satisfy a corresponding *Riccati difference system* (cf. [**2**, Lemma 2] or [**3**, Lemma 3]) $R_k[Q] = 0$, where the 'Riccati operator' R is defined by

$$R_k[Q] = Q_{k+1}(\mathcal{A}_k + \mathcal{B}_k Q_k) - (\mathcal{C}_k + \mathcal{D}_k Q_k). \tag{1.11}$$

Now we can formulate the *main result* of this paper.

Theorem 1.1. *Assume that the difference system (1.3) is symplectic, i.e. (1.2) holds, and let* $\binom{X}{U}$ *denote the principal solution of (1.1) at 0, i.e. (1.6) and (1.7) hold, and let* $\mathcal{F}$ *be defined by (1.4). Then* $\mathcal{F} \geqslant 0$*, i.e.* $\mathcal{F}(z) \geqslant 0$ *for every admissible sequence*

$$z = \begin{pmatrix} x_k \\ u_k \end{pmatrix}_{k=0}^{N+1},$$

i.e. z satisfies (1.5), if and only if the following two statements are true.

(i) $P_k \geqslant 0$ *for all* $0 \leqslant k \leqslant N$.

(ii) $x_k \in \operatorname{image} X_k$ *for all* $0 \leqslant k \leqslant N+1$ *and for every admissible sequence*

$$z = \begin{pmatrix} x_k \\ u_k \end{pmatrix}_{k=0}^{N+1}.$$

Remark 1.2. Note that, by [**8**, Lemma 1], the matrices P_k are always symmetric, and $M_k = 0$ (or equivalently $T_k = I$) if and only if $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$. Moreover, if $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$ for all $0 \leqslant k \leqslant N$, then statement (ii) is true (cf. [**2**, Remark 3 (iii)] or [**3**, Remark 1 (v)]). This kernel condition and statement (i) mean that $\binom{X}{U}$ has no focal point in the interval $(0, N+1]$ (cf. [**2**, Definition 3], [**3**, Definition 3] or [**8**, Definition 1]), and these two conditions are equivalent to the positivity of $\mathcal{F}$ by the Reid Roundabout Theorem (cf. [**2**, Theorem 2] or [**3**, Theorem 1]). Of course, our statement (ii) does not imply the kernel condition, and therefore our statements (i) and (ii) do not imply, for example, that $\binom{X}{U}$ has no focal points in the *open* interval $(0, N+1)$ (cf. [**8**, Definition 1]), as one might have expected in analogy to the continuous case [**7**, Remark 2.4.2]. This will be discussed in § 3 below (Corollary 3.1 and Remark 3.3). But the continuous results require *controllability* of the system, which is not needed here and also not needed for the discrete result on positivity.

The rest of this paper deals with the proof of Theorem 1.1. In the next section we show that (i) and (ii) imply $\mathcal{F} \geqslant 0$, using mainly a Generalized Picone Identity, i.e. Proposition 2.1. In the final section (§ 3) we prove the other direction by constructing examples with $\mathcal{F}(z) < 0$, if (i) or (ii) is violated.

2. Non-negativity

We need the following result (for special cases see [**2**, Lemma 2] or [**3**, Lemma 2]).

Proposition 2.1 (Generalized Picone Identity). *We use the notation presented in § 1 and assume (1.6), (1.2), (1.7) and (1.10). Let* $k \in [0, N] \cap \mathbb{Z}$ *and suppose that* $x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k$, $s_k = u_k - Q_k x_k$. *The following identities then hold.*

(i) $P_k = T_k^{\mathrm{T}} \tilde{P}_k T_k$.

(ii) $X_{k+1}^{\mathrm{T}} R_k[Q] X_k = 0$.

(iii) $M_k s_k = 0$ *if* $x_k \in \operatorname{image} X_k$ *and* $x_{k+1} \in \operatorname{image} X_{k+1}$.

(iv) $\mathcal{F}_k := x_k^{\mathrm{T}} \mathcal{A}_k^{\mathrm{T}} \mathcal{C}_k x_k + u_k^{\mathrm{T}} \mathcal{D}_k^{\mathrm{T}} \mathcal{B}_k u_k + 2 x_k^{\mathrm{T}} \mathcal{C}_k^{\mathrm{T}} \mathcal{B}_k u_k = x_{k+1}^{\mathrm{T}} Q_{k+1} x_{k+1} - x_k^{\mathrm{T}} Q_k x_k + s_k^{\mathrm{T}} P_k s_k$ *if* $x_k \in \operatorname{image} X_k$ *and* $x_{k+1} \in \operatorname{image} X_{k+1}$.

Proof. First, the formulae (1.9) imply that

$$\tilde{P}_k = \mathcal{B}_k^{\mathrm{T}} \mathcal{D}_k + M_k^{\mathrm{T}} Q_{k+1} M_k - \mathcal{B}_k^{\mathrm{T}} X_{k+1} X_{k+1}^{\dagger} Q_{k+1} X_{k+1} X_{k+1}^{\dagger} \mathcal{B}_k - \mathcal{B}_k^{\mathrm{T}} Q_{k+1} M_k - M_k^{\mathrm{T}} Q_{k+1} \mathcal{B}_k,$$

and we obtain from (1.8) and (1.10) that (using properties of the Moore–Penrose inverses)

$$\begin{aligned}\mathcal{B}_k^{\mathrm{T}}\mathcal{D}_k - \mathcal{B}_k^{\mathrm{T}}(X_{k+1}^{\dagger})^{\mathrm{T}}X_{k+1}^{\mathrm{T}}U_{k+1}X_{k+1}^{\dagger}\mathcal{B}_k \\ &= M_k^{\mathrm{T}}U_{k+1}X_{k+1}^{\dagger}\mathcal{B}_k + \mathcal{D}_k^{\mathrm{T}}M_k + (\mathcal{D}_k^{\mathrm{T}}X_{k+1} - \mathcal{B}_k^{\mathrm{T}}U_{k+1})X_{k+1}^{\dagger}\mathcal{B}_k \\ &= M_k^{\mathrm{T}}U_{k+1}X_{k+1}^{\dagger}\mathcal{B}_k + \mathcal{D}_k^{\mathrm{T}}M_k + X_kX_{k+1}^{\dagger}\mathcal{B}_k.\end{aligned}$$

Hence $T_k^{\mathrm{T}}\tilde{P}_kT_k = P_k$ because $M_kT_k = 0$, which proves (i).

To show (ii), note that

$$\begin{aligned}X_{k+1}^{\mathrm{T}}R_k[Q]X_k &= X_{k+1}^{\mathrm{T}}\{Q_{k+1}(\mathcal{A}_kX_k + \mathcal{B}_kU_k) - (\mathcal{C}_kX_k + \mathcal{D}_kU_k)\}X_k^{\dagger}X_k \\ &= \{U_{k+1}^{\mathrm{T}}X_{k+1} - X_{k+1}^{\mathrm{T}}U_{k+1}\}X_k^{\dagger}X_k = 0\end{aligned}$$

follows from (1.6), (1.10), (1.11), and the properties of Moore–Penrose inverses.

We have $x_\nu = X_\nu X_\nu^{\dagger}x_\nu$ for $\nu \in \{k, k+1\}$. Therefore

$$\begin{aligned}M_ku_k &= (I - X_{k+1}X_{k+1}^{\dagger})(x_{k+1} - \mathcal{A}_kx_k) \\ &= (I - X_{k+1}X_{k+1}^{\dagger})(X_{k+1}X_{k+1}^{\dagger}x_{k+1} - \mathcal{A}_kX_kX_k^{\dagger}x_k) \\ &= (I - X_{k+1}X_{k+1}^{\dagger})(\mathcal{B}_kU_k - X_{k+1})X_k^{\dagger}x_k \\ &= M_kU_kX_k^{\dagger}X_kX_k^{\dagger}x_k = M_kQ_kx_k,\end{aligned}$$

and hence (iii) holds.

Finally we show (iv). It follows from [**3**, Lemma 2] (see also [**2**, p. 812]) that

$$\begin{aligned}\tilde{\mathcal{F}}_k &:= -2u_k^{\mathrm{T}}\mathcal{B}_kR_k[Q]x_k + x_k^{\mathrm{T}}\{Q_k\mathcal{B}_k^{\mathrm{T}}R_k[Q] - R_k^{\mathrm{T}}[Q]\mathcal{A}_k\}x_k \\ &= \mathcal{F}_k - x_{k+1}^{\mathrm{T}}Q_{k+1}x_{k+1} + x_k^{\mathrm{T}}Q_kx_k - s_k^{\mathrm{T}}\tilde{P}_ks_k.\end{aligned}$$

Since $T_ks_k = s_k$ by (iii) and (1.9), we have that $s_k^{\mathrm{T}}\tilde{P}_ks_k = s_k^{\mathrm{T}}P_ks_k$ by (i). Moreover, using (ii),

$$u_k^{\mathrm{T}}\mathcal{B}_k^{\mathrm{T}}R_k[Q]x_k = (x_{k+1} - \mathcal{A}_kx_k)^{\mathrm{T}}R_k[Q]x_k = -x_k^{\mathrm{T}}\mathcal{A}_k^{\mathrm{T}}R_k[Q]x_k,$$

so that

$$\tilde{\mathcal{F}}_k = (\mathcal{A}_kx_k + \mathcal{B}_kQ_kx_k)^{\mathrm{T}}R_k[Q]x_k = x_k^{\mathrm{T}}(X_k^{\dagger})^{\mathrm{T}}X_{k+1}^{\mathrm{T}}R_k[Q]X_kX_k^{\dagger}x_k = 0,$$

which yields (iv). □

The next result shows one direction of Theorem 1.1, namely that (i) and (ii) imply $\mathcal{F} \geqslant 0$.

Proposition 2.2. *Under the assumptions of Theorem 1.1 suppose that statement (i) is true, and assume that*

$$z = \begin{pmatrix} x_k \\ u_k \end{pmatrix}_{k=0}^{N+1}$$

is admissible with $x_k \in \operatorname{image} X_k$ *for all* $2 \leqslant k \leqslant N$. *Then* $\mathcal{F}(z) \geqslant 0$.

Proof. Note that

$$x_0 = x_{N+1} = 0 \in \operatorname{image} X_0 = \{0\} \subset \operatorname{image} X_{N+1},$$

and

$$x_1 = \mathcal{B}_0 u_0 = X_1 u_0 \in \operatorname{image} X_1$$

for every admissible sequence z. It follows from the Generalized Picone Identity, Proposition 2.1 (iv), that

$$\begin{aligned}\mathcal{F}(z) &= \sum_{k=0}^{N} \{x_{k+1}^{\mathrm{T}} Q_{k+1} x_{k+1} - x_k^{\mathrm{T}} Q_k x_k + s_k^{\mathrm{T}} P_k s_k\} \\ &= \sum_{k=0}^{N} s_k^{\mathrm{T}} P_k s_k \\ &\geqslant 0.\end{aligned}$$

Hence $\mathcal{F} \geqslant 0$. □

3. Construction of examples

In the first two parts of this section we show the other direction of Theorem 1.1. First, we consider the case where statement (i) is violated and construct an admissible z with $\mathcal{F}(z) < 0$. Then we consider the case where statement (ii) is violated and also construct an admissible z with $\mathcal{F}(z) < 0$. We conclude this section with some remarks and consequences of Theorem 1.1.

3.1. Statement (i) is not true

Assume that $P_m \not\geqslant 0$ for some $m \in [1, N] \cap \mathbb{Z}$. Note that $P_0 = 0 \geqslant 0$ by (1.7) and (1.9). We use (with different notation) the construction in [**5**] (cf. also [**2**, p. 814]). To do so, let $c \in \mathbb{R}^n$ with $c^{\mathrm{T}} P_m c < 0$. We define $d := X_{m+1}^{\dagger} \mathcal{B}_m T_m c$ and

$$x_k := \begin{cases} X_k d & \text{for } 0 \leqslant k \leqslant m, \\ 0 & \text{for } m+1 \leqslant k \leqslant N+1, \end{cases}$$

$$u_k := \begin{cases} U_k d & \text{for } 0 \leqslant k \leqslant m-1, \\ U_k d - T_k c & \text{for } k = m, \\ 0 & \text{for } m+1 \leqslant k \leqslant N+1. \end{cases}$$

Then $x_0 = x_{N+1} = 0$, $x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k$ for $0 \leqslant k \leqslant N$ with $k \neq m$, and

$$\mathcal{A}_m x_m + \mathcal{B}_m u_m = X_{m+1} d - \mathcal{B}_m T_m c = 0 = x_{m+1},$$

because $X_{m+1} X_{m+1}^{\dagger} \mathcal{B}_m T_m = (\mathcal{B}_m - M_m) T_m = \mathcal{B}_m T_m$ by (1.9). Hence $z = \binom{x}{u}$ is admissible. Using [**3**, p. 711] and

$$\mathcal{C}_{m-1} x_{m-1} + \mathcal{D}_{m-1} u_{m-1} = U_m d = u_m + T_m c,$$

we obtain

$$\begin{aligned}\mathcal{F}(z) &= \sum_{k=0}^{N} x_{k+1}^{\mathrm{T}}\{\mathcal{C}_k x_k + \mathcal{D}_k u_k - u_{k+1}\}\\ &= x_m^{\mathrm{T}}\{\mathcal{C}_{m-1}x_{m-1} + \mathcal{D}_{m-1}u_{m-1} - u_m\}\\ &= c^{\mathrm{T}}T_m^{\mathrm{T}}X_m d = c^{\mathrm{T}}P_m c < 0.\end{aligned}$$

Hence $\mathcal{F} \not\geqslant 0$.

3.2. Statement (ii) is not true

Assume that there exists an admissible $z = \binom{x}{u}$ and $1 \leqslant m \leqslant N-1$ such that $x_m \in \operatorname{image} X_m$, but $x_{m+1} \notin \operatorname{image} X_{m+1}$. We put $x = x_{m+1}$, $X = X_m$, $x_m = X\alpha \in \operatorname{image} X_m$, $Y = X_{m+1}$, $U = U_{m+1}$ and $M = M_m$. Then $\operatorname{rank}(Y^{\mathrm{T}}, U^{\mathrm{T}}) = n$ and $Y^{\mathrm{T}}U = U^{\mathrm{T}}Y$, so that $K := Y^{\mathrm{T}}Y + U^{\mathrm{T}}U$ is invertible with

$$\begin{pmatrix} K^{-1}Y^{\mathrm{T}} & K^{-1}U^{\mathrm{T}} \\ -U^{\mathrm{T}} & Y^{\mathrm{T}} \end{pmatrix}\begin{pmatrix} Y & -UK^{-1} \\ U & YK^{-1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

(cf. the proof of [**7**, Corollary 3.3.9]). Then the matrix S', defined by

$$S' = K^{-1}U^{\mathrm{T}}M,$$

satisfies

$$YS' = 0 \quad \text{and} \quad US' = M \tag{3.1}$$

because

$$\begin{pmatrix} K^{-1}Y^{\mathrm{T}} & K^{-1}U^{\mathrm{T}} \\ -U^{\mathrm{T}} & Y^{\mathrm{T}} \end{pmatrix}\begin{pmatrix} 0 \\ M \end{pmatrix} = \begin{pmatrix} K^{-1}U^{\mathrm{T}}M \\ 0 \end{pmatrix},$$

since $Y^{\mathrm{T}}M = 0$ by (1.9).

Next we prove that

$$M^{\mathrm{T}}x \neq 0. \tag{3.2}$$

To this end, assume that $M^{\mathrm{T}}x = 0$. It follows that

$$\begin{aligned}0 &= \mathcal{B}_m^{\mathrm{T}}(I - YY^{\dagger})x\\ &= \mathcal{B}_m^{\mathrm{T}}(I - YY^{\dagger})(\mathcal{A}_k X\alpha + \mathcal{B}_m u_m)\\ &= \mathcal{B}_m^{\mathrm{T}}(I - YY^{\dagger})^2(Y\alpha + \mathcal{B}_m(u_m - U_m\alpha))\\ &= M^{\mathrm{T}}M(u_m - U_m\alpha).\end{aligned}$$

Hence,

$$\begin{aligned}0 &= M(u_m - U_m\alpha)\\ &= (I - YY^{\dagger})(\mathcal{B}_m u_m - \mathcal{B}_m U_m\alpha)\\ &= (I - YY^{\dagger})(x - Y\alpha) = (I - YY^{\dagger})x\end{aligned}$$

so that $x = YY^{\dagger}x$, which contradicts our assumption $x \notin \operatorname{image} Y$. Thus (3.2) holds.

Now we define a sequence

$$\tilde{z} = \begin{pmatrix} \tilde{x} \\ \tilde{u} \end{pmatrix} = \begin{pmatrix} \tilde{x}_k \\ \tilde{u}_k \end{pmatrix}_{k=0}^{N+1}$$

as follows:

$$\tilde{x}_k := \begin{cases} X_k(\alpha + \tilde{\alpha}) & \text{for } 0 \leqslant k \leqslant m, \\ x_k & \text{for } m+1 \leqslant k \leqslant N+1, \end{cases}$$

$$\tilde{u}_k := \begin{cases} U_k(\alpha + \tilde{\alpha}) & \text{for } 0 \leqslant k \leqslant m-1, \\ u_k + U_k\tilde{\alpha} & \text{for } k = m, \\ u_k & \text{for } m+1 \leqslant k \leqslant N+1, \end{cases}$$

where

$$\tilde{\alpha} := tS' M^{\mathrm{T}} x \text{ with a free parameter } t \in \mathbb{R}. \tag{3.3}$$

First, we show that $\tilde{z}$ is admissible: $\tilde{x}_0 = 0 = \tilde{x}_{N+1} = x_{N+1}$, $\tilde{x}_{k+1} = \mathcal{A}_k\tilde{x}_k + \mathcal{B}_k\tilde{u}_k$ for $0 \leqslant k \leqslant N$ with $k \neq m$, and

$$\mathcal{A}_m\tilde{x}_m + \mathcal{B}_m\tilde{u}_m = \mathcal{A}_m x_m + \mathcal{B}_m u_m + (\mathcal{A}_m X_m + \mathcal{B}_m U_m)\tilde{\alpha} = x_{m+1} + Y\tilde{\alpha} = x_{m+1}$$

because $Y\tilde{\alpha} = 0$ by (3.1) and (3.3). Hence $\tilde{z}$ is admissible. Next, using the same formula as in § 3.1, we get

$$\mathcal{F}(\tilde{z}) = \sum_{k=0}^{N} \tilde{x}_{k+1}^{\mathrm{T}}\{\mathcal{C}_k\tilde{x}_k + \mathcal{D}_k\tilde{u}_k - \tilde{u}_{k+1}\},$$

where $\mathcal{C}_k\tilde{x}_k + \mathcal{D}_k\tilde{u}_k - \tilde{u}_{k+1} = 0$ for $0 \leqslant k \leqslant m-2$ by (1.6),

$$\mathcal{C}_{m-1}\tilde{x}_{m-1} + \mathcal{D}_{m-1}\tilde{u}_{m-1} - \tilde{u}_m = U_m\alpha - u_m,$$

and

$$\mathcal{C}_m\tilde{x}_m + \mathcal{D}_m\tilde{u}_m - \tilde{u}_{m+1} = \mathcal{C}_m X_m\alpha + \mathcal{D}_m u_m + U\tilde{\alpha} - u_{m+1}.$$

Therefore, $\mathcal{F}(\tilde{z}) = \mathcal{F}_* + \mathcal{F}_{**}$, where the first summand

$$\mathcal{F}_* := \sum_{k=m+1}^{N} x_{k+1}^{\mathrm{T}}\{\mathcal{C}_k x_k + \mathcal{D}_k u_k - u_{k+1}\} + \alpha^{\mathrm{T}} X^{\mathrm{T}}(U_m\alpha - u_m) + x^{\mathrm{T}}(\mathcal{C}_m X\alpha + \mathcal{D}_m u_m - u_{m+1})$$

does not depend on $\tilde{\alpha}$, i.. not on the parameter t, and where

$$\mathcal{F}_{**} = \mathcal{F}_{**}(t) := \tilde{\alpha}^{\mathrm{T}} X^{\mathrm{T}}(U_m\alpha - u_m) + x^{\mathrm{T}} U\tilde{\alpha}.$$

Since $Y\tilde{\alpha} = 0$ by (3.1) and (3.3), we obtain from (1.8)

$$\begin{aligned} (U_m\alpha - u_m)^{\mathrm{T}} X\tilde{\alpha} &= (U_m\alpha - u_m)^{\mathrm{T}}(\mathcal{D}_m^{\mathrm{T}} Y - \mathcal{B}_m^{\mathrm{T}} U)\tilde{\alpha} \\ &= (\mathcal{B}_m u_m - \mathcal{B}_m U_m\alpha)^{\mathrm{T}} U\tilde{\alpha} \\ &= x^{\mathrm{T}} U\tilde{\alpha} - \alpha^{\mathrm{T}} Y^{\mathrm{T}} U\tilde{\alpha} = x^{\mathrm{T}} U\tilde{\alpha}. \end{aligned}$$

Hence $\mathcal{F}_{**}(t) = 2x^{\mathrm{T}} U\tilde{\alpha} = 2tx^{\mathrm{T}} US' M^{\mathrm{T}} x = 2t\|M^{\mathrm{T}} x\|^2$ by (3.1) and (3.3). Since $M^{\mathrm{T}} x \neq 0$ by (3.2), we obtain that $\mathcal{F}(\tilde{z}) = \mathcal{F}_* + 2t\|M^{\mathrm{T}} x\|^2 \to -\infty$ as $t \to -\infty$. Hence $\mathcal{F} \ngeqslant 0$, and this completes the proof of Theorem 1.1.

3.3. Some consequences

For every admissible sequence

$$z = \begin{pmatrix} x_k \\ u_k \end{pmatrix}_{k=0}^{N+1}$$

we have that $x_0 = 0 \in \operatorname{image} X_0$, $x_{N+1} = 0 \in \operatorname{image} X_{N+1}$, and $x_1 = \mathcal{B}_0 u_0 \in \operatorname{image} X_1$ because $X_1 = \mathcal{B}_0$ (see the proof of Proposition 2.2). Moreover, by (3.2), we have for $1 \leqslant k \leqslant N-1$ that

$$M_k^{\mathrm{T}} x_{k+1} \neq 0 \quad \text{if } x_k \in \operatorname{image} X_k \text{ and } x_{k+1} \notin \operatorname{image} X_{k+1}.$$

Therefore, condition (iii) of the following Corollary 3.1 implies condition (ii) of Theorem 1.1. The inverse implication is also true because

$$M_k^{\mathrm{T}} X_{k+1} = \mathcal{B}_k^{\mathrm{T}} (I - X_{k+1} X_{k+1}^{\dagger}) X_{k+1} = 0$$

so that $M_k^{\mathrm{T}} x_{k+1} = 0$ if $x_{k+1} \in \operatorname{image} X_{k+1}$. Hence, the subsequent Corollary 3.1 holds.

Corollary 3.1. *Assume (1.2), (1.6) and (1.7), and use the notation of Theorem 1.1. Then statement (ii) of Theorem 1.1 is true if and only if*

(iii) $M_k^{\mathrm{T}} x_{k+1} = 0$ *for all* $1 \leqslant k \leqslant N-1$ *and for every admissible sequence*

$$z = \begin{pmatrix} x_k \\ u_k \end{pmatrix}_{k=0}^{N+1}.$$

Next, if $\mathcal{B}_N$ is invertible, then every

$$z = \begin{pmatrix} x_k \\ u_k \end{pmatrix}_{k=0}^{N+1}$$

with $x_0 = 0$, $x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k$ for $0 \leqslant k \leqslant N$, and with $u_N := -\mathcal{B}_N^{-1} \mathcal{A}_N x_N$ is admissible. Hence, for $0 \leqslant k \leqslant N-1$, put $u_j = 0$ for $0 \leqslant j \leqslant k-1$ so that

$$x_0 = \cdots = x_k = 0, \qquad x_{k+1} = \mathcal{B}_k u_k.$$

Then condition (iii) of Corollary 3.1 implies that

$$0 = M_k^{\mathrm{T}} x_{k+1} = \mathcal{B}_k^{\mathrm{T}} (I - X_{k+1} X_{k+1}^{\dagger})^2 \mathcal{B}_k u_k = M_k^{\mathrm{T}} M_k u_k$$

for every $u_k \in \mathbb{R}^n$. Hence, $M_k = 0$, which means that $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$ for $0 \leqslant k \leqslant N-1$ by [**8**, Lemma 1]. Therefore, condition (iii) of Corollary 3.1 implies condition (iv) of the following Corollary 3.2, if $\mathcal{B}_N$ is invertible. The inverse implication is trivial. Therefore we have shown the following result.

Corollary 3.2. *Under the assumptions and notation of Theorem 1.1 or Corollary 3.1, suppose moreover that* $\mathcal{B}_N$ *is invertible. Then statement (ii) of Theorem 1.1 holds if and only if*

(iv) $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$ *for all* $0 \leqslant k \leqslant N-1$.

Note that condition (iii) does not imply $\operatorname{Ker} X_{N+1} \subset \operatorname{Ker} X_N$ in general (cf. [**6**, Proposition 1 or Theorem 2]).

Remark 3.3. Now, using the notion of focal points, Theorem 1.1 and Corollary 3.2 yield the following result. Under the assumptions of Theorem 1.1 suppose that $\mathcal{B}_N$ is invertible. Then the function $\mathcal{F}$ is non-negative if and only if the principal solution of the symplectic system possesses no focal point in the open interval $(0, N+1)$.

Acknowledgements. We thank the referee for pointing out the reference [**6**], which appeared when the paper was in the review process.

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