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Quadratic functionals for second order matrix equations on time scales

Ravi P. Agarwal *, Martin Bohner [1]

Department of Mathematics, National University of Singapore, Lower Kent Ridge Road, Singapore 119260, Singapore

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1. Introduction

We begin this paper by stating two well-known results concerning quadratic functionals. Result 1 deals with the "continuous" case while Result 2 represents the "discrete" counterpart. We let $n \in \mathbb{N}$.

Result 1 (*Jacobi's condition*). Let R and P be continuous $n \times n$-matrix valued functions on $\mathbb{R}$ such that $R(t)$ and $P(t)$ have real entries and are symmetric and such that $R(t)$ is positive definite for each $t \in \mathbb{R}$. Let $a, b \in \mathbb{R}$ with $a < b$. Then the quadratic functional

$$\mathscr{F}_C(y) = \int_a^b \{\dot{y}^{\mathrm{T}} R \dot{y} - y^{\mathrm{T}} P y\}(t)\,\mathrm{d}t$$

is positive definite, i.e., $\mathscr{F}_C(y) > 0$ for all nontrivial piecewise C^1-functions $y : [a, b] \to \mathbb{R}^n$ with $y(a) = y(b) = 0$, if and only if the solution Y of

$$\frac{\mathrm{d}}{\mathrm{d}t}[R(t)\dot{Y}] + P(t)Y = 0, \quad Y(a) = 0, \quad \dot{Y}(a) = R^{-1}(a)$$

satisfies

$$Y(t) \text{ invertible for all } t \in (a, b]. \tag{1}$$

* Corresponding author. E-mail: matravip@leonis.nus.sg.

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Result 2 (*Discrete version of Jacobi's condition*). Let R_k and P_k be symmetric $n \times n$-matrices with real entries such that R_k is invertible for each $k \in \mathbb{Z}$. Let $M, N \in \mathbb{Z}$ with $M < N - 1$. Then the discrete quadratic functional

$$\mathscr{F}_D(y) = \sum_{k=M}^{N-1} \{(\Delta y_k)^{\mathrm{T}} R_k (\Delta y_k) - y_{k+1}^{\mathrm{T}} P_k y_{k+1}\}$$

is positive definite, i.e., $\mathscr{F}_D(y) > 0$ for all nontrivial sequences $y : [M, N] \cap \mathbb{Z} \to \mathbb{R}^n$ with $y_M = y_N = 0$, if and only if the solution Y of

$$\Delta[R_k \Delta Y_k] + P_k Y_{k+1} = 0, \quad Y_M = 0, \quad \Delta Y_M = R_M^{-1}$$

satisfies

$$\begin{aligned} &Y_k \text{ invertible for all } k \in \{M+1, \ldots, N\}, \quad \text{and} \\ &Y_k Y_{k+1}^{-1} R_k^{-1} \text{ positive definite for all } k \in \{M+1, \ldots, N-1\}. \end{aligned} \tag{2}$$

The proofs of these results and many related topics can be found in recent monographs by Kratz [15] ("continuous" case; see also [11, 17]), and Ahlbrandt and Peterson [3] ("discrete" case; see also [2, 6–9, 16]). Besides featuring minor technical differences (note that $y^{\mathrm{T}}(t)P(t)y(t)$ in $\mathscr{F}_C$ is replaced by $y_{k+1}^{\mathrm{T}} P_k y_{k+1}$ in $\mathscr{F}_D$), the following two discrepancies occur in the first view:

- In Result 2 the matrices R_k need not be positive definite, while Result 1 does not hold without this assumption.
- The invertibility of the so-called principal solution in Eq. (1) is not enough for Result 2; in fact we need to supplement this condition by an additional condition in Eq. (2).

As such it is not clear why these strange differences occur. In this paper, we will present a theorem on a time scale $\mathbb{T}$ which covers both Results 1 and 2 in the sense that they are special cases of our theorem for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. Such a unification by the theory of time scales was initiated by Hilger in [12]. Besides this paper we shall refer to the work of Aulbach and Hilger [4] and the recently published monograph by Kaymakçalan et al. [13]. Some of our results extend the work of Erbe and Hilger [10] to systems. However, the main result of our paper, Theorem 5, is new even in the scalar case.

For the convenience of comparing our theorem from Results 1 and 2 above we first state our version of Jacobi's condition on time scales. For the notation needed we refer the reader to the next section on preliminaries about time scales. At this moment the only important thing for observing how Results 1 and 2 follow from our theorem is to know that $\rho(t) = \sigma(t) = t$ if $\mathbb{T} = \mathbb{R}$ and $\rho(t) = t - 1, \sigma(t) = t + 1$ if $\mathbb{T} = \mathbb{Z}$. Also, the theory has been prepared in such a manner that $y^{\Delta} = \dot{y}$ in the case $\mathbb{T} = \mathbb{R}$ and $y^{\Delta} = \Delta y$ in the case $\mathbb{T} = \mathbb{Z}$.

Result 3 (*Jacobi's condition on time scales*). Let R and P be rd-continuous $n \times n$-matrix valued functions on $\mathbb{T}$ such that $R(t)$ and $P(t)$ are symmetric and have real entries and such that $R(t)$ is invertible for each $t \in \mathbb{T}$. Let $a, b \in \mathbb{T}$ with $a < \rho(b)$. Then the quadratic functional

$$\mathscr{F}_T(y) = \int_a^b \{(y^\Delta)^{\mathrm{T}} R y^\Delta - (y^\sigma)^{\mathrm{T}} P y^\sigma\}(t)\Delta t$$

is positive definite, i.e., $\mathscr{F}_T(y) > 0$ for all nontrivial piecewise C^1_{rd}-functions $y:[a,b]\cap\mathbb{T} \to \mathbb{R}^n$ with $y(a)=y(b)=0$, if and only if the solution Y of

$$[R(t)Y^\Delta]^\Delta + P(t)Y^\sigma = 0, \quad Y(a)=0, \quad Y^\Delta(a) = R^{-1}(a)$$

satisfies

$$\begin{array}{ll} Y(t) & \text{invertible for all } t\in(a,b]\cap\mathbb{T}, \text{ and} \\ Y(t)Y^{-1}(\sigma(t))R^{-1}(t) & \text{positive definite for all } t\in(a,\rho(b)]\cap\mathbb{T}. \end{array} \tag{3}$$

The organisation of this paper is as follows: In the next section we introduce time scales and collect some of their basic properties which are needed later. The proofs of these properties and further details can be found in [4, 13]. Section 3 gives a short motivation why it is useful to examine quadratic functionals by considering certain variational problems on time scales. Section 4 contains preliminaries and several fundamental results (e.g. Picone's identity) for self-adjoined second order matrix equations and corresponding Riccati matrix equations on time scales. Section 5 proves the above mentioned time-scales-version of Jacobi's condition. We conclude this paper by giving Sturmian separation and comparison results on time scales in Section 6.

2. Preliminaries about time scales

A time scale $\mathbb{T}$ is defined to be any closed subset of $\mathbb{R}$. Hence the *jump operators* $\sigma,\rho:\mathbb{T}\to\mathbb{T}$

$$\sigma(t) = \inf\{s\in\mathbb{T}:\ s>t\} \quad \text{and} \quad \rho(t) = \sup\{s\in\mathbb{T}:\ s<t\}$$

(supplemented by $\inf\emptyset := \sup\mathbb{T}$ and $\sup\emptyset := \inf\mathbb{T}$) are well-defined. The point $t\in\mathbb{T}$ is called *left-dense, left-scattered, right-dense, right-scattered* if $\rho(t)=t$, $\rho(t)<t$, $\sigma(t)=t$, $\sigma(t)>t$, respectively. The *graininess* $\mu:\mathbb{T}\to\mathbb{R}_0^+$ is defined by $\mu(t)=\sigma(t)-t$.

Throughout we fix $a,b\in\mathbb{T}$ with $a<\rho(b)$ and put $\tilde{\mathscr{T}} = [a,b]\cap\mathbb{T}$. By $\mathscr{T}$ we shall always denote a nonempty subset of $\mathbb{T}$. Also, whenever we write $\mathscr{T}=[a_1,b_1]\cap\mathbb{T}$, we mean $a_1,b_1\in\mathbb{T}$ with $a_1<\rho(b_1)$. We define

$$\mathscr{T}^\kappa = \begin{cases} \mathscr{T} & \text{if } \mathscr{T} \text{ is unbounded above,} \\ \mathscr{T}\backslash(\rho(\max\mathscr{T}),\max\mathscr{T}] & \text{otherwise.} \end{cases}$$

We say that a mapping f from $\mathbb{T}$ into some real Banach space X is differentiable at $t\in\mathbb{T}$ provided (see [10] (Section 2))

$$f^\Delta(t) := \lim_{s\to t}\frac{f(\sigma(t))-f(s)}{\sigma(t)-s}, \quad \text{where } s\to t,\ s\in\mathbb{T}\backslash\{\sigma(t)\},$$

exists. The function f is called differentiable on $\mathcal{T}$ if $f^{\Delta}(t)$ exists for all $t \in \mathcal{T}$. The following lemma contains results for this derivative. We shall write f^{σ} for $f \circ \sigma$.

Lemma 1. *Let $f,g:\mathbb{T}\to X$ and $t\in\mathbb{T}^{\kappa}$. Then the following hold:*

(1) *If $f^{\Delta}(t)$ exists, then f is continuous in t;*
(2) *If t is right-scattered and f is continuous in t, then $f^{\Delta}(t)=[f(\sigma(t))-f(t)]/\mu(t)$;*
(3) *If $f^{\Delta}(t)$ exists, then $f(\sigma(t))=f(t)+\mu(t)f^{\Delta}(t)$;*
(4) *If $f^{\Delta}(t), g^{\Delta}(t)$ exist and $(fg)(t)$ is defined, then $(fg)^{\Delta}(t)=f(\sigma(t))g^{\Delta}(t)+f^{\Delta}(t)$ $g(t)$;*
(5) *If f^{Δ} exists on $\mathcal{T}^{\kappa}$ and f is invertible on $\mathcal{T}$, then $(f^{-1})^{\Delta}=-(f^{\sigma})^{-1}f^{\Delta}f^{-1}$ on $\mathcal{T}^{\kappa}$;*
(6) *If $|f^{\Delta}|\le g^{\Delta}$ on $\mathcal{T}^{\kappa}$, then $|f(s)-f(r)|\le g(s)-g(r)$ for all $r,s\in\mathcal{T}$ with $r\le s$;*
(7) *If $f^{\Delta}=0$ on $\mathcal{T}^{\kappa}$, then f is constant on $\mathcal{T}$.*

Proof. For (1) and (2) see [4] (Theorem 3) or [13] (Theorem 1.2.2). While (3) follows (note that $\mu(t)=0$ if t is right-dense) from (1) and (2), (4) and (5) are from [4] (Theorem 4) or [13] (Theorem 1.2.3). Finally, for (6) and (7) see [4] (Theorem 5) or [13] (Corollary 1.3.1). □

Let $t_1,\dots,t_m\in\mathbb{T}^{\kappa}$. A function $f:\mathbb{T}\backslash\{t_1,\dots,t_m\}\to X$ is called *regulated* on $\mathcal{T}$ provided f has a left-sided limit at all left-dense points and a right-sided limit at all right-dense points of $\mathcal{T}$. An f is called *rd-continuous* on $\mathcal{T}$ – we write $f\in C_{\mathrm{rd}}(\mathcal{T})=C_{\mathrm{rd}}$ $(\mathcal{T},X)$ – if it is regulated on $\mathcal{T}$ and continuous at all right-dense points of $\mathcal{T}$. We write $f\in C^1_{\mathrm{rd}}(\mathcal{T})$ if f is differentiable on $\mathcal{T}^{\kappa}$ with $f^{\Delta}\in C_{\mathrm{rd}}(\mathcal{T}^{\kappa})$. If f is continuous on $\mathcal{T}$ and if it is differentiable on $\mathcal{T}^{\kappa}$ except at $t_1,\dots,t_m\in\mathcal{T}^{\kappa}$ such that f^{Δ} is regulated on $\mathcal{T}^{\kappa}$, we abbreviate this piecewise rd-continuous differentiability by writing $f\in C^1_{\mathrm{p}}(\mathcal{T})$. Note that in this case (apply Lemma 1(2)) the points $t_1,\dots,t_m$ are necessarily left-dense and right-dense at the same time. Moreover, f is said to be *regressive* on $\mathcal{T}$ if $I+\mu(t)f(t)$ is invertible for each $t\in\mathcal{T}$, where I denotes the identity in X. Next, for any on $\mathcal{T}$ regulated function $f:\mathcal{T}\backslash\{t_1,\dots,t_m\}\to X$ there exists a unique *pre-antiderivative* F (see [13] (Theorem 1.4.2)), i.e., a continuous function $F:\mathcal{T}\to X$ that is differentiable on $\mathcal{T}^{\kappa}\backslash\{t_1,\dots,t_m\}$ and satisfies $F^{\Delta}(t)=f(t)$ for all $t\in\mathcal{T}^{\kappa}\backslash\{t_1,\dots,t_m\}$. We then define the *Cauchy integral* of f by $\int_r^s f(t)\Delta t=F(s)-F(r)$, where $r,s\in\mathcal{T}$. If F satisfies $F^{\Delta}(t)=f(t)$ for all $t\in\mathcal{T}^{\kappa}$, then F is said to be an *antiderivative* of f on $\mathcal{T}$. In the following lemma we collect properties of rd-continuous functions that are needed in our work. The terminology "right-sequence for t" is used for strictly decreasing sequences $\{t_m\}_{m\in\mathbb{N}}\subset\mathbb{T}$ with $\lim_{m\to\infty}t_m=t$. Note that due to the definition of σ there always exist right-sequences for any right-dense $t\in\mathbb{T}$ (except if $t=\max\mathbb{T}$ in case $\mathbb{T}$ is bounded above).

Lemma 2. *Let $f\in C_{\mathrm{rd}}(\mathcal{T})$. Then the following hold:*

(1) *f has an antiderivative;*
(2) *If f is regressive on $\mathcal{T}$ and if $\tilde{t}\in\mathcal{T}$, $\tilde{g}\in X$, then any initial value problem $g^{\Delta}=f(t)g$, $g(\tilde{t})=\tilde{g}$ has a unique solution on $\mathcal{T}$;*
(3) *$\int_t^{\sigma(t)} f(\tau)\Delta\tau=\mu(t)f(t)$ hold for all $t\in\mathcal{T}^{\kappa}$;*

(4) *If* $t_1 \in \mathbb{T}^\kappa$ *and* $f(t_1)>0$, *then there exists* $t_2 \in \mathbb{T}$ *with* $\int_{t_1}^{t_2} f(t)\Delta t>0$;
(5) *If* a *is right-dense and if* $\{a_m\}_{m\in\mathbb{N}} \subset \mathscr{T} = \tilde{\mathscr{T}}$ *is a right-sequence for* a, *then*

$$\lim_{m\to\infty} \int_{a_m}^{b} f(t)\,\Delta t = \int_a^b f(t)\,\Delta t;$$

(6) *If* $\mathscr{T} = [a_1, b_1] \cap \mathbb{T}$, $f_m \in C_{\mathrm{rd}}(\mathscr{T})$, *and* $\lim_{m\to\infty} f_m = f$ *uniformly on* $\mathscr{T}^\kappa$, *then*

$$\lim_{m\to\infty} \int_{a_1}^{b_1} f_m(t)\Delta t = \int_{a_1}^{b_1} f(t)\,\Delta t.$$

Proof. While (1) is from [4] (Theorem 6) or [13] (Theorem 1.4.4), see [4] (Theorem 8) or [13] (Theorem 2.5.1) for (2). Now, (3) follows from (1) and Lemma 1(3). For (4) note the following: Either t_1 is right-scattered, then by (3)

$$\int_{t_1}^{\sigma(t_1)} f(t)\,\Delta t = \mu(t_1)f(t_1)>0,$$

or t_1 is right-dense, then f is continuous in t_1 and there exists a $\delta>0$ and $t_2 \in (t_1,\infty)\cap \mathbb{T}$ such that $f(t)\geq\delta$ for all $t\in[t_1,t_2]\cap\mathbb{T}$. In this case there exist antiderivatives F (according to part (1)) and G (because of

$$\lim_{s\to t} \frac{G(\sigma(t)) - G(s)}{\sigma(t)-s} = \lim_{s\to t} \frac{\delta\sigma(t) - \delta s}{\sigma(t)-s} = \delta =: g(t)$$

actually $G(t)=\delta t$) of f and g, and an application of Lemma 1(4) yields

$$\int_{t_1}^{t_2} f(t)\,\Delta t = F(t_2) - F(t_1) \geq G(t_2) - G(t_1) = \delta(t_2 - t_1)>0.$$

Finally, (5) follows from (1), and (6) is from [12] (Theorem 4.3) or [13] (Theorem 1.4.3). $\square$

In this paper we shall consider matrix-valued functions. The set of $m\times n$-matrices with real entries will be abbreviated by $\mathbb{R}^{m\times n}$.

3. Variational problems on time scales

The motivation of our investigations in Sections 4–6 comes by examining variational problems of the form

$$\text{(V)} \quad \mathscr{L}(y) := \int_a^b L(t, y^\sigma(t), y^\Delta(t))\,\Delta t \to \min,$$
$$y(a)=\alpha, \quad y(b)=\beta,$$

where $\alpha,\beta\in\mathbb{R}^n$ and $L:\mathbb{T}\times\mathbb{R}^{2n}\to\mathbb{R}$ is a C^2-function in the last two variables (by L_x and L_u we denote the derivatives of L with respect to the second and the third variables, respectively). For simplicity we allow in this section only functions $y:\tilde{\mathscr{T}}\to\mathbb{R}^n$ that are differentiable on $\tilde{\mathcal{T}}$ with continuous derivative y^{Δ}, i.e., $y \in C^1(\tilde{\mathcal{T}})$. A function $\tilde{y}: \mathbb{T} \to \mathbb{R}^n$ with $\tilde{y} \in C^1(\tilde{\mathcal{T}})$, $\tilde{y}(a) = \alpha$, and $\tilde{y}(b) = \beta$ is called a (weak) *local minimum* of (V) provided there exists a $\delta > 0$ such that $\mathcal{L}(y) \geq \mathcal{L}(\tilde{y})$ for all $y \in C^1(\tilde{\mathcal{T}})$ with $y(a) = \alpha$, $y(b) = \beta$, and $\|y - \tilde{y}\| < \delta$; and the local minimum $\tilde{y}$ is called *proper* provided $\mathcal{L}(y) = \mathcal{L}(\tilde{y})$ if and only if $y = \tilde{y}$. Here we shall consider the (according to Lemma 1(1) well-defined) norm

$$\|f\| = \max_{t \in \tilde{\mathcal{T}}^{\kappa}} |f^{\sigma}(t)| + \max_{t \in \tilde{\mathcal{T}}^{\kappa}} |f^{\Delta}(t)| \quad \text{for } f \in C^1(\tilde{\mathcal{T}}).$$

Now let $\eta: \mathbb{T} \to \mathbb{R}^n$ be any *admissible variation*, i.e., $\eta \in C^1(\tilde{\mathcal{T}})$ with $\eta(a) = \eta(b) = 0$. We put

$$\phi(\varepsilon) = \phi(\varepsilon; y, \eta) = \mathcal{L}(y + \varepsilon\eta), \quad \varepsilon \in \mathbb{R}.$$

We suppose that L is such that we have

$$\mathcal{L}_1(y, \eta) := \dot{\phi}(0) = \int_a^b \{L_x(\cdot)\eta^{\sigma}(t) + L_u(\cdot)\eta^{\Delta}(t)\} \, \Delta t,$$

and

$$\mathcal{L}_2(y, \eta) := \ddot{\phi}(0) = \int_a^b \{(\eta^{\sigma}(t))^{\mathrm{T}} L_{xx}(\cdot)\eta^{\sigma}(t) + 2(\eta^{\sigma}(t))^{\mathrm{T}} L_{xu}(\cdot)\eta^{\Delta}(t) + (\eta^{\Delta}(t))^{\mathrm{T}} L_{uu}(\cdot)\eta^{\Delta}(t)\} \, \Delta t,$$

where $(\cdot) = (t, y^{\sigma}(t), y^{\Delta}(t))$. Then y is said to satisfy the *Euler equation* and $\mathcal{L}_2(y)$ is called *positive semidefinite* if $\mathcal{L}_1(y, \eta) = 0$ and $\mathcal{L}_2(y, \eta) \geq 0$, respectively, for all admissible variations η. If $\mathcal{L}_2(y, \eta) > 0$ for all nontrivial admissible variations η, then we say that $\mathcal{L}_2(y)$ is *positive definite*.

The following two results provide necessary and sufficient conditions for (weak) local minima of (V) in terms of positive definiteness of the quadratic functional $\mathcal{L}_2(y)$. Concerning the corresponding results for the discrete case we refer to [1] (Example 1.6.7), [5] (Theorems 1 and 2), and [14] (Theorems 8.1 and 8.9).

Theorem 1 (Necessary condition). *If $\tilde{y}$ is a local minimum of* (V), *then $\tilde{y}$ satisfies the Euler equation and $\mathcal{L}_2(\tilde{y})$ is positive definite.*

Proof. Let η be any admissible variation and define $\phi: \mathbb{R} \to \mathbb{R}$ as above, namely $\phi(\varepsilon) = \phi(\varepsilon; \tilde{y}, \eta)$. We then have $\tilde{y} + \varepsilon\eta \in C^1(\tilde{\mathcal{T}})$, $(\tilde{y} + \varepsilon\eta)(a) = \alpha$, $(\tilde{y} + \varepsilon\eta)(b) = \beta$, and, since there exists a $\delta > 0$ such that

$$\mathcal{L}(y) \geq \mathcal{L}(\tilde{y}) \quad \text{for all } y \in C^1(\tilde{\mathcal{T}}) \quad \text{with } y(a) = \alpha, \; y(b) = \beta, \; \|y - \tilde{y}\| < \delta,$$

we also have

$$\phi(\varepsilon) = \mathcal{L}(\tilde{y} + \varepsilon\eta) \geq \mathcal{L}(\tilde{y}) = \phi(0) \quad \text{for all } \varepsilon \in \mathbb{R} \quad \text{with } |\varepsilon| < \frac{\delta}{\|\eta\|}.$$

Hence ϕ has a local minimum at $\tilde{\varepsilon} = 0$, and thus our assertion follows. □

Theorem 2 (Sufficient condition). *If $\tilde{y} \in C^1(\tilde{\mathscr{T}})$ satisfies the Euler equation, $\tilde{y}(a) = \alpha$ and $\tilde{y}(b) = \beta$, and $\mathscr{L}_2(\tilde{y})$ is positive definite, then $\tilde{y}$ is a proper local minimum of* (V).

Proof. Since $\mathscr{L}_2(\tilde{y})$ is positive definite, it is also strongly positive (see, e.g., [11] (Footnote 4, p. 100)), i.e., there exists an $\varepsilon > 0$ with

$$\mathscr{L}_2(\tilde{y}, \eta) \geq 2\varepsilon(b-a)\|\eta\|^2.$$

Since L is a C^2-function in x and u, there exists a $\delta > 0$ such that $|L''(t,z) - L''(t,\tilde{z})| < \varepsilon$ whenever $z, \tilde{z} \in \mathbb{R}^{2n}$ with $|z - \tilde{z}| < \delta$. Let $y \in C^1(\tilde{\mathscr{T}})$ be such that $y \neq \tilde{y}$, $y(a) = \alpha$, $y(b) = \beta$, and $\|y - \tilde{y}\| < \delta$. We shall show $\mathscr{L}(y) > \mathscr{L}(\tilde{y})$. To do so, we put $\eta = y - \tilde{y}$ so that η is a nontrivial admissible variation. Now, using the above remarks and Taylor's theorem, we find that there exists a $\zeta \in (0,1)$ with

$$\begin{aligned}
\mathscr{L}(y) - \mathscr{L}(\tilde{y}) &= \phi(1) - \phi(0) = \dot{\phi}(0) + \frac{1}{2}\ddot{\phi}(\zeta) = \frac{1}{2}\ddot{\phi}(\zeta) \\
&= \frac{1}{2}\ddot{\phi}(0) + \frac{1}{2}\{\ddot{\phi}(\zeta) - \ddot{\phi}(0)\} \geq \frac{1}{2}\ddot{\phi}(0) - \frac{1}{2}|\ddot{\phi}(\zeta) - \ddot{\phi}(0)| \\
&= \frac{1}{2}\ddot{\phi}(0) - \frac{1}{2}\left| \int_a^b \begin{pmatrix} \eta^\sigma(t) \\ \eta^\Delta(t) \end{pmatrix}^{\mathrm{T}} \{L''(t,z(t)) - L''(t,\tilde{z}(t))\} \begin{pmatrix} \eta^\sigma(t) \\ \eta^\Delta(t) \end{pmatrix} \Delta t \right| \\
&\geq \varepsilon(b-a)\|\eta\|^2 - \frac{1}{2}\int_a^b \|\eta\|\varepsilon\|\eta\|\Delta t = \varepsilon(b-a)\|\eta\|^2 - \frac{1}{2}\varepsilon(b-a)\|\eta\|^2 \\
&= \varepsilon\frac{b-a}{2}\|\eta\|^2 > 0,
\end{aligned}$$

where we put

$$z = \begin{pmatrix} \tilde{y}^\sigma + \zeta\eta^\sigma \\ \tilde{y}^\Delta + \zeta\eta^\Delta \end{pmatrix} \quad \text{and} \quad \tilde{z} = \begin{pmatrix} \tilde{y}^\sigma \\ \tilde{y}^\Delta \end{pmatrix}$$

so that $|z(t) - \tilde{z}(t)| \leq \zeta\|\eta\| < \|\eta\| < \delta$ holds for all $t \in \tilde{\mathscr{T}}^\kappa$. Hence $\tilde{y}$ is indeed a proper local minimum of (V). $\square$

4. Self-adjoint second order matrix equations on time scales

Let $R, P \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ be such that $R(t)$ is symmetric and invertible and $P(t)$ is symmetric for each $t \in \mathbb{T}$. Consider the self-adjoint second order matrix equation

$$(E) \quad [R(t)Y^\Delta]^\Delta + P(t)Y^\sigma = 0.$$

We call $Y \in C^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ a solution of (E) provided

$$\{[RY^\Delta]^\Delta + PY^\sigma\}(t) = 0 \quad \text{holds for all } t \in (\mathbb{T}^\kappa)^\kappa.$$

Note that any solution Y of (E) satisfies $RY^\Delta \in C^1_{\mathrm{rd}}(\mathbb{T}^\kappa)$. We shall prove first the following basic result.

Lemma 3 (Wronskian identity). *Let Y and Y_1 be solutions of* (E). *Then*

$$Y^{\mathrm{T}}RY_1^\Delta - (Y^\Delta)^{\mathrm{T}}RY_1 \quad \text{is a constant function on } \mathbb{T}^\kappa.$$

Proof. We apply the product rule Lemma 1(4) to obtain

$$\begin{aligned}[Y^{\mathrm{T}}RY_1^\Delta - (Y^\Delta)^{\mathrm{T}}RY_1]^\Delta &= (Y^\sigma)^{\mathrm{T}}[RY_1^\Delta]^\Delta + (Y^\Delta)^{\mathrm{T}}RY_1^\Delta - [(Y^\Delta)^{\mathrm{T}}R]^\Delta Y_1^\sigma - (Y^\Delta)^{\mathrm{T}}RY_1^\Delta \\ &= -(Y^\sigma)^{\mathrm{T}}PY_1^\sigma + (Y^\sigma)^{\mathrm{T}}PY_1^\sigma = 0\end{aligned}$$

on $(\mathbb{T}^\kappa)^\kappa$. Now the claim follows from Lemma 1(7) with $\mathscr{T} = \mathbb{T}^\kappa$. □

Next, putting

$$Z = \begin{pmatrix} Y \\ RY^\Delta \end{pmatrix} \quad \text{on } \mathbb{T}^\kappa \quad \text{and} \quad S = \begin{pmatrix} 0 & R^{-1} \\ -P & -\mu PR^{-1} \end{pmatrix} \quad \text{on } \mathbb{T},$$

it is easy to see that Y solves (E) if and only if (note that $Y^\sigma = Y + \mu Y^\Delta$ by Lemma 1(3)) Z solves

$$Z^\Delta = S(t)Z. \tag{4}$$

Now $S \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{2n\times 2n})$ is regressive on $\mathbb{T}$ because

$$\{I + \mu S\}^{-1} = \begin{pmatrix} I & \mu R^{-1} \\ -\mu P & I - \mu^2 PR^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} I - \mu^2 R^{-1}P & -\mu R^{-1} \\ \mu P & I \end{pmatrix}$$

holds on $\mathbb{T}$. Hence, by Lemma 2(2), any initial value problem

$$(4), \quad Z(a) = Z_a, \qquad \text{i.e.,} \quad (E), \quad Y(a) = Y_a, \quad Y^\Delta(a) = Y_a'$$

with any choice of a $2n \times n$-matrix Z_a, i.e., of $n \times n$-matrices Y_a and Y_a', has a unique solution. We let $\tilde{Y}$ be the unique solution of the initial value problem

$$(E), \quad Y(a) = 0, \quad Y^\Delta(a) = R^{-1}(a).$$

This $\tilde{Y}$ is called the *principal solution* of (E) (at a), and according to the Wronskian identity, Lemma 3, it satisfies

$$\{\tilde{Y}^{\mathrm{T}}R\tilde{Y}^\Delta - (\tilde{Y}^\Delta)^{\mathrm{T}}R\tilde{Y}\}(t) \equiv \{\tilde{Y}^{\mathrm{T}}R\tilde{Y}^\Delta - (\tilde{Y}^\Delta)^{\mathrm{T}}R\tilde{Y}\}(a) = 0$$

for all $t \in \mathbb{T}^\kappa$, i.e., $\{\tilde{Y}^{\mathrm{T}}R\tilde{Y}^\Delta\}(t)$, and hence (where these are defined) both

$$\tilde{Q}(t) = \{R\tilde{Y}^\Delta \tilde{Y}^{-1}\}(t) \quad \text{and} \quad \tilde{D}(t) = \{\tilde{Y}(\tilde{Y}^\sigma)^{-1}R^{-1}\}(t)$$

are symmetric matrices for each $t \in \mathbb{T}^\kappa$. Furthermore, we call Y a *conjoined solution* of (E) provided $Y^{\mathrm{T}}RY^\Delta$ is symmetric on $\mathbb{T}^\kappa$, and two conjoined solutions Y and Y_1

of (E) are called *normalized* if $Y^{\mathrm{T}}RY_1^{\Delta}-(Y^{\Delta})^{\mathrm{T}}RY_1=I$ on $\mathbb{T}^{\kappa}$. Besides the principal solution $\tilde{Y}$ we will also need the *associated solution* $\tilde{Y}_1$ of (E) (at a) which is defined to be the solution of the initial value problem

$$(E), \quad Y(a)=-I, \quad Y^{\Delta}(a)=0.$$

Of course, $\tilde{Y}$ and $\tilde{Y}_1$ are normalized conjoined solutions of (E).

We shall now consider the matrix *Riccati equation*

$$(R) \quad Q^{\Delta}+P(t)+Q^{\mathrm{T}}\{R(t)+\mu(t)Q\}^{-1}Q=0.$$

Lemma 4 (Riccati equation). *If* (E) *has a conjoined solution Y such that $Y(t)$ is invertible for all $t\in\mathscr{T}$, then Q defined by*

$$Q(t)=R(t)Y^{\Delta}(t)Y^{-1}(t) \quad \text{for } t\in\mathscr{T} \tag{5}$$

is a symmetric solution of (R) *on* $\mathscr{T}^{\kappa}$. *Conversely, if* (R) *has a symmetric solution Q on $\mathscr{T}^{\kappa}$, then there exists a conjoined solution Y of* (E) *such that $Y(t)$ is invertible for all $t\in\mathscr{T}$ and relation* (5) *holds.*

Proof. First we assume that Y solves (E) and is invertible on $\mathscr{T}$. We then define Q by Eq. (5) so that on $\mathscr{T}^{\kappa}$,

$$\begin{aligned}
&Q^{\Delta}+P+Q^{\mathrm{T}}\{R+\mu Q\}^{-1}Q\\
&=Q^{\Delta}+P+Q^{\mathrm{T}}\{I+(Y^{\sigma}-Y)Y^{-1}\}^{-1}R^{-1}Q\\
&=[RY^{\Delta}]^{\Delta}(Y^{\sigma})^{-1}-RY^{\Delta}(Y^{\sigma})^{-1}Y^{\Delta}Y^{-1}+P+RY^{\Delta}Y^{-1}(Y^{\sigma}Y^{-1})^{-1}R^{-1}RY^{\Delta}Y^{-1}\\
&=-PY^{\sigma}(Y^{\sigma})^{-1}-RY^{\Delta}(Y^{\sigma})^{-1}Y^{\Delta}Y^{-1}+P+RY^{\Delta}(Y^{\sigma})^{-1}Y^{\Delta}Y^{-1}=0
\end{aligned}$$

holds according to Lemma 1(3)/(4), i.e., Q is a symmetric solution of (R) on $\mathscr{T}^{\kappa}$.

Conversely, let Q be a symmetric solution of (R) on $\mathscr{T}^{\kappa}$, let $t_0\in\mathscr{T}$, and let Y be the solution of the initial value problem (note that $I+\mu R^{-1}Q=R^{-1}(R+\mu Q)$ is invertible on $\mathscr{T}^{\kappa}$ and apply Lemma 2(2))

$$Y^{\Delta}=R^{-1}(t)Q(t)Y, \qquad Y(t_0)=I.$$

Then Y is a fundamental solution of $Y^{\Delta}=R^{-1}(t)Q(t)Y$ and hence invertible on $\mathscr{T}$ (see [4] (Theorem 9)). Furthermore, it satisfies on $(\mathscr{T}^{\kappa})^{\kappa}$,

$$\begin{aligned}
[RY^{\Delta}]^{\Delta}=[QY]^{\Delta}&=Q^{\Delta}Y^{\sigma}+QY^{\Delta}\\
&=-PY^{\sigma}-Q(R+\mu Q)^{-1}QY^{\sigma}+QR^{-1}QY\\
&=-PY^{\sigma}+Q(R+\mu Q)^{-1}\{(R+\mu Q)R^{-1}QY-QY^{\sigma}\}=-PY^{\sigma}.
\end{aligned}$$

We extend Y to a solution of (E) so that, since $\{Y^{\mathrm{T}}RY^{\Delta}\}(t_0)=Q(t_0)$ is symmetric, Y is indeed a conjoined solution of (E). □

Our next result is the main tool for proving Jacobi's condition in the next section.

Theorem 3 (Picone's identity). *Let $\alpha \in \mathbb{R}^n$ and suppose Y and Y_1 are normalized conjoined solutions of* (E) *such that Y is invertible on $\mathcal{T}$. We put*

$$Q = RY^{\Delta}Y^{-1} \quad \text{on } \mathcal{T} \quad \text{and} \quad D = Y(Y^{\sigma})^{-1}R^{-1} \quad \text{on } \mathcal{T}^{\kappa}.$$

Let $t \in \mathcal{T}^{\kappa}$ and $y : \mathcal{T} \to \mathbb{R}^n$ be differentiable at t. Then we have at t

$$\begin{aligned}&\{y^{\mathrm{T}}Qy + 2\alpha^{\mathrm{T}}Y^{-1}y - \alpha^{\mathrm{T}}Y^{-1}Y_1\alpha\}^{\Delta}\\ &= (y^{\Delta})^{\mathrm{T}}Ry^{\Delta} - (y^{\sigma})^{\mathrm{T}}Py^{\sigma} - \{Ry^{\Delta} - Qy - (Y^{-1})^{\mathrm{T}}\alpha\}^{\mathrm{T}}D\{Ry^{\Delta} - Qy - (Y^{-1})^{\mathrm{T}}\alpha\}.\end{aligned}$$

Proof. According to the differentiation rules, Lemma 1(4)/(5), we have at t

$$\begin{aligned}[Y^{-1}Y_1]^{\Delta} &= -(Y^{\sigma})^{-1}Y^{\Delta}Y^{-1}Y_1 + (Y^{\sigma})^{-1}Y_1^{\Delta}\\ &= -(Y^{\sigma})^{-1}R^{-1}QY_1 + (Y^{\sigma})^{-1}R^{-1}RY_1^{\Delta}\\ &= (Y^{\sigma})^{-1}R^{-1}(Y^{-1})^{\mathrm{T}}\{Y^{\mathrm{T}}RY_1^{\Delta} - (Y^{\Delta})^{\mathrm{T}}RY_1\}\\ &= (Y^{\sigma})^{-1}R^{-1}(Y^{-1})^{\mathrm{T}},\end{aligned}$$

and (also using Lemmas 1(3) and 4),

$$\begin{aligned}&[y^{\mathrm{T}}Qy]^{\Delta} + \{Ry^{\Delta} - Qy\}^{\mathrm{T}}Y(Y^{\sigma})^{-1}R^{-1}\{Ry^{\Delta} - Qy\}\\
&= (y^{\Delta})^{\mathrm{T}}Qy + (y^{\sigma})^{\mathrm{T}}Q^{\Delta}y^{\sigma} + (y^{\sigma})^{\mathrm{T}}Qy^{\Delta} + (y^{\Delta})^{\mathrm{T}}RY(Y^{\sigma})^{-1}y^{\Delta}\\
&\quad - y^{\mathrm{T}}QY(Y^{\sigma})^{-1}y^{\Delta} - (y^{\Delta})^{\mathrm{T}}RY(Y^{\sigma})^{-1}R^{-1}Qy + y^{\mathrm{T}}QY(Y^{\sigma})^{-1}R^{-1}Qy\\
&= (y^{\Delta})^{\mathrm{T}}Qy - (y^{\sigma})^{\mathrm{T}}Py^{\sigma} - (y^{\sigma})^{\mathrm{T}}QY(Y^{\sigma})^{-1}R^{-1}Qy^{\sigma} + (y^{\sigma})^{\mathrm{T}}Qy^{\Delta}\\
&\quad + (y^{\Delta})^{\mathrm{T}}RY(Y^{\sigma})^{-1}y^{\Delta} - y^{\mathrm{T}}QY(Y^{\sigma})^{-1}y^{\Delta} - (y^{\Delta})^{\mathrm{T}}RY(Y^{\sigma})^{-1}R^{-1}Qy\\
&\quad + y^{\mathrm{T}}QY(Y^{\sigma})^{-1}R^{-1}Qy\\
&= (y^{\Delta})^{\mathrm{T}}Qy - (y^{\sigma})^{\mathrm{T}}Py^{\sigma} - (y^{\sigma})^{\mathrm{T}}QY(Y^{\sigma})^{-1}R^{-1}Qy^{\sigma} + (y^{\sigma})^{\mathrm{T}}Qy^{\Delta}\\
&\quad + (y^{\Delta})^{\mathrm{T}}R\{Y^{\sigma} - \mu Y^{\Delta}\}(Y^{\sigma})^{-1}y^{\Delta} - y^{\mathrm{T}}QY(Y^{\sigma})^{-1}y^{\Delta}\\
&\quad - (y^{\Delta})^{\mathrm{T}}R\{Y^{\sigma} - \mu Y^{\Delta}\}(Y^{\sigma})^{-1}R^{-1}Qy + y^{\mathrm{T}}QY(Y^{\sigma})^{-1}R^{-1}Qy\\
&= (y^{\Delta})^{\mathrm{T}}Ry^{\Delta} - (y^{\sigma})^{\mathrm{T}}Py^{\sigma} + (y^{\sigma})^{\mathrm{T}}Qy^{\Delta} - \mu(y^{\Delta})^{\mathrm{T}}RY^{\Delta}(Y^{\sigma})^{-1}y^{\Delta}\\
&\quad - y^{\mathrm{T}}QY(Y^{\sigma})^{-1}y^{\Delta} + \mu(y^{\Delta})^{\mathrm{T}}RY^{\Delta}(Y^{\sigma})^{-1}R^{-1}Qy + y^{\mathrm{T}}QY(Y^{\sigma})^{-1}R^{-1}Qy\\
&\quad - (y^{\sigma})^{\mathrm{T}}QY(Y^{\sigma})^{-1}R^{-1}Qy^{\sigma}\end{aligned}$$

$$
\begin{aligned}
&= (y^\Delta)^T R y^\Delta - (y^\sigma)^T P y^\sigma + (y^\sigma)^T Q y^\Delta - (y^\sigma)^T Q Y (Y^\sigma)^{-1} R^{-1} Q y^\sigma \\
&\quad - \{\mu y^\Delta + y\}^T Q Y (Y^\sigma)^{-1} y^\Delta + \{\mu y^\Delta + y\}^T Q Y (Y^\sigma)^{-1} R^{-1} Q y \\
&= (y^\Delta)^T R y^\Delta - (y^\sigma)^T P y^\sigma + (y^\sigma)^T Q y^\Delta - (y^\sigma)^T Q Y (Y^\sigma)^{-1} R^{-1} Q y^\sigma \\
&\quad - (y^\sigma)^T Q Y (Y^\sigma)^{-1} y^\Delta + (y^\sigma)^T Q Y (Y^\sigma)^{-1} R^{-1} Q y \\
&= (y^\Delta)^T R y^\Delta - (y^\sigma)^T P y^\sigma + (y^\sigma)^T Q y^\Delta \\
&\quad - (y^\sigma)^T Q \{Y^\sigma - \mu Y^\Delta\} (Y^\sigma)^{-1} y^\Delta + (y^\sigma)^T Q Y (Y^\sigma)^{-1} R^{-1} Q \{y - y^\sigma\} \\
&= (y^\Delta)^T R y^\Delta - (y^\sigma)^T P y^\sigma + \mu (y^\sigma)^T Q \{Y^\Delta (Y^\sigma)^{-1} - Y (Y^\sigma)^{-1} Y^\Delta Y^{-1}\} y^\Delta \\
&= (y^\Delta)^T R y^\Delta - (y^\sigma)^T P y^\sigma
\end{aligned}
$$

so that

$$
\begin{aligned}
&[y^T Q y + 2\alpha^T Y^{-1} y - \alpha^T Y^{-1} Y_1 \alpha]^\Delta - (y^\Delta)^T R y^\Delta + (y^\sigma)^T P y^\sigma \\
&\quad = -\{R y^\Delta - Q y\}^T Y (Y^\sigma)^{-1} R^{-1} \{R y^\Delta - Q y\} - 2\alpha^T (Y^\sigma)^{-1} Y^\Delta Y^{-1} y \\
&\qquad + 2\alpha^T (Y^\sigma)^{-1} y^\Delta + \alpha^T (Y^\sigma)^{-1} R^{-1} (Y^{-1})^T \alpha \\
&\quad = -\{R y^\Delta - Q y - (Y^{-1})^T \alpha\}^T Y (Y^\sigma)^{-1} R^{-1} \{R y^\Delta - Q y - (Y^{-1})^T \alpha\},
\end{aligned}
$$

and hence our claimed identity follows. □

5. Jacobi's condition on time scales

In this section we will prove the main result of this paper. As it is motivated by the investigations of Section 3, we will consider a quadratic functional of the form

$$\mathcal{F}(y) = \int_a^b \{(y^\Delta)^T R y^\Delta - (y^\sigma)^T P y^\sigma\}(t)\,\Delta t.$$

This functional $\mathcal{F}$ is called *positive definite* – we write $\mathcal{F} > 0$ – provided

$$\mathcal{F}(y) > 0 \quad \text{for all } y \in C_p^1(\tilde{\mathcal{T}}, \mathbb{R}^n) \setminus \{0\} \quad \text{with } y(a) = y(b) = 0.$$

Lemma 5 (Quadratic functional). *If* $\mathcal{T} = [a_1, b_1] \cap \mathbb{T}$, $y \in C_{rd}^1(\mathcal{T}, \mathbb{R}^n)$, *and*

$$\{[R y^\Delta]^\Delta + P y^\sigma\}(t) = 0 \quad \textit{for all } t \in (\mathcal{T}^\kappa)^\kappa,$$

then

$$\int_{a_1}^{\rho(b_1)} \{(y^\Delta)^T R y^\Delta - (y^\sigma)^T P y^\sigma\}(t)\,\Delta t = \{y^T R y^\Delta\}(\rho(b_1)) - \{y^T R y^\Delta\}(a_1).$$

Proof. We have on $(\mathcal{T}^\kappa)^\kappa$

$$[y^T R y^\Delta]^\Delta = (y^\sigma)^T [R y^\Delta]^\Delta + (y^\Delta)^T R y^\Delta = (y^\Delta)^T R y^\Delta - (y^\sigma)^T P y^\sigma$$

so that the assertion follows. □

Our next result, Theorem 4, offers a sufficient condition for the positive definiteness of $\mathscr{F}$. A speical case of this result yields one direction of Jacobi's condition, which we shall prove in Theorem 5. Theorem 4 also finds application to establish Sturm's separation theorem, this we shall present in Theorem 6.

Theorem 4 (Sufficient condition for positive definiteness). *A sufficient condition for $\mathscr{F} > 0$ is that there exist normalized conjoined solutions Y and Y_1 of (E) with*

$$Y \textit{ invertible on } (a,b]\cap\mathbb{T} \quad\text{and}\quad Y(Y^\sigma)^{-1}R^{-1} \textit{ positive definite on } (a,\rho(b)]\cap\mathbb{T}.$$

Proof. We let Y and Y_1 as above and put $D = Y(Y^\sigma)^{-1}R^{-1}$ on $\tilde{\mathscr{T}}^\kappa$. Pick any $y \in C_p^1(\tilde{\mathscr{T}},\mathbb{R}^n)$ with $y(a) = y(b) = 0$. First we shall consider the case that a is right-scattered so that we have $\sigma(a) > a$ and $\mu(a) > 0$. By Lemma 2(3) and Theorem 3 (with $\alpha = 0$) we have

$$\begin{aligned}
\mathscr{F}(y) &= \int_a^{\sigma(a)} \{(y^\Delta)^{\mathrm{T}} R y^\Delta - (y^\sigma)^{\mathrm{T}} P y^\sigma\}(t)\,\Delta t + \int_{\sigma(a)}^b \{(y^\Delta)^{\mathrm{T}} R y^\Delta - (y^\sigma)^{\mathrm{T}} P y^\sigma\}(t)\,\Delta t \\
&= \{(y^\Delta)^{\mathrm{T}} R y^\Delta - (y^\sigma)^{\mathrm{T}} P y^\sigma\}(a)\mu(a) \\
&\quad + \int_{\sigma(a)}^b \{[y^{\mathrm{T}} Q y]^\Delta + (Ry^\Delta - Qy)^{\mathrm{T}} D (Ry^\Delta - Qy)\}(t)\,\Delta t \\
&= \{(y^\sigma)^{\mathrm{T}} R y^\sigma \mu^{-1} - (y^\sigma)^{\mathrm{T}} P y^\sigma \mu\}(a) + \{y^{\mathrm{T}} Q y\}(b) \\
&\quad - \{y^{\mathrm{T}} Q y\}(\sigma(a)) + \int_{\sigma(a)}^b \{(Ry^\Delta - Qy)^{\mathrm{T}} D (Ry^\Delta - Qy)\}(t)\,\Delta t \\
&\geq \{(y^\sigma)^{\mathrm{T}} R y^\sigma \mu^{-1} - (y^\sigma)^{\mathrm{T}} P y^\sigma \mu\}(a) + \{y^{\mathrm{T}} Q y\}(b) - \{y^{\mathrm{T}} Q y\}(\sigma(a)) \\
&= \{(y^\sigma)^{\mathrm{T}} R y^\sigma \mu^{-1} - (y^\sigma)^{\mathrm{T}} P y^\sigma \mu - (y^\sigma)^{\mathrm{T}} [RY^\Delta Y^{-1}]^\sigma y^\sigma\}(a) \\
&= (y^\sigma)^{\mathrm{T}} \{R\mu^{-1} - P\mu - (RY^\Delta + \mu[RY^\Delta]^\Delta)(Y^\sigma)^{-1}\} y^\sigma(a) \\
&= (y^\sigma)^{\mathrm{T}} \{R\mu^{-1} - P\mu - (RY^\sigma\mu^{-1} - \mu P Y^\sigma)(Y^\sigma)^{-1}\} y^\sigma(a) = 0.
\end{aligned}$$

If however, $\mathscr{F}(y) = 0$, then because of Lemma 2(4)

$$\{Ry^\Delta - Qy\}(t) = 0 \quad \text{for all } t \in [\sigma(a),\rho(b)]\cap\mathbb{T},$$

i.e., $y^\Delta(t) = \{Y^\Delta Y^{-1} y\}(t)$ for all $t \in [\sigma(a),\rho(b)]\cap\mathbb{T}$. Since

$$I + \mu Y^\Delta Y^{-1} = I + [Y^\sigma - Y]Y^{-1} = Y^\sigma Y^{-1},$$

the map $Y^\Delta Y^{-1}$ is regressive on $[\sigma(a),\rho(b)]\cap\mathbb{T}$ so that by Lemma 2(2) the initial value problem

$$y^\Delta = \{Y^\Delta Y^{-1}\}(t) y \quad \text{on } [\sigma(a),\rho(b)]\cap\mathbb{T}, \quad y(b) = 0$$

admits only one, namely the trivial solution. Hence $y = 0$ on $\tilde{\mathscr{T}}$ so that $\mathscr{F} > 0$ follows.

Next, let a be right-dense and pick a right-sequence $\{a_m\}_{m\in\mathbb{N}} \subset \tilde{\mathscr{T}}$ for a. With $\lim_{m\to\infty} a_m = a$. Let $m \in \mathbb{N}$. We put $\alpha_m = -\{(Y^\Delta)^\mathrm{T} Ry\}(a_m)$ and apply Picone's identity (with $\alpha = \alpha_m$) to obtain the following:

$$\begin{aligned}
&\int_{a_m}^b \{(y^\Delta)^\mathrm{T} R y^\Delta - (y^\sigma)^\mathrm{T} P y^\sigma\}(t)\,\Delta t\\
&\quad = \int_{a_m}^b [y^\mathrm{T} Q y + 2\alpha_m^\mathrm{T} Y^{-1} y - \alpha_m^\mathrm{T} Y^{-1} Y_1 \alpha_m]^\Delta(t)\,\Delta t\\
&\qquad + \int_{a_m}^b \{Ry^\Delta - Qy - (Y^{-1})^\mathrm{T}\alpha_m\}^\mathrm{T} D\{Ry^\Delta - Qy - (Y^{-1})^\mathrm{T}\alpha_m\}(t)\,\Delta t\\
&\quad \ge \int_{a_m}^b [y^\mathrm{T} Q y + 2\alpha_m^\mathrm{T} Y^{-1} y - \alpha_m^\mathrm{T} Y^{-1} Y_1 \alpha_m]^\Delta(t)\,\Delta t\\
&\quad = -\{\alpha_m^\mathrm{T} Y^{-1} Y_1 \alpha_m\}(b) - \{y^\mathrm{T} Q y + 2\alpha_m^\mathrm{T} Y^{-1} y - \alpha_m^\mathrm{T} Y^{-1} Y_1 \alpha_m\}(a_m)\\
&\quad = -\{\alpha_m^\mathrm{T} Y^{-1} Y_1 \alpha_m\}(b) + \{y^\mathrm{T} Q y + y^\mathrm{T} Q Y_1 (Y^\Delta)^\mathrm{T} R y\}(a_m)\\
&\quad = -\{\alpha_m^\mathrm{T} Y^{-1} Y_1 \alpha_m\}(b) + \{y^\mathrm{T} Q y + y^\mathrm{T} (Y^{-1})^\mathrm{T} [Y^\mathrm{T} R Y_1^\Delta - I](Y^\Delta)^\mathrm{T} R y\}(a_m)\\
&\quad = -\{\alpha_m^\mathrm{T} Y^{-1} Y_1 \alpha_m\}(b) + \{y^\mathrm{T} R Y_1^\Delta (Y^\Delta)^\mathrm{T} R y\}(a_m),
\end{aligned}$$

and hence by Lemma 2(5)

$$\begin{aligned}
\mathscr{F}(y) &= \lim_{m\to\infty} \int_{a_m}^b \{(y^\Delta)^\mathrm{T} R y^\Delta - (y^\sigma)^\mathrm{T} P y^\sigma\}(t)\,\Delta t\\
&\ge -\lim_{m\to\infty} \{\alpha_m^\mathrm{T}(Y^{-1}Y_1)(b)\alpha_m + y^\mathrm{T}(RY_1^\Delta)(a_m)\alpha_m\} = 0.
\end{aligned}$$

Finally, if $\mathscr{F}(y) = 0$, we know that $\lim_{m\to\infty} \int_{a_m}^b \{z_m^\mathrm{T} D z_m\}(t)\,\Delta$ exists and equals 0, where

$$z_m := Ry^\Delta - Qy - (Y^{-1})^\mathrm{T}\alpha_m \to Ry^\Delta - Qy =: z, \quad m \to \infty$$

holds uniformly (observe also Lemma 1(1)) on $[a_k, b] \cap \mathbb{T}$ for each $k \in \mathbb{N}$. Now let $k \in \mathbb{N}$. We first note

$$\int_{a_m}^b \{z_m^\mathrm{T} D z_m\}(t)\,\Delta t \ge \int_{a_k}^b \{z_m^\mathrm{T} D z_m\}(t)\,\Delta t \quad \text{for all } m \ge k.$$

Hence, by applying Lemma 2(6) we have

$$\lim_{m\to\infty} \int_{a_m}^b \{z_m^\mathrm{T} D z_m\}(t)\,\Delta t \ge \int_{a_k}^b \{z^\mathrm{T} D z\}(t)\,\Delta t.$$

Now we let $k \to \infty$ and apply Lemma 2(5) to obtain

$$0 = \lim_{m\to\infty} \int_{a_m}^b \{z_m^\mathrm{T} D z_m\}(t)\,\Delta t \ge \int_a^b \{z^\mathrm{T} D z\}(t)\,\Delta t \ge 0.$$

Hence, again by Lemma 2(4), we have $z=0$ on $\tilde{\mathcal{T}}^{\kappa}$ so that as in the first part of this proof the triviality of y follows. Therefore, $\mathcal{F}$ is again positive definite and the proof of our result is complete. □

We say that the equation (E) is *disconjugate* (on $\tilde{\mathcal{T}}$) if the principal solution $\tilde{Y}$ of (E) satisfies

$$\tilde{Y} \text{ invertible on } (a,b]\cap\mathbb{T} \quad\text{and}\quad \tilde{Y}(\tilde{Y}^{\sigma})^{-1}R^{-1} \text{ positive definite on } (a,\rho(b)]\cap\mathbb{T}.$$

Now we are ready to prove the main result of this paper (see also Result 3 of Section 1).

Theorem 5 (Jacobi's condition).

$$\mathcal{F}>0 \quad \textit{if and only if } (\mathrm{E}) \textit{ is disconjugate.}$$

Proof. First, if (E) is disconjugate, then we may apply our Theorem 4 with $\tilde{Y}$ and $\tilde{Y}_1$, where $\tilde{Y}$ is the principal solution of (E) and $\tilde{Y}_1$ is the associated solution of (E), and hence disconjugacy implies $\mathcal{F}>0$.

Conversely, suppose that (E) is not disconjugate. Then there exists a $t_0\in\mathbb{T}$ with exactly one of the following two properties:

(1) $t_0\in\tilde{\mathcal{T}}\backslash\{a\}$ such that $\tilde{Y}(t)$ is invertible for all $t\in(a,t_0)\cap\mathbb{T}$ and $\tilde{Y}(t_0)$ is singular,

(2) $t_0\in\tilde{\mathcal{T}}^{\kappa}\backslash\{a\}$ such that $\tilde{Y}(t)$ is invertible for all $t\in(a,b]\cap\mathbb{T}$ and $\tilde{D}(t_0)=\{\tilde{Y}(\tilde{Y}^{\sigma})^{-1}R^{-1}\}(t_0)$ is not positive definite.

Let $d\in\mathbb{R}^n\backslash\{0\}$ with $\tilde{Y}(t_0)d=0$ in Case (1) and $d^{\mathrm{T}}(\tilde{Y}^{\mathrm{T}}R\tilde{Y}^{\sigma})(t_0)d=(R\tilde{Y}^{\sigma}d)^{\mathrm{T}}\tilde{D}(R\tilde{Y}^{\sigma}d)(t_0)\le 0$ in Case (2). Putting

$$y(t)=\begin{cases}\tilde{Y}(t)d & \text{for } t\le t_0,\\ 0 & \text{otherwise}\end{cases}$$

yields $y(a)=y(b)=0$, $y(t)\neq 0$ for all $t\in(a,t_0)$, and, moreover, $y\in C^1_{\mathrm{p}}(\tilde{\mathcal{T}})$ except for Case (2) with t_0 right-dense. Hence we assume for the moment not being in this situation, and we will deal with this special case later. Now it is easy to see that both $y^{\Delta}(t)$ and $y^{\sigma}(t)$ are zero for all $t>t_0$ so that

$$\{(y^{\Delta})^{\mathrm{T}}Ry^{\Delta}-(y^{\sigma})^{\mathrm{T}}Py^{\sigma}\}(t)=0 \quad \text{for all } t>t_0.$$

In the following we use Lemmas 5 and 2(3), and for $\alpha\in\mathbb{R}$ we put for convenience $\alpha^{\dagger}=0$ if $\alpha=0$ and $\alpha^{\dagger}=\alpha^{-1}$ if $\alpha\neq 0$. We have

$$\begin{aligned}\mathcal{F}(y)&=\int_a^{\sigma(t_0)}\{(y^{\Delta})^{\mathrm{T}}Ry^{\Delta}-(y^{\sigma})^{\mathrm{T}}Py^{\sigma}\}(t)\,\Delta t\\ &=\int_a^{t_0}\{(y^{\Delta})^{\mathrm{T}}Ry^{\Delta}-(y^{\sigma})^{\mathrm{T}}Py^{\sigma}\}(t)\,\Delta t+\int_{t_0}^{\sigma(t_0)}\{(y^{\Delta})^{\mathrm{T}}Ry^{\Delta}-(y^{\sigma})^{\mathrm{T}}Py^{\sigma}\}(t)\,\Delta t\end{aligned}$$

$$
\begin{aligned}
&= \{d^{\mathrm{T}}\tilde{Y}^{\mathrm{T}}R\tilde{Y}^{\Delta}d\}(t_0) + \{(y^{\Delta})^{\mathrm{T}}Ry^{\Delta} - (y^{\sigma})^{\mathrm{T}}Py^{\sigma}\}(t_0)\mu(t_0)\\
&= d^{\mathrm{T}}\{\tilde{Y}^{\mathrm{T}}R\tilde{Y}^{\Delta} + \tilde{Y}^{\mathrm{T}}R\tilde{Y}\mu^{\dagger}\}(t_0)d\\
&= d^{\mathrm{T}}\{\mu^{\dagger}\tilde{Y}^{\mathrm{T}}R\tilde{Y}^{\sigma} + [1-\mu^{\dagger}\mu]\tilde{Y}^{\mathrm{T}}R\tilde{Y}^{\Delta}\}(t_0)d \le 0.
\end{aligned}
$$

Therefore $\mathscr{F} \not> 0$.

Now we will discuss the remaining Case 2 with right-dense t_0. This situation implies $R(t_0) \not> 0$, i.e., there exists a $d \in \mathbb{R}^n$ such that $d^{\mathrm{T}}R(t_0)d < 0$. We have to show $\mathscr{F} \not> 0$, and for the sake of achieving a contradiction we assume $\mathscr{F} > 0$. First we suppose that t_0 is left-scattered. Then $t_0 \neq b$. Let $\{t_m\}_{m\in\mathbb{N}} \subset \tilde{\mathscr{T}}$ be a right-sequence for t_0. For $m \in \mathbb{N}$ we put

$$
y_m(t) = \begin{cases} \dfrac{t_m - t}{\sqrt{t_m - t_0}}d & \text{if } t \in (t_0, t_m] \cap \mathbb{T},\\ 0 & \text{otherwise.}\end{cases}
$$

Hence $y_m(a) = y_m(b) = 0$, $y_m(t_0) = \sqrt{t_m - t_0}\,d \neq 0$, and $y \in C_{\mathrm{p}}^1(\tilde{\mathscr{T}})$. Therefore

$$
\begin{aligned}
0 < \mathscr{F}(y_m) &= \int_{t_0}^{\sigma(t_m)} \{(y^{\Delta})^{\mathrm{T}}Ry^{\Delta} - (y^{\sigma})^{\mathrm{T}}Py^{\sigma}\}(t)\,\Delta t\\
&= \int_{t_0}^{\sigma(t_m)} \left\{ \frac{d^{\mathrm{T}}}{\sqrt{t_m - t_0}}R(t)\frac{d}{\sqrt{t_m - t_0}} - d^{\mathrm{T}}\frac{t_m - \sigma(T)}{\sqrt{t_m - t_0}}P(t)\frac{t_m - \sigma(t)}{\sqrt{t_m - t_0}}d \right\}\Delta t\\
&= d^{\mathrm{T}}\left\{\frac{1}{t_m - t_0}\int_{t_0}^{\sigma(t_m)} R(t)\Delta t\right\}d - d^{\mathrm{T}}\left\{\int_{t_0}^{\sigma(t_m)} \frac{(t_m - \sigma(t))^2}{t_m - t_0}P(t)\Delta t\right\}d\\
&\to d^{\mathrm{T}}R(t_0)d < 0, \quad m \to \infty,
\end{aligned}
$$

which yields our desired contradiction. The remaining case of left-dense t_0 can be treated the same way with

$$
y_m(t) = \begin{cases} \frac{t - t_m^*}{\sqrt{t_0 - t_m^*}}d & \text{if } t \in [t_m^*, t_0) \cap \mathbb{T},\\ 0 & \text{otherwise,}\end{cases}
$$

or

$$
y_m(t) = \begin{cases} \frac{t - t_m^*}{\sqrt{t_0 - t_m^*}}d & \text{if } t \in [t_m^*, t_0) \cap \mathbb{T},\\ \frac{t_m - t}{\sqrt{t_m - t_0}}d & \text{if } t \in (t_0, t_m] \cap \mathbb{T},\\ 0 & \text{otherwise,}\end{cases}
$$

if $t_0 = b$ or otherwise, respectively, using a strictly increasing sequence $\{t_m^*\}_{m\in\mathbb{N}} \subset \tilde{\mathscr{T}}$ with $\lim_{m\to\infty} t_m^* = t_0$. $\square$

6. Sturmian theory on time scales

We call a solution Y of (E) *basis* whenever

$$\operatorname{rank}\begin{pmatrix} Y(a) \\ Y^{\Delta}(a) \end{pmatrix} = n.$$

A conjoined solution is said to have *no focal points* in $(a,b]$ provided it satisfies

$$Y \text{ invertible on } \tilde{\mathcal{T}}\backslash\{a\} \quad \text{and} \quad Y(Y^{\sigma})^{-1}R^{-1} \text{ positive definite on } \tilde{\mathcal{T}}^{\kappa}\backslash\{a\}.$$

Using this terminology, equation (E) is disconjugate (on $\tilde{\mathcal{T}}$) if and only if the principal solution of (E) (at a) has no focal points in $(a,b]$. Concerning conjoined bases of (E) we have the following version of Sturm's separation result on time scales.

Theorem 6 (Sturm's separation theorem). *Suppose there exists a conjoined basis of* (E) *with no focal points in* $(a,b]$. *Then equation* (E) *is disconjugate (on* $\tilde{\mathcal{T}}$*).*

Proof. Let Y be a conjoined basis of (E) satisfying

$$Y \text{ invertible on } (a,b]\cap\mathbb{T} \quad \text{and} \quad Y(Y^{\sigma})^{-1}R^{-1} \text{ positive definite on } (a,\rho(b)]\cap\mathbb{T}.$$

Since Y is a basis, we note that the matrix

$$K = Y^{\mathrm{T}}(a)Y(a) + (Y^{\Delta})^{\mathrm{T}}(a)R^2(a)Y^{\Delta}(a) \quad \text{is invertible.}$$

Let Y_1 be the solution of

$$\text{(E)}, \quad Y_1(a) = -R(a)Y^{\Delta}(a)K^{-1}, \quad Y_1^{\Delta}(a) = R^{-1}(a)Y(a)K^{-1}.$$

Then in view of the Wronskian identity, Lemma 3, Y_1 satisfies

$$\begin{aligned}
\{Y_1^{\mathrm{T}}RY_1^{\Delta} - (Y_1^{\Delta})^{\mathrm{T}}RY_1\}(t) &\equiv \{Y_1^{\mathrm{T}}RY_1^{\Delta} - (Y_1^{\Delta})^{\mathrm{T}}RY_1\}(a) \\
&= -(K^{-1})^{\mathrm{T}}(Y^{\Delta})^{\mathrm{T}}(a)R(a)Y(a)K^{-1} + (K^{-1})^{\mathrm{T}}Y^{\mathrm{T}}(a)R(a)Y^{\Delta}(a)K^{-1} \\
&= (K^{-1})^{\mathrm{T}}\{Y^{\mathrm{T}}RY^{\Delta} - (Y^{\Delta})^{\mathrm{T}}RY\}(a)K^{-1} = 0,
\end{aligned}$$

and

$$\begin{aligned}
\{Y^{\mathrm{T}}RY_1^{\Delta} - (Y^{\Delta})^{\mathrm{T}}RY_1\}(t) &\equiv \{Y^{\mathrm{T}}RY_1^{\Delta} - (Y^{\Delta})^{\mathrm{T}}RY_1\}(a) \\
&= Y^{\mathrm{T}}(a)Y(a)K^{-1} + (Y^{\Delta})^{\mathrm{T}}(a)R^2(a)Y^{\Delta}(a)K^{-1} = KK^{-1} = I,
\end{aligned}$$

and hence Y and Y_1 are normalized conjoined solutions of (E). An application of Theorem 4 now yields that $\mathcal{F} > 0$. However, now Jacobi's condition, Theorem 5, in turn shows that (E) is disconjugate, i.e., the principal solution of (E) has no focal points in $(a,b]$. $\square$

In the final result of this paper we shall also consider the equation

$$(\tilde{\mathrm{E}})\quad [\tilde{R}(t)Y^{\Delta}]^{\Delta}+\tilde{P}(t)Y^{\sigma}=0,$$

where $\tilde{R}$ and $\tilde{P}$ satisfy the same assumptions as R and P.

Theorem 7 (Sturm's comparison theorem). *Suppose we have for all $t\in\mathbb{T}$*

$$\tilde{R}(t)\le R(t) \quad\text{and}\quad \tilde{P}(t)\ge P(t).$$

Then, if $(\tilde{\mathrm{E}})$ *is disconjugate, then* (E) *is also disconjugate.*

Proof. Suppose $(\tilde{\mathrm{E}})$ is disconjugate. Then by Jacobi's condition, Theorem 5,

$$\tilde{\mathscr{F}}(y)=\int_a^b \{(y^{\Delta})^{\mathrm{T}}\tilde{R}y^{\Delta}-(y^{\sigma})^{\mathrm{T}}\tilde{P}y^{\sigma}\}(t)\,\Delta t>0$$

for all nontrivial $y\in C_p^1(\tilde{\mathscr{T}})$ with $y(a)=y(b)=0$. For such a y we also have

$$\begin{aligned}\mathscr{F}(y)&=\int_a^b \{(y^{\Delta})^{\mathrm{T}}Ry^{\Delta}-(y^{\sigma})^{\mathrm{T}}Py^{\sigma}\}(t)\,\Delta t\\&\ge\int_a^b \{(y^{\Delta})^{\mathrm{T}}\tilde{R}y^{\Delta}-(y^{\sigma})^{\mathrm{T}}\tilde{P}y^{\sigma}\}(t)\,\Delta t=\tilde{\mathscr{F}}(y)>0.\end{aligned}$$

Hence $\mathscr{F}>0$ and thus (E) is disconjugate by Theorem 5. □

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