RICCATI MATRIX DIFFERENCE EQUATIONS AND LINEAR HAMILTONIAN DIFFERENCE SYSTEMS

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Abstract. Making use of a recently proved Reid Roundabout Theorem for linear Hamiltonian difference systems in the so-called singular case, we derive a Sturm Separation Theorem, a Sturm Comparison Theorem, and a characterization of a discrete quadratic functional being positive definite subject to separated boundary conditions. These results enable us to prove the main theorem of this paper which states the equivalence of positive definiteness of a certain discrete quadratic functional and solvability of some Riccati matrix difference equation together with a certain matrix inequality.

Key Words: Discrete Riccati equation, Hamiltonian difference system, discrete quadratic functional, Reid Roundabout Theorem.

AMS Subject Classifications: 39A10, 93C55, 34C10.

1. Introduction

Strongly related to positive definiteness of the discrete quadratic functional

$$\mathcal{F}_0(x,u) = \sum_{k=0}^{N} \left\{ x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k \right\}$$

are linear Hamiltonian difference systems

$$\text{(H)} \qquad \Delta X_k = A_k X_{k+1} + B_k U_k, \quad \Delta U_k = C_k X_{k+1} - A_k^T U_k, \quad 0 \le k \le N$$

as well as Riccati matrix difference equations of the form

$$\text{(R)} \qquad Q_{k+1} = C_k + (I - A_k^T) Q_k (I + B_k Q_k)^{-1} (I - A_k), \quad 0 \le k \le N.$$

Here, $A_k, B_k, C_k, X_k, U_k, Q_k$ are $n \times n$-matrices with $I - A_k$ invertible and B_k, C_k, Q_k symmetric; $x_k, u_k \in I\!R^n$; and $\mathcal{F}_0$ is called positive definite if

$$\begin{cases} \mathcal{F}_0(x,u) > 0 \text{ for all } (x,u) \text{ with} \\ x_0 = x_{N+1} = 0,\ x \ne 0, \quad \text{and} \quad \Delta x_k = A_k x_{k+1} + B_k u_k,\ 0 \le k \le N. \end{cases}$$

In [9], L. Erbe and P. Yan who introduced systems (H) in [8], conjectured that positive definiteness of $\mathcal{F}_0$ and the matrices B_k imply the existence of symmetric matrices Q_k satisfying (R) such that the matrices $(I + B_k Q_k)^{-1} B_k$ are positive semidefinite. Such a result would be of great interest in optimal control theory, where this question arises in applications including the discrete regulator problem, discrete Kalman filtering, robust control, and H$^\infty$ control, as is pointed out by C. Ahlbrandt and M. Heifetz in [2] (see also [1]). In the present paper, we prove the following central result.

Theorem A. *Let B_k and C_k be* **any** *symmetric matrices, $0 \le k \le N$. Then $\mathcal{F}_0$ is positive definite if and only if there exist symmetric matrices Q_k, $0 \le k \le N$, which satisfy both*

$$Q_{k+1} = C_k + (I - A_k^T) Q_k (I + B_k Q_k)^{-1} (I - A_k), \quad 0 \le k \le N$$

and

$$(I + B_k Q_k)^{-1} B_k \ge 0, \quad 0 \le k \le N.$$

This result comes much harder than the corresponding result when the B_k are assumed to be invertible (compare e.g. [8, Theorem 2.5]). Besides the theory for systems (H) which allows the B_k to be **singular** matrices (so that the important case of Sturm-Liouville difference equations of higher order is included) and which has been presented by the author in [4, 5, 6] (see also an extension to symplectic systems [7] in a joint work with O. Došlý), the proof of Theorem A requires a Sturm Separation Theorem as well as a result on a more general discrete quadratic functional subject to so-called separated boundary conditions. These results are contained in Sections 3 and 4 below, and a Sturm Comparison Theorem is added to Section 3. Section 2 contains some preliminaries, e.g., Picone's identity and Jacobi's condition from [4], while Section 5 is devoted to the proof of our main result, Theorem A above.

2. Preliminary Results

Let $n \in \mathbb{N}$, $N \in \mathbb{N} \cup \{0\}$, $J := [0, N] \cap \mathbb{Z}$, $J^* := [0, N+1] \cap \mathbb{Z}$, and let be given $n \times n$-matrices A_k, B_k, C_k, $k \in J$, such that $\tilde{A}_k := (I - A_k)^{-1}$ exists and such that B_k, C_k are symmetric. We say that (X, U) is a *conjoined basis* of

$$\text{(H)} \qquad \Delta X_k = A_k X_{k+1} + B_k U_k, \quad \Delta U_k = C_k X_{k+1} - A_k^T U_k$$

whenever the $n \times n$-matrices X_k, U_k, $k \in J^*$, solve (H) with $X_0^T U_0$ symmetric and $\operatorname{rank}(X_0^T \quad U_0^T) = n$, and this implies as is very well know that

$$X_k^T U_k = U_k^T X_k \quad \text{and} \quad \operatorname{rank}(X_k^T \quad U_k^T) = n \quad \text{for all} \quad k \in J^*$$

hold. Here, Δ is the forward difference operator, i.e., $\Delta X_k = X_{k+1} - X_k$, $k \in J$. Furthermore, for any given conjoined basis (X, U) of (H) there exists another conjoined basis $(\tilde{X}, \tilde{U})$ of (H) such that $X_k^T \tilde{U}_k - U_k^T \tilde{X}_k \equiv I$ (I being the identity matrix) holds, and then we say that (X, U), $(\tilde{X}, \tilde{U})$ are *normalized.* Also, (H) together with any initial values for X and U is uniquely solvable due to our assumptions on the matrices A_k. The solutions of (H) with

$$X_0 = 0, \quad U_0 = I \quad \text{and} \quad \tilde{X}_0 = -I, \quad \tilde{U}_0 = 0$$

are called the *special* normalized conjoined bases of (H) at 0, and (X, U) is said to be the *principal solution* of (H) at 0.

Now, if $x_k, u_k \in \mathbb{R}^n$, $k \in J^*$, with $\Delta x_k = A_k x_{k+1} + B_k u_k$ for all $k \in J$, then (x,u) is referred to as being *admissible*, and

$$\mathcal{F}_0(x,u)=\sum_{k=0}^{N}\{x_{k+1}^TC_kx_{k+1}+u_k^TB_ku_k\}$$

is called *positive definite* whenever $\mathcal{F}_0(x,u)>0$ for all admissible (x,u) with $x_0=x_{N+1}=0$ and $x\neq 0$. Moreover, for $2n\times 2n$-matrices R and S such that S is symmetric, we say that the functional

$$\mathcal{F}(x,u)=\sum_{k=0}^{N}\{x_{k+1}^TC_kx_{k+1}+u_k^TB_ku_k\}+\begin{pmatrix}-x_0\\ x_{N+1}\end{pmatrix}^TS\begin{pmatrix}-x_0\\ x_{N+1}\end{pmatrix}$$

is positive definite if

$$\mathcal{F}(x,u)>0 \text{ holds for all admissible } (x,u) \text{ with } \begin{pmatrix}-x_0\\ x_{N+1}\end{pmatrix}\in \mathrm{Im}R^T \text{ and } x\neq 0.$$

For a conjoined basis (X,U) of (H) we use the notation

$$D_k=X_kX_{k+1}^{\dagger}\tilde{A}_kB_k,$$

where $M^{\dagger}$ stands for the *Moore-Penrose inverse* of the (arbitrary) matrix M, see e.g. [3, Lemma 1.5], and, of course, the notations

$$\mathrm{KerM},\quad \mathrm{Im}M,\quad M\geq 0,\quad M>0$$

stand for the kernel of M, the image of M, M is positive semidefinite, and M is positive definite, respectively.

We now state some elementary results which will be used in Sections 3, 4, and 5 below. The reader may find these results and their proofs (respectively results that differ only in minor inessential details from the stated ones) in the given references.

Lemma 1 ([5, Lemma 1] or [8, (1.7)]). *Let (x,u) be admissible such that $\Delta u_k=C_kx_{k+1}-A_k^Tu_k$ for all $k\in J$. Then $\mathcal{F}_0(x,u)=x_{N+1}^Tu_{N+1}-x_0^Tu_0$.*

Lemma 2 ([4, Lemma 4] or [5, Lemma 2 (ii)]). *If (X,U) is a conjoined basis of (H) and if $\mathrm{Ker}X_{m+1}\subset\mathrm{Ker}X_m$ for some $m\in J$, then D_m is symmetric.*

Lemma 3 ([8, Proposition 1.3] or [9, Theorem 6]). *There exist symmetric matrices Q_k, $k\in J^*$, satisfying (R) iff there is a conjoined basis (X,U) of (H) such that X_k are invertible for all $k\in J^*$. In this case, $Q_k:=U_kX_k^{-1}$, $k\in J^*$, are symmetric matrices satisfying (R), and $D_k=(I+B_kQ_k)^{-1}B_k$ holds.*

Proposition 1 ([5, Propositon 1]). *Let (X, U) be a conjoined basis of (H). Let $m \in J$. If* $\text{Ker}X_{m+1} \not\subset \text{Ker}X_m$*, then there exists an admissible (x, u) with*

$$x_0 = X_0 d \in \text{Im}X_0, \quad x_{N+1} = 0, \quad x \neq 0, \quad \textit{and} \quad \mathcal{F}_0(x, u) = -d^T X_0^T U_0 d.$$

Otherwise, for any $c \in I\!R^n$, there exists an admissible (x, u) with

$$x_0 = X_0 d \in \text{Im}X_0, \quad x_{N+1} = 0, \quad \textit{and} \quad \mathcal{F}_0(x, u) = c^T D_m c - d^T X_0^T U_0 d.$$

Proposition 2 (Picone's Identity; [5, Propositon 4 and Lemma 2]). *Let (X, U), $(\tilde{X}, \tilde{U})$ be normalized conjoined bases with* $\text{Ker}X_{k+1} \subset \text{Ker}X_k$ *for all $k \in J$. Then, for admissible (x, u) and $\alpha \in I\!R^n$ with $\alpha + U_0^T x_0 \in \text{Im}X_0^T$, we have*

$$\begin{cases} \Delta\left\{\binom{\alpha}{x_k}^T Q_k^* \binom{\alpha}{x_k}\right\} = x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k - z_k^T D_k z_k, \\ x_k + D_k z_k = \left\{I - A_k - B_k \tilde{A}_k^T (Q_{k+1} - C_k)\right\} x_{k+1} - B_k \tilde{A}_k^T \tilde{Q}_{k+1}^T \alpha \end{cases}$$

for all $k \in J$, where

$$Q_k = U_k X_k^\dagger + (U_k X_k^\dagger \tilde{X}_k - \tilde{U}_k)(I - X_k^\dagger X_k) U_k^T \;\; \textit{and} \;\; \tilde{Q}_k = X_k^\dagger + X_k^\dagger \tilde{X}_k (I - X_k^\dagger X_k) U_k^T,$$

$$Q_k^* = \begin{pmatrix} -X_k^\dagger \tilde{X}_k X_k^\dagger X_k & \tilde{Q}_k \\ \tilde{Q}_k^T & Q_k \end{pmatrix}$$ *is symmetric, and where $z_k = u_k - Q_k x_k - \tilde{Q}_k^T \alpha$.*

Now, as is done in [4, Definition 4] or in [5, Definition 3], it makes sense to say that a conjoined basis (X, U) of (H) does not have a *focal point* in $(0, N+1]$ iff

$$\text{Ker}X_{k+1} \subset \text{Ker}X_k \quad \text{and} \quad D_k \geq 0 \quad \text{for all} \quad k \in J$$

hold. The following result then is an immediate consequence of Proposition 2 (with $\alpha = 0$) and of Proposition 1, while Proposition 2 (with $\alpha = -x_0$) gives proof for the subsequent result.

Proposition 3 (Jacobi's Condition). *$\mathcal{F}_0$ is positive definite if and only if the principal solution of (H) at 0 does not have a focal point in $(0, N+1]$.*

Proposition 4. *If the principal solution of (H) at 0 does not have a focal point in $(0, N+1]$ and if $R(S + Q_{N+1}^*)R^T > 0$ on* $\text{Im}R$*, then $\mathcal{F}$ is positive definite.*

3. Sturmian Theory

Theorem 1 (Sturm Separation Theorem). *If the principal solution of (H) at* 0 *has a focal point in* $(0, N+1]$*, then* **any** *conjoined basis of (H) does as well.*

Proof. Let (X, U) be a conjoined basis of (H) without focal points in $(0, N+1]$. For $k \in J^*$ we put

$$Q_k = U_k X_k^\dagger + (U_k X_k^\dagger \tilde{X}_k - \tilde{U}_k)(I - X_k^\dagger X_k)U_k^T \quad \text{and} \quad z_k = u_k - Q_k x_k,$$

and then Proposition 2 with $\alpha = 0$ yields that for any admissible (x, u) with $x_0 = x_{N+1} = 0$

$$\mathcal{F}_0(x, u) = \sum_{k=0}^{N} z_k^T D_k z_k \geq 0$$

holds. If $\mathcal{F}_0(x, u)$ vanishes, then we have $D_k z_k = 0$ for all $k \in J$, and the second formula in Proposition 2 yields $x_N = x_{N-1} = \ldots = x_1 = 0$ so that $\mathcal{F}_0$ is positive definite. This in turn shows by Proposition 3 that the principal solution of (H) at 0 does not have a focal point in $(0, N+1]$, and this proves our result. ■

While we make use of the Sturm Separation Theorem when proving our main result, the Sturm Comparison Theorem is not needed later on. However, since it fits into the context, we give it here for completeness. Observe first that we have

$$\begin{aligned} -&\begin{pmatrix} x_{k+1} \\ \Delta x_k \end{pmatrix}^T \begin{pmatrix} -C_k - A_k^T B_k^\dagger A_k & A_k^T B_k^\dagger \\ B_k^\dagger A_k & -B_k^\dagger \end{pmatrix} \begin{pmatrix} x_{k+1} \\ \Delta x_k \end{pmatrix} - x_{k+1}^T C_k x_{k+1} \\ =\ & x_{k+1}^T A_k^T B_k^\dagger A_k x_{k+1} - x_{k+1}^T A_k^T B_k^\dagger \Delta x_k - \Delta x_k^T B_k^\dagger A_k x_{k+1} + \Delta x_k^T B_k^\dagger \Delta x_k \\ =\ & \{\Delta x_k - A_k x_{k+1}\}^T B_k^\dagger \{\Delta x_k - A_k x_{k+1}\} = u_k^T B_k B_k^\dagger B_k u_k = u_k^T B_k u_k \end{aligned}$$

for all $k \in J$ whenever (x, u) is admissible.

Theorem 2 (Sturm Comparison Theorem). *Let* $\underset{\sim}{A}_k, \underset{\sim}{B}_k, \underset{\sim}{C}_k$ *satisfy our general assumptions as well and put*

$$H_k = \begin{pmatrix} -C_k - A_k^T B_k^\dagger A_k & A_k^T B_k^\dagger \\ B_k^\dagger A_k & -B_k^\dagger \end{pmatrix} \ \textit{and}\ \underset{\sim}{H}_k = \begin{pmatrix} -\underset{\sim}{C}_k - \underset{\sim}{A}_k^T \underset{\sim}{B}_k^\dagger \underset{\sim}{A}_k & \underset{\sim}{A}_k^T \underset{\sim}{B}_k^\dagger \\ \underset{\sim}{B}_k^\dagger \underset{\sim}{A}_k & -\underset{\sim}{B}_k^\dagger \end{pmatrix}$$

for $k \in J$. Then, if the principal solution of

$$(\underset{\sim}{H}) \qquad \Delta X_k = \underset{\sim}{A}_k X_{k+1} + \underset{\sim}{B}_k U_k, \quad \Delta U_k = \underset{\sim}{C}_k X_{k+1} - \underset{\sim}{A}_k^T U_k$$

at 0 has a focal point in $(0, N+1]$ and if

$$\underset{\sim}{H}_k \le H_k \quad \text{and} \quad \mathrm{Im}(\underset{\sim}{A}_k - A_k \quad \underset{\sim}{B}_k) \subset \mathrm{Im} B_k \quad \text{for all} \quad k \in J$$

hold, then **any** *conjoined basis of*

$$(\mathrm{H}) \qquad \Delta X_k = A_k X_{k+1} + B_k U_k, \quad \Delta U_k = C_k X_{k+1} - A_k^T U_k$$

has also a focal point in $(0, N+1]$.

Proof. Again we assume that there exists a conjoined basis of (H) without focal points in $(0, N+1]$. Hence,

$$\begin{cases} \sum_{k=0}^{N} \left\{ x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k \right\} > 0 \\ \text{for all (w.r.t. (H)) admissible } (x,u) \text{ with } x_0 = x_{N+1} = 0 \text{ and } x \ne 0 \end{cases}$$

by Theorem 1 and Proposition 3. Now let $(x, \underset{\sim}{u})$ be admissible w.r.t. $(\underset{\sim}{H})$ such that $x_0 = x_{N+1} = 0$ and $x \ne 0$ hold. Due to our assumptions, there exist u_k, $k \in J$, with

$$B_k u_k = \left\{ \underset{\sim}{A}_k - A_k \right\} x_{k+1} + \underset{\sim}{B}_k \underset{\sim}{u}_k = \underset{\sim}{A}_k x_{k+1} + \underset{\sim}{B}_k \underset{\sim}{u}_k - A_k x_{k+1} = \Delta x_k - A_k x_{k+1}$$

so that (x, u) turns out to be admissible w.r.t. (H). Thus,

$$\begin{aligned} \sum_{k=0}^{N} \left\{ x_{k+1}^T \underset{\sim}{C}_k x_{k+1} + \underset{\sim}{u}_k^T \underset{\sim}{B}_k \underset{\sim}{u}_k \right\} &= -\sum_{k=0}^{N} \begin{pmatrix} x_{k+1} \\ \Delta x_k \end{pmatrix}^T \underset{\sim}{H}_k \begin{pmatrix} x_{k+1} \\ \Delta x_k \end{pmatrix} \\ &\ge -\sum_{k=0}^{N} \begin{pmatrix} x_{k+1} \\ \Delta x_k \end{pmatrix}^T H_k \begin{pmatrix} x_{k+1} \\ \Delta x_k \end{pmatrix} \\ &= \sum_{k=0}^{N} \left\{ x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k \right\} > 0. \end{aligned}$$

Hence,

$$\begin{cases} \sum_{k=0}^{N} \left\{ x_{k+1}^T \underset{\sim}{C}_k x_{k+1} + u_k^T \underset{\sim}{B}_k u_k \right\} > 0 \\ \text{for all (w.r.t. } (\underset{\sim}{H})) \text{ admissible } (x,u) \text{ with } x_0 = x_{N+1} = 0 \text{ and } x \ne 0 \end{cases}$$

so that the principal solution of $(\underset{\sim}{H})$ at 0 does not have a focal point in $(0, N+1]$ again by Proposition 3. ∎

4. Separated Boundary Conditions

The proof of our main theorem still requires one more result which we give in this section. We say that the boundary conditions are separated whenever the matrices R and S are of the form

$$R = \begin{pmatrix} R_0 & 0 \\ 0 & R_{N+1} \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} -S_0 & 0 \\ 0 & S_{N+1} \end{pmatrix}$$

with $n \times n$-matrices $R_0, R_{N+1}, S_0, S_{N+1}$ such that both S_0 and S_{N+1} are symmetric. These assumptions yield

$$\mathcal{F}(x,u) = \mathcal{F}_0(x,u) + x_{N+1}^T S_{N+1} x_{N+1} - x_0^T S_0 x_0$$

and $\binom{-x_0}{x_{N+1}} \in \mathrm{Im} R^T$ iff $x_0 \in \mathrm{Im} R_0^T$ and $x_{N+1} \in \mathrm{Im} R_{N+1}^T$. Now we choose invertible matrices T and $\tilde{T}$ such that $\tilde{T}^{-1} R_0 T = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ holds (I being the $\mathrm{rank} R_0 \times \mathrm{rank} R_0$-identity matrix) and put

$$R_0^* = \tilde{T} T^T + R_0 (S_0 - T T^T) \quad \text{and} \quad S_0^* = \tilde{T}(I - \tilde{T}^{-1} R_0 T) T^T.$$

Then we have $R_0^* = R_0 S_0 + S_0^*$, $R_0 S_0^{*T} = 0$, $R_0 R_0^{*T} = R_0 S_0 R_0^T = R_0^* R_0^T$, and (see also [10, Corollary 3.1.3])

$$\begin{aligned} &\mathrm{rank}(R_0^* \quad R_0) = \mathrm{rank}(S_0^* \quad R_0) \\ &= \mathrm{rank}\left\{ \tilde{T}^{-1}(S_0^* \quad R_0) \begin{pmatrix} (T^{-1})^T & 0 \\ 0 & T \end{pmatrix} \right\} = \mathrm{rank} \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \end{pmatrix} = n. \end{aligned}$$

Now, let (X, U) be the solution of (H) with

$$X_0 = -R_0^T \quad \text{und} \quad U_0 = R_0^{*T}.$$

By what we have shown, (X, U) is a conjoined basis of (H). In our central result concerning separated boundary conditions we use the notation introduced so far.

Theorem 3 (Separated Boundary Conditions). *Let (X, U) be the conjoined basis of (H) with $X_0 = -R_0^T$ and $U_0 = R_0^{*T}$, put*

$$Q_k = U_k X_k^\dagger + (U_k X_k^\dagger \tilde{X}_k - \tilde{U}_k)(I - X_k^\dagger X_k) U_k^T, \quad k \in J^*,$$

and consider the two conditions

(i) (X, U) *does not have a focal point in* $(0, N+1]$*;*

(ii) $M := R_{N+1}(S_{N+1} + Q_{N+1})R_{N+1}^T > 0$ *on* $\mathrm{Im}R_{N+1}$.

Then (i) and (ii) imply the positive definiteness of $\mathcal{F}$*. If* $\mathcal{F}$ *is positive definite and if* $\mathrm{Im}R_{N+1}^T \subset \mathrm{Im}X_{N+1}$ *holds, then (i) and (ii) hold as well.*

Proof. First assume (i) and (ii) hold. Let (x, u) be admissible with $x_0 = R_0^T c_0$ and $x_{N+1} = R_{N+1}^T c_{N+1}$. This yields $x_0 \in \mathrm{Im}X_0$ so that we may apply Picone's identity, Proposition 2, with $\alpha = 0$. Hence,

$$\begin{aligned}
\mathcal{F}(x, u) &= x_{N+1}^T Q_{N+1} x_{N+1} - x_0^T Q_0 x_0 + \sum_{k=0}^{N} z_k^T D_k z_k \\
&\quad + x_{N+1}^T S_{N+1} x_{N+1} - x_0^T S_0 x_0 \\
&= c_{N+1}^T M c_{N+1} + \sum_{k=0}^{N} z_k^T D_k z_k - c_0^T \left\{ R_0 U_0 X_0^\dagger R_0^T + R_0 S_0 R_0^T \right\} c_0 \\
&= c_{N+1}^T M c_{N+1} + \sum_{k=0}^{N} z_k^T D_k z_k \geq 0
\end{aligned}$$

since

$$R_0 U_0 X_0^\dagger R_0^T = -R_0 R_0^{*^T} (R_0^\dagger)^T R_0^T = -R_0^* R_0^T = -R_0 S_0 R_0^T.$$

If $\mathcal{F}(x, u)$ vanishes, then we have $c_{N+1} = 0$ and $D_k z_k = 0$ for all $k \in J$ so that $x_{N+1} = 0$ follows which implies $x_N = x_{N-1} = \ldots = x_1 = x_0 = 0$ by the second part of Proposition 2. Thus $\mathcal{F}$ is positive definite.

Now there are two possibilities that (i) doesn't happen: First there might be $m \in J$ with $\mathrm{Ker}X_{m+1} \not\subset \mathrm{Ker}X_m$ in which case Proposition 1 yields the existence of an admissible (x, u) with $x_0 = X_0 d \in \mathrm{Im}R_0^T$, $x_{N+1} = 0 \in \mathrm{Im}R_{N+1}^T$, $x \neq 0$, and

$$\mathcal{F}(x, u) = d^T R_0 R_0^{*^T} d - x_0^T S_0 x_0 = 0.$$

Otherwise, if $\mathrm{Ker}X_{k+1} \subset \mathrm{Ker}X_k$ for all $k \in J$ but $D_m \not\geq 0$ for some $m \in J$, there exists $c \in \mathbb{R}^n$ with $c^T D_m c < 0$ (see Lemma 2), and then Proposition 1 shows the existence of an admissible (x, u) with

$$\left\{ \begin{array}{l} \quad x_0 = X_0 d \in \mathrm{Im}R_0^T, \ x_{N+1} = 0 \in \mathrm{Im}R_{N+1}^T, \quad \text{and} \\ \mathcal{F}(x, u) = c^T D_m c + d^T R_0 R_0^{*^T} d - x_0^T S_0 x_0 = c^T D_m c < 0. \end{array} \right.$$

It follows from what we have shown so far that $\mathcal{F}$ is not positive definite whenever (i) doesn't hold.

For the remaining part of the proof we suppose $\mathrm{Im}R_{N+1}^T \subset \mathrm{Im}X_{N+1}$ and assume that (ii) does not hold. Pick $c \in \mathrm{Im}R_{N+1} \setminus \{0\}$ and let $d = R_{N+1}^T c$ so that $d \neq 0$. We define

$$x_k = X_k X_{N+1}^\dagger d \quad \text{and} \quad u_k = U_k X_{N+1}^\dagger d, \qquad k \in J^*.$$

Thus $x_0 \in \mathrm{Im}R_0^T$, $x_{N+1} = d \in \mathrm{Im}R_{N+1}^T \setminus \{0\}$, and (by Lemma 1)

$$\begin{aligned} \mathcal{F}(x,u) &= x_{N+1}^T u_{N+1} - x_0^T u_0 + x_{N+1}^T S_{N+1} x_{N+1} - x_0^T S_0 x_0 \\ &= c^T M c - d^T (X_{N+1}^\dagger)^T \left\{ X_0^T U_0 + X_0^T S_0 X_0 \right\} X_{N+1}^\dagger d \\ &= c^T M c \leq 0, \end{aligned}$$

which proves the desired result. ∎

5. Solvability of Riccati Equations

Now most of the work is done but there is still one crucial result missing. However, this result then will yield Theorem A right away by the aid of Proposition 3 and Lemma 3, and it reads as follows.

Theorem 4. *The following conditions are equivalent:*

(i) The principal solution (X,U) of (H) at 0 does not have a focal point in $(0, N+1]$, i.e.,

$$\mathrm{Ker}X_{k+1} \subset \mathrm{Ker}X_k \quad \textit{and} \quad X_k X_{k+1}^\dagger \tilde{A}_k B_k \geq 0 \quad \textit{for all} \quad 0 \leq k \leq N$$

hold;

(ii) There exists a conjoined basis $(\underset{\sim}{X}, \underset{\sim}{U})$ of (H) without focal points in $(0, N+1]$ and such that $\underset{\sim}{X}_0$ is invertible, i.e.,

$$\begin{cases} \underset{\sim}{X}_k \textit{ invertible} & \textit{for all} \quad 0 \leq k \leq N+1, \\ \underset{\sim}{X}_k \underset{\sim}{X}_{k+1}^{-1} \tilde{A}_k B_k \geq 0 & \textit{for all} \quad 0 \leq k \leq N \end{cases}$$

holds.

Proof. Of course, (ii) implies (i) by our Sturm Separation Theorem, Theorem 1 from Section 3. Now we assume (i) and let (X,U), $(\tilde{X}, \tilde{U})$ be the special normalized conjoined bases of (H) at 0. Since $\hat{Q} := -X_{N+1}^\dagger \tilde{X}_{N+1} X_{N+1}^\dagger X_{N+1}$ is symmetric (compare Proposition 2), there exists an orthogonal matrix P such that

$$P\hat{Q}P^T = \hat{P} := \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n),$$

where $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$ are the eigenvalues of $\hat{Q}$. We define

$$\varepsilon := \frac{1}{1+|\lambda|} > 0 \quad \text{with} \quad \lambda := \min_{1 \le i \le n} \lambda_i,$$

and then we have for all $x \in \mathbb{R}^n \setminus \{0\}$ (put $y = Px$)

$$\begin{aligned} x^T(I+\varepsilon\hat{Q})x &= \|x\|^2 + \varepsilon x^T P^T \hat{P} P x = \|y\|^2 + \varepsilon y^T \hat{P} y \\ &= \|y\|^2 + \varepsilon \sum_{i=1}^n y_i^2 \lambda_i \ge \|y\|^2 + \varepsilon\lambda \sum_{i=1}^n y_i^2 \\ &= \|y\|^2 \left\{ 1 + \frac{\lambda}{1+|\lambda|} \right\} \ge \|y\|^2 \left\{ 1 - \frac{|\lambda|}{1+|\lambda|} \right\} > 0 \end{aligned}$$

so that $I + \varepsilon\hat{Q} > 0$. We now let

$$R = \begin{pmatrix} -\varepsilon I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} \frac{1}{\varepsilon} I & 0 \\ 0 & 0 \end{pmatrix}.$$

With Q^*_{N+1} from Proposition 2 we have

$$R(S + Q^*_{N+1})R^T = \begin{pmatrix} \varepsilon^2(\frac{1}{\varepsilon}I + \hat{Q}) & 0 \\ 0 & 0 \end{pmatrix} > 0 \quad \text{on} \quad \operatorname{Im} R$$

from what we have shown above. Proposition 4 (which is part of our main result from [5]) now shows

$$\begin{cases} \sum_{k=0}^{N} \left\{ x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k \right\} + \frac{1}{\varepsilon}\|x_0\|^2 > 0 \\ \text{for all admissible } (x,u) \text{ with } x_{N+1} = 0 \text{ and } x \ne 0. \end{cases}$$

On the other hand, the present boundary conditions are of course separated with

$$R_0 = -\varepsilon I, \quad S_0 = -\frac{1}{\varepsilon} I, \quad R_0^* = I, \quad S_0^* = 0, \quad \text{and} \quad R_{N+1} = S_{N+1} = 0,$$

so we could as well make use of our central result concerning separated boundary conditions, Theorem 3 from Section 4. To do so, let $(\underset{\sim}{X}, \underset{\sim}{U})$ be the conjoined basis of (H) with

$$\underset{\sim}{X}_0 = -R_0^T = \varepsilon I \quad \text{and} \quad \underset{\sim}{U}_0 = R_0^{*T} = I.$$

Since $R_{N+1} = 0$, we have $\mathrm{Im} R^T_{N+1} \subset \mathrm{Im} X_{N+1}$ so that $(\underset{\sim}{X}, \underset{\sim}{U})$ does not have a focal point in $(0, N+1]$ by condition (i) of Theorem 3. Since $\underset{\sim}{X}_0 = \varepsilon I$ is invertible, this yields

$$\begin{cases} \underset{\sim}{X}_k \text{ invertible} & \text{for all} \quad 0 \le k \le N+1, \\ \underset{\sim}{X}_k \underset{\sim}{X}^{-1}_{k+1} \tilde{A}_k B_k \ge 0 & \text{for all} \quad 0 \le k \le N \end{cases}$$

so that we are done. ■

Concluding Remarks. The solution $\underset{\sim}{Q}$ of

$$\text{(R)} \qquad Q_{k+1} = C_k + (I - A^T_k) Q_k (I + B_k Q_k)^{-1} (I - A_k), \quad k \in J,$$

which has been constructed in Theorem 4 above, satisfies the initial condition

$$\underset{\sim}{Q}_0 = \underset{\sim}{U}_0 \underset{\sim}{X}^{-1}_0 = \frac{1}{\varepsilon} I = \{1 + |\lambda|\} I,$$

where λ is the maximal eigenvalue of $X^{\dagger}_{N+1} \tilde{X}_{N+1} X^{\dagger}_{N+1} X_{N+1}$ with the special normalized conjoined bases (X, U), $(\tilde{X}, \tilde{U})$ of (H) at 0. Of course, any Q with initial condition $Q_0 = \mu I$ such that $\mu > 1 + |\lambda|$ will do as well. However, sometimes it may be difficult to compute λ at all.

In [5, Theorem 2] it was shown that $\mathcal{F}_0 > 0$ iff

$$Q_k = U_k X^{\dagger}_k + (U_k X^{\dagger}_k \tilde{X}_k - \tilde{U}_k)(I - X^{\dagger}_k X_k) U^T_k, \quad k \in J^*$$

satisfies the "implicit Riccati equation"

$$(\mathrm{R_I}) \qquad \left\{ \tilde{A}^T_k (Q_{k+1} - C_k) \tilde{A}_k (I + B_k Q_k) - Q_k \right\} G_k = 0, \quad k \in J$$

and the inequalities $B_k - B_k \tilde{A}^T_k (Q_{k+1} - C_k) \tilde{A}_k B_k \ge 0$. Here, the *controllability matrices* G_k are defined by

$$G_k = \left(\tilde{A}_{k-1} \tilde{A}_{k-2} \ldots \tilde{A}_0 B_0 \quad \tilde{A}_{k-1} \tilde{A}_{k-2} \ldots \tilde{A}_1 B_1 \quad \ldots\ldots \quad \tilde{A}_{k-1} \tilde{A}_{k-2} B_{k-2} \quad \tilde{A}_{k-1} B_{k-1} \right),$$

and if the system (H) is *controllable* on J^* with controllability index $\kappa \in J^*$ (see [4, Definition 3] or [5, Definition 5]), then we have $\mathrm{rank} G_k = n$ for all $\kappa \le k \le N+1$, and $(\mathrm{R_I})$ will become (R) in case of $\mathcal{F}_0 > 0$ and for $\kappa \le k \le N$. The advantage of equation $(\mathrm{R_I})$ is that we know the solution very well and also the initial value of the solution which is just $Q_0 = 0$. However, the price for this advantage is a rather complicated equation (R_I) for $0 \le k \le \kappa - 1$, while equation (R) is much easier to deal with.

Finally, we wish to discuss a special situation with separated boundary conditions that has been entitled by "*C-disfocality*" (see e.g. the papers [11, 12, 13] by T. Peil, A. Peterson, and J. Ridenhour). We may call a system (H) C-disfocal on J^* if

$$\begin{cases} \sum_{k=0}^{N} \{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\} - x_{N+1}^T C_N x_{N+1} > 0 \\ \text{for all admissible } (x, u) \text{ with } x_0 = 0 \text{ and } x \neq 0 \end{cases}$$

holds. Using our notation we then have

$$R_0 = S_0 = 0, \quad R_{N+1} = I, \quad S_{N+1} = -C_N, \quad \text{and} \quad S_0^* = R_0^* = I,$$

and the conjoined basis of (H) from Theorem 3 is the solution of (H) with $X_0 = -R_0^T = 0$ and $U_0 = R_0^{*T} = I$, i.e., the principal solution of (H) at 0. If (H) is controllable on J (and this assumption is satisfied e.g. when the B_k are invertible as is assumed in the main result of [12], Theorem 2), then C-disfocality of (H) on J^* is equivalent to each of the following two conditions:

- The principal solution (X, U) of (H) at 0 has no focal point in $(0, N+1]$ and

$$U_{N+1} X_{N+1}^{-1} > C_N \quad \text{(i.e.,} \quad U_N^T (X_N + B_N U_N) > 0);$$

- There exists a solution Q of (R_I) with $Q_0 = 0$ and

$$Q_{N+1} > C_N \quad \text{(i.e.,} \quad Q_N (I + B_N Q_N) > 0).$$

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