# QUALITATIVE ANALYSIS OF A SOLOW MODEL ON TIME SCALES

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ABSTRACT. In this paper, we further analyze the Solow model on time scales. This model was recently introduced by the authors and it combines the continuous and discrete Solow models and extends them to different time scales. Assuming constant labor force growth in the Solow model, we establish a comparison theorem. Then, under the more realistic assumption that the labor force growth rate is a monotonically decreasing function, we discuss the comparison theorem as well as stability and monotonicity of the solutions of the Solow model. The economic meanings are also indicated in some remarks.

## 1. INTRODUCTION

The neoclassical growth model, developed by Solow [17] and Swan [18], had a great impact on how the economists think about economic growth. Since then, it has stimulated an enormous amount of work [2, 12, 20]. Since differential equation systems are usually more easily handled than difference systems from the analytical point of view, some of the economic models have used continuous timing [1, 9, 13–15] while others are given in difference models because some people think economic data are collected at discrete intervals and transformation of capital into investments depends on the length of time lag, etc. [8, 10, 21].

Hence, in economic modeling, either continuous timing or discrete timing is present, and there is not a common view among economists on which representation of time is better for economic models [14]. Meanwhile, many results concerning differential equations may carry over quite easily to corresponding results for difference equations, while other results seems to be completely different in nature from their continuous counterparts [6].

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The blanket assumption that economic processes are either solely continuous or solely discrete, while convenient for traditional mathematical approaches, may sometimes be inappropriate, because in reality many economic phenomena do feature both continuous and discrete elements. In biology, a familiar example is a "seasonal breeding population in which generations do not overlap" [6, 19]. A similar typical example in economics is the "seasonally changing investment and revenue in which seasons play an important effect on this kind of economic activity". In addition, option pricing and stock dynamics in finance [11] and the frequency and duration of market trading in economics [19] also contain this hybrid continuous-discrete processes.

Therefore, there is a great need to find a more flexible mathematical framework to accurately model the dynamical blend of such systems, so that they are precisely described and better understood. To meet this requirement, an emerging, progressive and modern area of mathematics, known as "dynamic equations on time scales", has been introduced. This calculus has the capacity to act as the framework to effectively describe the above phenomena and to make advances in their associate fields, see e.g., [3–5, 19].

This theory was introduced by Stefan Hilger in 1988 in his Ph.D. thesis [16] in order to unify continuous and discrete analysis, and has been developed by many mathematicians. A time scale $\mathbb{T}$ is defined as any nonempty closed subset of $\mathbb{R}$. In the time scales setting, once a result is established, special cases include the result for the differential equation when the time scale is the set of all real numbers $\mathbb{R}$ and the result for the difference equation when the time scale is the set of all integers $\mathbb{Z}$. The induction principle plays an important rôle in the proofs of some of our results, so we give it here.

**Theorem 1.1** (See [6, Theorem 1.7]). *Let $t_0 \in \mathbb{T}$ and assume that*

$$\{S(t): \ t \in [t_0, \infty)\}$$

*is a family of statements satisfying:*

*A. The statement $S(t_0)$ is true.*
*B. If $t \in [t_0, \infty)$ is right-scattered and $S(t)$ is true, then $S(\sigma(t))$ is also true.*
*C. If $t \in [t_0, \infty)$ is right-dense and $S(t)$ is true, then there is a neighborhood $U$ of $t$ such that $S(r)$ is true for all $r \in U \cap (t, \infty)$.*
*D. If $t \in [t_0, \infty)$ is left-dense and $S(r)$ is true for all $r \in [t_0, t)$, then $S(\sigma(t))$ is true.*

*Then $S(t)$ is true for all $t \in [t_0, \infty)$.*

For other notations and a systematic introduction to time scales theory, we refer the reader to [6, 7].

## 2. The Solow Model on Time Scales

In this section, we will first recall some elements of the Solow model on general time scales as introduced by the authors in [5]. In the original Solow model [1, 17], the key elements are the production function, i.e., how the inputs of capital $K$ and labor $L$ are transformed into outputs, and how capital and labor force change over time. Here we still assume the following:

1. The production function $F$ satisfies:
   (a) $F(\lambda K, \lambda L) = \lambda F(K, L)$ for any $\lambda, K, L \in \mathbb{R}^+$ (constant return to scales);
   (b) $F(K, 0) = F(0, L) = 0$ for any $K, L \in \mathbb{R}^+$;
   (c) $\dfrac{\partial F}{\partial K} > 0$, $\dfrac{\partial F}{\partial L} > 0$, $\dfrac{\partial^2 F}{\partial K^2} < 0$, $\dfrac{\partial^2 F}{\partial L^2} < 0$;
   (d) $\lim\limits_{K \to 0^+} \dfrac{\partial F}{\partial K} = \lim\limits_{L \to 0^+} \dfrac{\partial F}{\partial L} = \infty$, $\lim\limits_{K \to \infty} \dfrac{\partial F}{\partial K} = \lim\limits_{L \to \infty} \dfrac{\partial F}{\partial L} = 0$ (Inada conditions).
2. The capital stock changes are equal to the gross investment $I = sF(K, L)$ minus the capital depreciation $\delta K$, where $s$ and $\delta$ are the savings rate and the depreciation factor of goods, respectively.
3. The labor force $L$ changes at a constant rate $n$.

The three assumptions give, for any $t \in \mathbb{T}$

$$\begin{cases} Y(t) &= F(K(t), L(t)), \\ K^\Delta(t) &= I(t) - \delta K(t), \\ I(t) &= sY(t), \\ L^\Delta(t) &= nL(t). \end{cases} \tag{2.1}$$

From (2.1), we obtain

$$K^\Delta(t) = sY(t) - \delta K(t) = sF(K(t), L(t)) - \delta K(t). \tag{2.2}$$

Define

$$k(t) := \frac{K(t)}{L(t)} \quad \text{and} \quad y(t) := \frac{Y(t)}{L(t)},$$

which are regarded as the capital stock per worker and the production per worker, respectively. Let

$$f(k) := F\left(\frac{K}{L}, 1\right) = F(k, 1)$$

be the production function in intensive form. Then condition 1 changes to

$$
\begin{cases} f(0) = 0; \\ f'(k) > 0 \quad \text{and} \quad f''(k) < 0 \quad \text{for all} \quad k \in \mathbb{R}^+ \\ \lim_{k \to 0^+} f'(k) = \infty \quad \text{and} \quad \lim_{k \to \infty} f'(k) = 0. \end{cases} \tag{2.3}
$$

Applying the time scales quotient rule [6, Theorem 1.20], we use (2.2) to find

$$
\begin{aligned}
k^{\Delta}(t) = \left(\frac{K}{L}\right)^{\Delta}(t) &= \frac{K^{\Delta}(t)L(t) - K(t)L^{\Delta}(t)}{L(t)L^{\sigma}(t)} \\
&= \frac{K^{\Delta}(t)}{L^{\sigma}(t)} - \frac{K(t)L^{\Delta}(t)}{L(t)L^{\sigma}(t)} \\
&= \frac{K^{\Delta}(t)}{L(t)(1+\mu(t)n)} - \frac{K(t)n}{L(t)(1+\mu(t)n)} \\
&= \frac{sF(K(t),L(t)) - \delta K(t)}{L(t)(1+\mu(t)n)} - \frac{K(t)n}{L(t)(1+\mu(t)n)} \\
&= \frac{s}{1+\mu(t)n} f(k(t)) - \frac{\delta + n}{1+\mu(t)n} k(t),
\end{aligned}
$$

i.e.,

$$
k^{\Delta}(t) = \frac{s}{1+\mu(t)n} f(k(t)) - \frac{\delta + n}{1+\mu(t)n} k(t), \tag{2.4}
$$

which describes the Solow model on time scales. When $\mathbb{T} = \mathbb{R}$, equation (2.4) is the continuous Solow model in [1], whereas when $\mathbb{T} = \mathbb{Z}$, equation (2.4) is the discrete Solow model discussed in [10].

Equation (2.4) has a nontrivial equilibrium, denoted by $\hat{k}_n$, which is the unique positive solution of the equation

$$
sf(k) = (\delta + n)k.
$$

For $n = 0$, we denote by $\hat{k}_0$ the nontrivial steady state of equation (2.4). Obviously, we have

$$
\lim_{n \to 0^+} \hat{k}_n = \hat{k}_0,
$$

and $\hat{k}_n$ increases to $\hat{k}_0$ as $n$ decreases to zero.

Next we will discuss some sufficient conditions for the existence and uniqueness of solutions of initial value problems for equation (2.4). Some comparison theorems between two solutions with different initial conditions will be given.

**Theorem 2.1.** *Assume* (2.3). *For* $t_0 \in \mathbb{T}$ *and* $k_0 \in \mathbb{R}^+$, *the initial value problem*

$$\begin{cases} k^{\Delta}(t) &= \dfrac{s}{1+\mu(t)n} f(k(t)) - \dfrac{\delta+n}{1+\mu(t)n} k(t), \\ k(t_0) &= k_0, \end{cases} \tag{2.5}$$

*has a unique solution on* $\mathbb{T}_{t_0}^+ = \{t \in \mathbb{T} : t \geq t_0\}$.

*Proof.* Let

$$u(t,k) = \frac{s}{1+\mu(t)n} f(k) - \frac{\delta+n}{1+\mu(t)n} k.$$

Then $u(\cdot, k)$ is rd-continuous and regressive on $\mathbb{T}$, and

$$\begin{aligned} |u(t,k_1) - u(t,k_2)| &= \left| \frac{\partial u}{\partial k}(t,\xi) \right| |k_1 - k_2| \\ &= \left| \frac{s}{1+\mu(t)n} f'(\xi) - \frac{\delta+n}{1+\mu(t)n} \right| |k_1 - k_2| \\ &\leq (sf'(t_0) + \delta + n)|k_1 - k_2|, \end{aligned}$$

where $\xi \in (k_1, k_2)$. With the theorem of global existence and uniqueness in [6, Theorem 8.20], we can deduce that the solution of the problem (2.5) exists uniquely. □

Hence in the following, with condition (2.3), we always have the existence and uniqueness of solutions for initial value problems (2.5).

**Theorem 2.2.** *Assume* (2.3) *and let* $\delta > 0$ *be such that* $-\delta \in \mathcal{R}^+$. *Let* $k_1$ *and* $k_2$ *be solutions of equation* (2.4) *on* $\mathbb{T}_{t_0}^+$ *with initial conditions* $k_1(t_0) = k_{01}$ *and* $k_2(t_0) = k_{02}$, *respectively. If* $0 < k_{01} < k_{02}$, *then*

$$k_1 < k_2 \quad on \quad \mathbb{T}_{t_0}^+.$$

*Proof.* We use the induction principle Theorem 1.1.

A. If $t = t_0$, then the result is obvious from the hypothesis.
B. If $t \in \mathbb{T}_{t_0}^+$ is right-scattered and $k_1(t) < k_2(t)$, then

$$\begin{aligned} k_1(\sigma(t)) - k_2(\sigma(t)) &= k_1(t) - k_2(t) + \mu(t)(k_1^{\Delta}(t) - k_2^{\Delta}(t)) \\ &= k_1(t) - k_2(t) + \frac{\mu(t)}{1+\mu(t)n}\left[sf(k_1(t)) - (\delta+n)k_1(t)\right] \\ &\quad - \frac{\mu(t)}{1+\mu(t)n}\left[sf(k_2(t)) - (\delta+n)k_2(t)\right] \\ &= k_1(t) - k_2(t) + \frac{s\mu(t)}{1+\mu(t)n}\left[f(k_1(t)) - f(k_2(t))\right] \end{aligned}$$

$$
\begin{aligned}
&-\frac{(\delta+n)\mu(t)}{1+\mu(t)n}\left[k_1(t)-k_2(t)\right] \\
=\;& \frac{1-\mu(t)\delta}{1+\mu(t)n}\left[k_1(t)-k_2(t)\right]+\frac{\mu(t)s}{1+\mu(t)n}\left[f(k_1(t))-f(k_2(t))\right] \\
<\;& 0,
\end{aligned}
$$

so

$$k_1(\sigma(t)) < k_2(\sigma(t)).$$

C. If $t$ is right-dense and $k_1(t) < k_2(t)$, then there exists a right neighborhood $\mathring{U}^+(t)\cap\mathbb{T}$ of $t$ such that $k_1(r) < k_2(r)$ for any $r \in \mathring{U}^+(t)\cap\mathbb{T}$. For if such a neighborhood does not exist, then there must exist a decreasing series $\{t_n\} \subset \mathring{U}^+(t)\cap\mathbb{T}$ and $\lim_{n\to\infty} t_n = t$, such that

$$k_1(t_n) \ge k_2(t_n)$$

and taking limit on both sides, we obtain $k_1(t) \ge k_2(t)$, a contradiction.

D. If $t$ is left-dense and $k_1(r) < k_2(r)$ for any $r \in [t_0, t)\cap\mathbb{T}$, then from the continuity of the solutions, $k_1(t) \le k_2(t)$. Uniqueness of solutions of initial value problems yields $k_1(t) < k_2(t)$.

Now an application of Theorem 1.1 concludes the proof. □

Note that Theorem 2.2 implies that the solution for the initial value problem (2.5) is always positive provided $k(t_0) > 0$.

**Corollary 2.3.** *Assume* (2.3) *and let* $\delta > 0$ *be such that* $-\delta \in \mathcal{R}^+$. *Then all solutions of equation* (2.4) *converge to the nontrivial steady state* $\hat{k}_n$ *monotonically, and the equilibrium point* $\hat{k}_n$ *is asymptotically stable and hence is a global attractor.*

*Proof.* If $k(t_0) < \hat{k}_n$, then Theorem 2.2 implies that

$$k(t) < \hat{k}_n \quad \text{for any} \quad t \in \mathbb{T}^+_{t_0}.$$

Hence

$$k^\Delta(t) = \frac{s}{1+\mu(t)n} f(k(t)) - \frac{\delta+n}{1+\mu(t)n} k(t) > 0 \quad \text{for any} \quad t \in \mathbb{T}^+_{t_0}.$$

This means that $k$ is increasing to the equilibrium point. Similar arguments apply to the case with $k(t_0) > \hat{k}_n$, which concludes the proof. □

*Remark* 2.4. Corollary 2.3 means that for two countries or districts with constant population growth rates, the one with the smaller population growth rate has a bigger capital per worker in the long run.

## 3. Improved Solow Model on Time Scales

In Section 2, we assumed that the labor force $L$ grows at a constant rate $n$ on the time scale, i.e.,

$$L^{\Delta}(t) = nL(t), \tag{3.1}$$

which implies that the labor force grows exponentially, that is,

$$L(t) = L_0 e_n(t, t_0),$$

where $L_0$ is the initial labor level at $t_0 \in \mathbb{T}$. With the properties of the exponential function on time scales and the fact that $n > 0$, we have

$$\lim_{t\to\infty} L(t) = \infty.$$

This means the labor force approaches infinity when $t$ goes to infinity, which is unrealistic, because in reality the environment has a carrying capacity. So the simple growth model of labor in equation (3.1) can provide an adequate approximation to such growth only for an initial period, but does not accommodate growth reductions due to competition for environmental resources such as food, habitat and the policy factor etc. [1].

Since the 1950s, developing countries have recognized that the high population growth rate has seriously hampered the economic growth and adopted the population control policy. As a result, the population growth rates of many countries decreased fast in the last 40 years, such as in China. Also due to the aging of the population and, consequently, a dramatic increase in the number of deaths, the population growth rate decreased below zero in some developed countries, and is projected to decrease to zero during the next few decades in the developing countries [1].

So to incorporate the numerical upper bound on the growth size, on the reference of [10], we revise Condition 3 from Section 2 as follows.

3.′ The labor force $L$ satisfies the following properties:

(a) The population is strictly increasing and bounded, i.e.,

$$L > 0,\ L^{\Delta} > 0 \quad \text{on} \quad \mathbb{T}^{+}_{t_0} \quad \text{and} \quad \lim_{t\to\infty} L(t) = L_{\infty} < \infty.$$

(b) The population growth rate is decreasing to 0, i.e.,

$$\text{If} \quad n = \frac{L^{\Delta}}{L}, \quad \text{then} \quad \lim_{t\to\infty} n(t) = 0 \quad \text{and} \quad n^{\Delta} < 0 \quad \text{on} \quad \mathbb{T}^{+}_{t_0}.$$

Hence equation (2.4) takes the form

$$k^{\Delta}(t) = \frac{s}{1+\mu(t)n(t)} f(k(t)) - \frac{\delta + n(t)}{1+\mu(t)n(t)} k(t). \tag{3.2}$$

Note that this is a nonautonomous dynamic equation on a time scale. Next we give the theorem of existence and uniqueness for solutions of initial value problems for (3.2).

**Theorem 3.1.** *Assume* (2.3). *For* $t_0 \in \mathbb{T}$ *and* $k_0 \in \mathbb{R}^+$, *the initial value problem*

$$\begin{cases} k^\Delta(t) &= \dfrac{s}{1+\mu(t)n(t)} f(k(t)) - \dfrac{\delta + n(t)}{1+\mu(t)n(t)} k(t) \\ k(t_0) &= k_0, \end{cases} \tag{3.3}$$

*has a unique solution on* $\mathbb{T}^+_{t_0}$.

*Proof.* Following the same way as in the proof of Theorem 2.1, we let

$$u(t,k) = \frac{s}{1+\mu(t)n(t)} f(k) - \frac{\delta + n(t)}{1+\mu(t)n(t)} k.$$

So $u(\cdot, k)$ is rd-continuous and regressive, and

$$\left| \frac{\partial u}{\partial k}(t,\xi) \right| \le s f'(k_0) + \delta + n(t_0).$$

From the theorem of global existence and uniqueness in [6], the solution of the problem (3.3) exists uniquely. □

**Theorem 3.2.** *Assume* (2.3) *and let* $\delta > 0$ *be such that* $-\delta \in \mathcal{R}^+$. *Let* $k_1$ *and* $k_2$ *be solutions of equation* (3.2) *with initial conditions* $k_1(t_0) = k_{01}$ *and* $k_2(t_0) = k_{02}$, *respectively. If* $0 < k_{01} < k_{02}$, *then*

$$k_1 < k_2 \quad \text{on} \quad \mathbb{T}^+_{t_0}.$$

*Proof.* The proof is similar to the proof of Theorem 2.2. □

*Remark* 3.3. Theorem 3.2 means that if two economies have the same fundamentals, then the one with the bigger initial capital per worker will always have the bigger capital per worker for ever on any time scale. The result in Theorem 3.2 includes the results in [1] and [10] as special cases.

**Theorem 3.4.** *Assume* (2.3) *and let* $\delta > 0$ *be such that* $-\delta \in \mathcal{R}^+$. *Let* $k_1$ *and* $k_2$ *be solutions of the dynamic equations on the same time scale*

$$k^\Delta(t) = \frac{s}{1+\mu(t)n_1(t)} f(k(t)) - \frac{\delta + n_1(t)}{1+\mu(t)n_1(t)} k(t) =: u(k(t),t) \tag{3.4}$$

*and*

$$k^\Delta(t) = \frac{s}{1+\mu(t)n_2(t)} f(k(t)) - \frac{\delta + n_2(t)}{1+\mu(t)n_2(t)} k(t) =: v(k(t),t), \tag{3.5}$$

*respectively, with the same initial condition* $k_1(t_0) = k_2(t_0)$*. If* $n_1 < n_2$ *on* $\mathbb{T}^+_{t_0}$*, then*

$$k_1 \geq k_2 \quad on \quad \mathbb{T}^+_{t_0}.$$

*Proof.* From $n_1(t) < n_2(t)$, we have $u(k(t), t) > v(k(t), t)$ for all $t \in \mathbb{T}^+_{t_0}$. Let $z := k_1 - k_2$. Obviously, we have $z(t_0) = k_1(t_0) - k_2(t_0) = 0$ and

$$z^{\Delta}(t_0) = k_1^{\Delta}(t_0) - k_2^{\Delta}(t_0) = u(k_1(t_0), t_0) - v(k_2(t_0), t_0) > 0.$$

So $z$ is right-increasing at $t_0$, i.e., if $t_0$ is right-scattered, then we have $z(\sigma(t_0)) > z(t_0) = 0$; if $t_0$ is right-dense, then there exists a nonempty neighborhood $\mathring{U}^+(t_0) \cap \mathbb{T}$ of $t_0$ such that $z(t) > 0$ for any $t \in \mathring{U}^+(t_0) \cap \mathbb{T}$. We now show that $z \geq 0$ holds on $\mathbb{T}^+_{t_0}$. If this is not the case, then there must be a point $t_1 > t_0$, $t_1 \in \mathbb{T}$ such that $z(t_1) < 0$ and $z(t) \geq 0$ when $t \in (t_0, t_1) \cap \mathbb{T}$. If $t_1$ is left-dense, then continuity of $z$ gives that $z(t_1) \geq 0$, which contradicts the assumption. Hence $t_1$ is left-scattered. Let $\rho(t_1) = t_2$. Then $z(t_2) \geq 0$, i.e., $k_1(t_2) \geq k_2(t_2)$. Let $k_2'$ be the solution of equation (3.5) satisfying the initial condition $k_2'(t_2) = k_1(t_2)$. From the discussion in the beginning of this proof, we obtain that $k_1 - k_2'$ is also right-increasing at $t_2$, i.e.,

$$k_1(t) > k_2'(t) \quad \text{for} \quad t \in \mathring{U}^+(t_2) \cap \mathbb{T}, \tag{3.6}$$

where $\mathring{U}^+(t_2) \cap \mathbb{T}$ is a nonempty right neighborhood of $t_2$ (at least including $t_1$). Taking into account that $k_2(t_2) \leq k_2'(t_2)$, Theorem 3.2 gives

$$k_2(t) \leq k_2'(t) \quad \text{for all} \quad t \in \mathbb{T}^+_{t_2}. \tag{3.7}$$

From (3.6) and (3.7), we have

$$k_1(t) > k_2(t) \quad \text{for} \quad t \in \mathring{U}^+(t_2) \cap \mathbb{T},$$

and thus $k_1(t_1) > k_2(t_1)$, which contradicts the fact $z(t_1) < 0$. This concludes the proof. □

*Remark* 3.5. Theorem 3.4 implies that, on any economic domain, for two economies with the same initial capital per worker, the economy with the smaller population growth rate will always have the bigger capital per worker on any time scale. The result here also includes the results in [1, 13] and [10] as special cases.

**Theorem 3.6.** *Assume* (2.3) *and let* $\delta > 0$ *be such that* $-\delta \in \mathcal{R}^+$*. If* $k$ *solves* (3.2)*, then* $\lim_{t\to\infty} k(t) = \hat{k}_0$*.*

*Proof.* We want to prove that for any $\varepsilon > 0$, there exists $T > 0$, such that if $t > T$, $t \in \mathbb{T}$, we have $|k(t) - \hat{k}_0| < \varepsilon$. Now let $\varepsilon > 0$. Since

$$\lim_{n \to 0^+} \hat{k}_n = \hat{k}_0,$$

we know that there exists $\overline{n} > 0$ such that

$$|\hat{k}_n - \hat{k}_0| < \frac{\varepsilon}{3} \quad \text{for all} \quad n \in (0, \overline{n}).$$

Let $t_1 \in \mathbb{T}_{t_0}^+$ such that $n_{t_1} = n(t_1) < \overline{n}$, i and let $k_{n_{t_1}}$ and $k_0$ be the solutions of

$$k^\Delta(t) = \frac{s}{1 + \mu(t) n_{t_1}} f(k(t)) - \frac{\delta + n_{t_1}}{1 + \mu(t) n_{t_1}} k(t)$$

and

$$k^\Delta(t) = s f(k(t)) - \delta k(t),$$

respectively, with the initial conditions

$$k_{n_{t_1}}(t_1) = k_0(t_1) = k(t_1).$$

Then Theorem 3.4 implies that

$$k_{n_{t_1}}(t) \le k(t) \le k_0(t) \quad \text{for all} \quad t \in \mathbb{T}_{t_1}^+.$$

Since $\lim_{t \to \infty} k_0(t) = \hat{k}_0$, there exists $T_1 > 0$ such that

$$|k_0(t) - \hat{k}_0| < \frac{\varepsilon}{3} \quad \text{for all} \quad t > T_1.$$

Moreover, since $\lim_{t \to \infty} k_{n_{t_1}}(t) = \hat{k}_{n_{t_1}}$, there exists $T_2 > 0$ such that

$$|k_{n_{t_1}}(t) - \hat{k}_{n_{t_1}}| < \frac{\varepsilon}{3} \quad \text{for all} \quad t > T_2.$$

Hence for $t > T := \max\{T_1, T_2, t_1\}$, we have

$$\hat{k}_0 - \frac{2}{3}\varepsilon < \hat{k}_{n_{t_1}} - \frac{\varepsilon}{3} < k_{n_{t_1}}(t) \le k(t) \le k_0(t) < \hat{k}_0 + \frac{\varepsilon}{3},$$

which implies that $|k(t) - \hat{k}_0| < \varepsilon$ for any $t \in \mathbb{T}_T^+$. □

*Remark* 3.7. Theorem 3.6 says that for any economic domain $\mathbb{T}$, the population growth rate $n(t)$ has no influence on the level of per worker output in the long run. That is, provided that the economy possesses a population growth rate strictly decreasing to zero, the capital per worker always converges to the positive steady state of the Solow model on a time scale with a population growth rate of zero.

**Theorem 3.8.** *Assume* (2.3) *and let* $\delta > 0$ *be such that* $-\delta \in \mathcal{R}^+$. *Then the solution* $k$ *of* (3.2) *with* $k(t_0) = k_0$ *is asymptotically stable.*

*Proof.* To prove the Lyapunov stability of $k$ in equation (3.2) with initial condition $k(t_0) = k_0$, we have to show that for any $\varepsilon > 0$, there exists $\eta > 0$ such that for any solution $q$ of equation (3.2) with initial condition $q(t_0) = q_0$ and such that $|k(t_0) - q(t_0)| < \eta$, we have

$$|k(t) - q(t)| < \varepsilon \quad \text{for any} \quad t \in \mathbb{T}^+_{t_0}.$$

Let $\varphi_1$ and $\varphi_2$ be the solutions of equation (3.2) with initial conditions $\varphi_1(t_0) = \frac{3}{2}k(t_0)$ and $\varphi_2(t_0) = \frac{1}{2}k(t_0)$, respectively. From Theorem 3.6, we have

$$\lim_{t\to\infty} \varphi_1(t) = \lim_{t\to\infty} \varphi_2(t) = \hat{k}_0 = \lim_{t\to\infty} k(t).$$

Thus, for any $\varepsilon > 0$, there exists $t_1 > t_0$, $t_1 \in \mathbb{T}^+_{t_0}$, such that

$$|\varphi_1(t) - k(t)| < \frac{\varepsilon}{2} \quad \text{and} \quad |\varphi_2(t) - k(t)| < \frac{\varepsilon}{2} \quad \text{for all} \quad t \in \mathbb{T}^+_{t_1}.$$

Let $q$ solve (3.2) with the initial condition $q_0 \in \left(\frac{1}{2}k(t_0), \frac{3}{2}k(t_0)\right)$. From Theorem 3.2, we have

$$\varphi_1(t) < q(t) < \varphi_2(t) \quad \text{for any} \quad t \in \mathbb{T}^+_{t_0}.$$

Thus

$$|q(t) - k(t)| < \varepsilon \quad \text{for any} \quad t \in \mathbb{T}^+_{t_1}.$$

Next we choose $\eta$ such that for any solution $q$ with initial value $q_0$, $|q_0 - k_0| < \eta$ implies $|k - q| < \varepsilon$ on $[t_0, t_1] \cap \mathbb{T}$. Following the proof of the theorem of continuous dependence on initial conditions, making use of the finite covering theorem, we can obtain that for any $\varepsilon > 0$, there exists $\eta < k_0/2$ such that $|q_0 - k_0| < \eta$ implies $|k(t) - q(t)| < \varepsilon$ for all $t \in [t_0, t_1] \cap \mathbb{T}$. From Theorem 3.6, for any solutions $k$ and $q$ of equation (3.2), we have that

$$\lim_{t\to\infty} k(t) = \lim_{t\to\infty} q(t) = \hat{k}_0,$$

and then

$$\lim_{t\to\infty} |q(t) - k(t)| = 0.$$

So the solution of equation (3.2) is asymptotically stable. □

*Remark* 3.9. Theorem 3.8 says that under the same fundamentals, if two economies operating on the same time domain have nearly the same initial capital per worker, the following capitals per worker will take on similar behavior.

Next we will present the monotonicity of the solutions of (3.2).

**Theorem 3.10.** *Assume* (2.3) *and let* $\delta > 0$ *be such that* $-\delta \in \mathcal{R}^+$. *Let* $t_0 \in \mathbb{T}$ *and* $k$, $k_{n_{t_0}}$, $k_0$ *be solutions of the dynamic equation* (3.2),

$$k^\Delta(t) = \frac{s}{1+\mu(t_0)n_{t_0}} f(k(t)) - \frac{\delta + n_{t_0}}{1+\mu(t_0)n_{t_0}} k(t), \tag{3.8}$$

*and*

$$k^\Delta(t) = sf(k(t)) - \delta k(t), \tag{3.9}$$

*respectively, with the initial values*

$$k(t_0) = k_{n_{t_0}}(t_0) = k_0(t_0).$$

*Then*

1. $k_{n_{t_0}} \le k \le k_0$ *on* $\mathbb{T}^+_{t_0}$;
2. *If* $k(t_0) \le \hat{k}_{n_0}$, *then* $k$ *is strictly increasing on* $\mathbb{T}^+_{t_0}$;
3. *If* $\hat{k}_{n_0} < k(t_0) \le \hat{k}_0$, *then there exists* $\widetilde{t} \in \mathbb{T}$ *such that* $k$ *is decreasing on* $[t_0, \widetilde{t}] \cap \mathbb{T}$ *and is increasing on* $\mathbb{T}^+_{\widetilde{t}}$;
4. *If* $\hat{k}_0 < k(t_0)$, *then* $k$ *is increasing on* $\mathbb{T}^+_{t_0}$, *or there exists* $\widetilde{t} \in \mathbb{T}$ *such that* $k$ *is decreasing on* $[t_0, \widetilde{t}] \cap \mathbb{T}$ *and is increasing on* $\mathbb{T}^+_{\widetilde{t}}$.

*Here* $\hat{k}_{n_0}$ *and* $\hat{k}_0$ *are the equilibria of* (3.8) *and* (3.9), *respectively.*

*Proof.* 1. For $t > t_0$, $t \in \mathbb{T}$, we have $n(t_0) > n(t) > 0$. So from Theorem 3.4, we obtain the result easily.

2. We want to prove the statement $S(t)$ given by $k^\Delta(t) > 0$ is true for any $t \in \mathbb{T}^+_{t_0}$. To do this, we use Theorem 1.1.
   A. Since $k(t_0) < \hat{k}_{n_0}$, we have $k^\Delta(t_0) > 0$. So $S(t)$ holds at $t = t_0$.
   B. If $t$ is right-scattered and $k^\Delta(t) > 0$, then

$$\begin{aligned}
k(\sigma(t)) &= k(t) + \mu(t)k^\Delta(t) \\
&= k(t) + \mu(t)\frac{sf(k(t)) - (\delta + n(t))k(t)}{1+\mu(t)n(t)} \\
&= \frac{(1-\mu(t)\delta)k(t) + s\mu(t)f(k(t))}{1+\mu(t)n(t)} \\
&< \frac{(1-\mu(t)\delta)k(\sigma(t)) + s\mu(t)f(k(\sigma(t)))}{1+\mu(t)n(\sigma(t))} \\
&= \frac{[1+\mu(t)n(\sigma(t))]k(\sigma(t))}{1+\mu(t)n(\sigma(t))} \\
&\quad + \frac{\mu(t)[sf(k(\sigma(t))) - (\delta + n(\sigma(t)))k(\sigma(t))]}{1+\mu(t)n(\sigma(t))} \\
&= k(\sigma(t)) + \frac{\mu(t)}{1+\mu(t)n(\sigma(t))}[1+\mu(\sigma(t))n(\sigma(t))]k^\Delta(\sigma(t)),
\end{aligned}$$

so $k^{\Delta}(\sigma(t)) > 0$.

C. If $t$ is right-dense and $k^{\Delta}(t) > 0$, then there exists a neighborhood $\mathring{U}^{+}(t) \cap \mathbb{T}$ such that $k^{\Delta}(r) > 0$ for any $r \in \mathring{U}^{+}(t) \cap \mathbb{T}$. To prove this, we assume that there does not exist such a neighborhood. Then there must exist a decreasing sequence $\{t_n\} \subset \mathring{U}^{+}(t) \cap \mathbb{T}$ such that $\lim_{n\to\infty} t_n = t$ and $k^{\Delta}(t_n) \leq 0$. From the properties of $f$, taking limit on both sides, we obtain $k^{\Delta}(t) \leq 0$, which is a contradiction.

D. Assume that $t$ is left-dense and $k^{\Delta}(r) > 0$ for any $r \in [t_0, t) \cap \mathbb{T}$. From continuity, we can get $k^{\Delta}(t) \geq 0$. If $k^{\Delta}(t) = 0$, then for any $r \in [t_0, t) \cap \mathbb{T}$, from the chain rule in [6], we have

$$\begin{aligned}[(1+\mu n)k^{\Delta}]^{\Delta}(r) &= [s(f \circ k) - (\delta + n)k]^{\Delta}(r) \\ &= sf'(k(r))k^{\Delta}(r) - n^{\Delta}(r)k(r) \\ &\quad -(\delta + n^{\sigma}(r))k^{\Delta}(r).\end{aligned}$$

Taking limit on both sides when $r \to t^{-}$, we obtain

$$[(1+\mu n)k^{\Delta})]^{\Delta}(t) = -n^{\Delta}(t)k(t) > 0.$$

So since $t$ is left-dense and from the continuity, we have

$$(1+\mu(t)n(t))k^{\Delta}(t) > (1+\mu(r)n(r))k^{\Delta}(r) > 0$$

for all $r \in \mathring{U}^{-}(t) \cap \mathbb{T}$. Hence $k^{\Delta}(t) > 0$.

3. If $\hat{k}_{n_0} < k(t_0) \leq \hat{k}_0$, then

$$\begin{aligned}k^{\Delta}(t_0) &= \frac{s}{1+\mu(t_0)n(t_0)} f(k(t_0)) - \frac{\delta + n(t_0)}{1+\mu(t_0)n(t_0)} k(t_0) \\ &= \frac{s}{1+\mu(t_0)n(t_0)} f(k_{n_{t_0}}(t_0)) - \frac{\delta + n(t_0)}{1+\mu(t_0)n(t_0)} k_{n_{t_0}}(t_0) \\ &= k^{\Delta}_{n_{t_0}}(t_0) < 0.\end{aligned}$$

Hence $k$ is right-decreasing at $t_0$, i.e., if $t_0$ is right-scattered, then $k(\sigma(t_0)) < k(t_0)$; if $t_0$ is right-dense, then there exists a nonempty neighborhood $\mathring{U}^{+}(t_0) \cap \mathbb{T}$ of $t_0$ such that $k(t) < k(t_0)$ for any $t \in \mathring{U}^{+}(t_0) \cap \mathbb{T}$. If $k^{\Delta} \leq 0$ is true on $\mathbb{T}^{+}_{t_0}$, then $k$ is decreasing on $\mathbb{T}^{+}_{t_0}$. Considering $\lim_{t\to\infty} k(t) = \hat{k}_0$ in Theorem 3.6, we have

$$\hat{k}_0 \leq k(t) < k(t_0) \leq \hat{k}_0 \quad \text{for} \quad t \in \mathbb{T}^{+}_{t_0},$$

which is a contradiction. So there must exist $\widetilde{t} \in \mathbb{T}^{+}_{t_0}$ such that $k^{\Delta}(\widetilde{t}) > 0$, and for simplicity we assume $\widetilde{t}$ is the first point that verifies the inequality. So it must be proved that $k^{\Delta}(t) > 0$ for all $t \in \mathbb{T}^{+}_{\widetilde{t}}$, which is similar to the proof of Statement 2.

4. Following the same proof as in Statement 3, we can obtain the monotonicity.

This completes the proof. □

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