Available online at www.sciencedirect.com

SCIENCE DIRECT®

Mathematical and Computer Modelling 43 (2006) 695–703

MATHEMATICAL AND COMPUTER MODELLING

www.elsevier.com/locate/mcm

Spectral analysis of q-difference equations with spectral singularities

Murat Adıvar[a], Martin Bohner[b,*]

[a] Department of Mathematics, Faculty of Science and Literature, Izmir University of Economics, TR-35330 Balcova, Izmir, Turkey
[b] University of Missouri-Rolla, Department of Mathematics and Statistics, 1870 Miner Circle, 106 Rolla Building, 65409 Rolla, MO, United States

Received 28 March 2005; accepted 4 April 2005

Abstract

In this paper we investigate the eigenvalues and the spectral singularities of non-selfadjoint q-difference equations of second order with spectral singularities.

Keywords: Spectral analysis; q-difference equation; Spectral singularity; Eigenvalue

---

1. Introduction

Spectral analysis of Sturm–Liouville, Schrödinger and Klein–Gordon differential equations with spectral singularities have been investigated intensively since 1960 [1–6]. Therefore, spectral analysis of discrete equations became an interesting subject in this field. Some problems of spectral theory of non-selfadjoint discrete Schrödinger and Dirac equations with spectral singularities were studied in [7–10]. Note that spectral analysis of selfadjoint equations has been investigated by Agarwal [11] and Agarwal and Wong [12].

In recent years, quantum calculus received a lot of attention, and most of the published work has been interested in some problems of q-difference equations. In particular, some q-analogues of definitions and theorems (for instance, q-derivative, q-integration, q-exponential function, q-trigonometric function, q-Taylor formula, q-Beta and Gamma functions, Euler–Maclaurin formula, etc.) of ordinary calculus have been introduced in [13]. It should be mentioned that, in [14,15], important generalizations and results were given for dynamic equations defined on an arbitrary time scale, hence including q-difference equations as a special case. Furthermore, several problems of q-difference equations have been treated by various authors [16–19]. But spectral analysis of non-selfadjoint q-difference equations with spectral singularities has not been investigated yet.

In this paper we let $q > 1$ and use the notations

$$q^{\mathbb{Z}} := \left\{q^n : n \in \mathbb{Z}\right\}, \quad q^{\mathbb{N}} := \left\{q^n : n \in \mathbb{N}\right\}, \quad q^{-\mathbb{N}} := \left\{q^{-n} : n \in \mathbb{N}\right\},$$

where $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of integers and positive integers, respectively. A q-difference equation is an equation that contains q-derivatives of a function defined on $q^{\mathbb{Z}}$.

---

[*] Corresponding author. Tel.: +1 573 341 4129.
E-mail addresses: murat.adivar@izmirekonomi.edu.tr (M. Adıvar), bohner@umr.edu (M. Bohner).

0895-7177/$ - see front matter © 2005 Elsevier Ltd. All rights reserved.
doi:10.1016/j.mcm.2005.04.014

Definition 1 (*[14, p. 8/11]*). The q-derivative of a function $f : q^{\mathbb{Z}} \to \mathbb{C}$ is defined by

$$f^{\Delta}(t) := \frac{f(qt) - f(t)}{\mu(t)} \quad \text{for all } t \in q^{\mathbb{Z}},$$

where $\mu(t) = (q-1)t$ is the graininess function. For $f, g : q^{\mathbb{Z}} \to \mathbb{C}$, the product rule is given by

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(qt)g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(qt).$$

Definition 2 (*[14, p. 29]*). The q-integral of a function $f : q^{\mathbb{Z}} \to \mathbb{C}$ is defined by

$$\int_a^b f(t)\Delta t := \begin{cases} \sum\limits_{t \in [a,b) \cap q^{\mathbb{Z}}} \mu(t) f(t) & \text{if } a < b \\ 0 & \text{if } a = b \\ \sum\limits_{t \in [b,a) \cap q^{\mathbb{Z}}} \mu(t) f(t) & \text{if } a > b. \end{cases}$$

Hereafter, we will denote the Hilbert space of complex-valued functions with the inner product

$$\langle f, g \rangle_q := \int_{q^{\mathbb{Z}}} f(t)\overline{g(t)}\Delta t \quad \text{for } f, g : q^{\mathbb{Z}} \to \mathbb{C}$$

and the norm

$$\|f\|_q := \left(\int_{q^{\mathbb{Z}}} |f(t)|^2 \Delta t \right)^{\frac{1}{2}} \quad \text{for } f : q^{\mathbb{Z}} \to \mathbb{C}$$

by $\ell^2(q^{\mathbb{Z}})$.

Let L denote the q-difference operator of second order generated in $\ell^2(q^{\mathbb{Z}})$ by the q-difference expression

$$(ly)(t) := q\gamma(t)y(qt) + \beta(t)y(t) + \gamma(t/q)y(t/q) \quad \text{for } t \in q^{\mathbb{Z}},$$

where $\{\gamma(t)\}_{t \in q^{\mathbb{Z}}}$ and $\{\beta(t)\}_{t \in q^{\mathbb{Z}}}$ are complex sequences and $\gamma(t) \neq 0$ for all $t \in q^{\mathbb{Z}}$. Note that we can write the difference expression $(ly)(t)$ in Sturm–Liouville form as

$$(ly)(t) = \left[ay^{\Delta}\right]^{\Delta}(t/q) + b(t)y(t) \quad \text{for } t \in q^{\mathbb{Z}},$$

where $a(t) = \gamma(t)\mu^2(t)$ and $b(t) = \beta(t) + q\gamma(t) + \gamma(t/q)$, and y^{Δ} denotes the q-derivative of y.

In this paper we investigate the eigenvalues and the spectral singularities of the q-difference operator L. We prove that the operator L has a finite number of eigenvalues and spectral singularities by using boundary uniqueness theorems of analytic functions. The purpose of this paper is to extend some results for difference equations obtained in [7] to the case of q-difference equations.

2. Jost solutions of $(ly)(t) = \lambda y(t)$

Let us consider the q-difference equation

$$q\gamma(t)y(qt) + \beta(t)y(t) + \gamma(t/q)y(t/q) = \lambda y(t) \quad \text{for } t \in q^{\mathbb{Z}}, \tag{2.1}$$

where λ is a spectral parameter and $\gamma(t) \neq 0$ for all $t \in q^{\mathbb{Z}}$. Suppose that the complex sequences $\{\gamma(t)\}_{t \in q^{\mathbb{Z}}}$ and $\{\beta(t)\}_{t \in q^{\mathbb{Z}}}$ satisfy

$$\sum_{t \in q^{\mathbb{Z}}} \left| \frac{\ln t}{\ln q} \right| (|1 - \gamma(t)| + |\beta(t)|) < \infty. \tag{2.2}$$

We now give two special solutions $e^+(t,z)$ and $e^-(t,z)$ of (2.1), which are asymptotically equal to the solutions

$$\widetilde{e}^{\,+}(t,z) := \frac{\exp\left(\mathrm{i}\frac{\ln t}{\ln q}z\right)}{\sqrt{\mu(t)}} \quad \text{and} \quad \widetilde{e}^{\,-}(t,z) := \frac{\exp\left(-\mathrm{i}\frac{\ln t}{\ln q}z\right)}{\sqrt{\mu(t)}}$$

of the equation

$$qy(qt) + y(t/q) = \lambda y(t) \quad \text{for } t \in q^{\mathbb{Z}},$$

where $\lambda = 2\sqrt{q}\cos z$.

Theorem 1. *Under the condition (2.2), Eq. (2.1) has unique solutions*

$$e^+(t,z) = \alpha^+(t)\frac{\exp\left(\mathrm{i}\frac{\ln t}{\ln q}z\right)}{\sqrt{\mu(t)}}\left(1 + \int_{q^{\mathbb{N}}} A^+(t,r)\frac{\exp\left(\mathrm{i}\frac{\ln r}{\ln q}z\right)}{\mu(r)}\Delta r\right) \tag{2.3}$$

and

$$e^-(t,z) = \alpha^-(t)\frac{\exp\left(-\mathrm{i}\frac{\ln t}{\ln q}z\right)}{\sqrt{\mu(t)}}\left(1 + \int_{q^{-\mathbb{N}}} A^-(t,r)\frac{\exp\left(-\mathrm{i}\frac{\ln r}{\ln q}z\right)}{\mu(r)}\Delta r\right) \tag{2.4}$$

for $\lambda = 2\sqrt{q}\cos z$, where $z \in \overline{\mathbb{C}}_+ := \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$ and $\alpha^\pm(t)$, $A^\pm(t,r)$ are expressed in terms of $\{\gamma(t)\}_{t\in q^{\mathbb{Z}}}$ and $\{\beta(t)\}_{t\in q^{\mathbb{Z}}}$ as

$$\alpha^+(t) = \left(\prod_{s\in[t,\infty)\cap q^{\mathbb{Z}}} \gamma(s)\right)^{-1}, \quad A^+(t,q) = -\frac{1}{\sqrt{q}}\sum_{s\in[qt,\infty)\cap q^{\mathbb{Z}}} \beta(s),$$

$$A^+\left(t,q^2\right) = \sum_{s\in[qt,\infty)\cap q^{\mathbb{Z}}}\left\{1 - \gamma^2(s) + \frac{1}{q}\beta(s)\sum_{p\in[qs,\infty)\cap q^{\mathbb{Z}}}\beta(p)\right\},$$

$$A^+\left(t,q^2 r\right) = A^+(qt,r) + \sum_{s\in[qt,\infty)\cap q^{\mathbb{Z}}}\left\{\left(1-\gamma^2(s)\right)A^+(qs,r) - \frac{\beta(s)}{\sqrt{q}}A^+(s,qr)\right\}$$

for $r \in q^{\mathbb{N}}$ and $t \in q^{\mathbb{Z}}$,

$$\alpha^-(t) = \left(\prod_{s\in\left(0,\frac{t}{q}\right]\cap q^{\mathbb{Z}}} \gamma(s)\right)^{-1}, \quad A^-\left(t,\frac{1}{q}\right) = -\frac{1}{\sqrt{q}}\sum_{s\in\left(0,\frac{t}{q}\right]\cap q^{\mathbb{Z}}} \beta(s),$$

$$A^-\left(t,\frac{1}{q^2}\right) = \sum_{s\in\left(0,\frac{t}{q}\right]\cap q^{\mathbb{Z}}}\left\{1 - \gamma^2(s/q) + \frac{1}{q}\beta(s)\sum_{s\in\left(0,\frac{t}{q}\right]\cap q^{\mathbb{Z}}}\beta(p)\right\},$$

$$A^-\left(t,\frac{r}{q^2}\right) = A^-\left(\frac{t}{q},r\right) + \sum_{s\in\left(0,\frac{t}{q}\right]\cap q^{\mathbb{Z}}}\left\{\left(1-\gamma^2(s/q)\right)A^-\left(\frac{s}{q},r\right) - \frac{\beta(s)}{\sqrt{q}}A^-\left(s,\frac{r}{q}\right)\right\}$$

for $r \in q^{-\mathbb{N}}$ and $t \in q^{\mathbb{Z}}$.

Proof. If we put $e^+(t,z)$ and $e^-(t,z)$ into (2.1), then we have the relations

$$\alpha^+(qt) = \gamma(t)\alpha^+(t), \quad A^+(t,q) - A^+\left(\frac{t}{q},q\right) = \frac{\beta(t)}{\sqrt{q}},$$

$$A^+\left(t,q^2\right)-A^+\left(\frac{t}{q},q^2\right)=\gamma^2(t)-1+\frac{1}{\sqrt{q}}\beta(t)A^+(t,q),$$
$$A^+\left(t,q^2\right)-A^+\left(\frac{t}{q},q^2\right)=\gamma^2(t)A^+(qt,r)+\frac{1}{\sqrt{q}}\beta(t)A^+(t,q)-A^+(t,r),$$
$$\alpha^-(t)=\gamma(t)\alpha^-(qt),\quad A^-\left(qt,\frac{1}{q}\right)-A^-\left(t,\frac{1}{q}\right)=-\frac{\beta(t)}{\sqrt{q}},$$
$$A^-\left(qt,\frac{1}{q^2}\right)-A^-\left(t,\frac{1}{q^2}\right)=1-\gamma^2\left(\frac{t}{q}\right)-\frac{1}{\sqrt{q}}\beta(t)A^-\left(t,\frac{1}{q}\right),$$
$$A^-\left(qt,\frac{r}{q^2}\right)-A^-\left(t,\frac{r}{q^2}\right)=A^-(t,r)-\gamma^2\left(\frac{t}{q}\right)A^-\left(\frac{t}{q},r\right)-\frac{\beta(t)}{\sqrt{q}}A^-\left(t,\frac{r}{q}\right),$$

and, using them, one easily obtains $\alpha^\pm(t)$ and $A^\pm(t,r)$ as convergent series. □

Under (2.2), $A^\pm(t,r)$ satisfy

$$\begin{cases}|A^+(t,r)|\le c\displaystyle\sum_{s\in\left[tq^{\left[\frac{\ln r}{2\ln q}\right]},\infty\right)\cap q^{\mathbb{Z}}}\{|1-\gamma(s)|+|\beta(s)|\},\\[2ex] |A^-(t,r)|\le c\displaystyle\sum_{s\in\left(0,tq^{\left[\frac{\ln r}{2\ln q}\right]+1},\infty\right]\cap q^{\mathbb{Z}}}\{|1-\gamma(s)|+|\beta(s)|\},\end{cases}\tag{2.5}$$

where $\left[\frac{\ln r}{2\ln q}\right]$ is the integer part of $\frac{\ln r}{2\ln q}$, and $c>0$ is a constant. Therefore, $e^+(t,z)$ and $e^-(t,z)$ ($t\in q^{\mathbb{Z}}$) are continuous in $\overline{\mathbb{C}}_+$ and analytic with respect to z in $\mathbb{C}_+:=\{z:z\in\mathbb{C},\ \operatorname{Im}z>0\}$, and

$$\begin{cases}e^+(t,z)=\dfrac{\exp\left(\mathrm{i}\frac{\ln t}{\ln q}z\right)}{\sqrt{\mu(t)}}(1+\mathrm{o}(1)),\quad t\to\infty,z\in\overline{\mathbb{C}}_+,\\[2ex] e^-(t,z)=\dfrac{\exp\left(-\mathrm{i}\frac{\ln t}{\ln q}z\right)}{\sqrt{\mu(t)}}(1+\mathrm{o}(1)),\quad t\to 0,z\in\overline{\mathbb{C}}_+,\end{cases}\tag{2.6}$$

and

$$e^\pm(t,z)=\alpha^\pm(t)\exp\left(\pm\mathrm{i}\frac{\ln t}{\ln q}z\right)(1+\mathrm{o}(1)),\quad t\in q^{\mathbb{Z}},\operatorname{Im}z\to\infty\tag{2.7}$$

hold. Analogously to the Sturm–Liouville equation, the solutions

$$e^+(z):=\left\{e^+(t,z)\right\}_{t\in q^{\mathbb{Z}}}\quad\text{and}\quad e^-(z):=\left\{e^-(t,z)\right\}_{t\in q^{\mathbb{Z}}}$$

are called *Jost solutions* of (2.1).

The *Wronskian* of two solutions $y=\{y(t,\lambda)\}_{t\in q^{\mathbb{Z}}}$ and $u=\{u(t,\lambda)\}_{t\in q^{\mathbb{Z}}}$ of (2.1) is defined by

$$W[y,u](t)=\mu(t)\gamma(t)\{y(t,\lambda)u(qt,\lambda)-y(qt,\lambda)u(t,\lambda)\}\quad\text{for } t\in q^{\mathbb{Z}}.$$

Hereafter, we will denote by ξ a real parameter. It is obvious that $e^+(-\xi)$ and $e^-(-\xi)$ are also solutions of (2.1) for $\lambda=2\sqrt{q}\cos\xi$. Using (2.6), we get that

$$W\left[e^\pm(\xi),e^\pm(-\xi)\right]=\mp\frac{2\mathrm{i}}{\sqrt{q}}\sin\xi.$$

So the pairs $e^+(\xi),e^+(-\xi)$ and $e^-(\xi),e^-(-\xi)$ form two fundamental systems of solutions of (2.1) for $\lambda=2\sqrt{q}\cos\xi$ and $\xi\neq k\pi$, $k\in\mathbb{Z}$. We have the relation

$$e^+(t,\xi)=\zeta(\xi)e^-(t,\xi)+\eta(\xi)e^-(t,-\xi)\quad\text{for } t\in q^{\mathbb{Z}}$$

with $\xi \neq k\pi, k \in \mathbb{Z}$, where

$$\zeta(\xi) = \sqrt{q}\frac{W\left[e^{+}(\xi), e^{-}(-\xi)\right]}{2\mathrm{i}\sin\xi}, \quad \eta(\xi) = \sqrt{q}\frac{W\left[e^{-}(\xi), e^{+}(\xi)\right]}{2\mathrm{i}\sin\xi}. \tag{2.8}$$

Moreover, the function η has an analytic continuation to the half-plane $\mathbb{C}_{+}$.

3. Continuous spectrum of L

Let L_1 denote the q-difference operator generated in $\ell^2(q^{\mathbb{Z}})$ by

$$(l_1 y)(t) := qy(qt) + y(t/q) \quad \text{for } t \in q^{\mathbb{Z}}. \tag{3.1}$$

Since

$$\begin{aligned}\langle l_1 y, u\rangle_q &= \sum_{t\in q^{\mathbb{Z}}} \mu(t)\left(qy(qt) + y(t/q)\right)\overline{u(t)} \\ &= \sum_{t\in q^{\mathbb{Z}}} \mu(t)y(t)\overline{(qu(qt) + u(t/q))} \\ &= \langle y, l_1 u\rangle_q\end{aligned}$$

and, since L_1 is bounded in the Hilbert space $l^2(q^{\mathbb{Z}})$, the operator L_1 is selfadjoint. It is obvious that

$$\widetilde{e}^{\,+}(t,z) = \frac{\exp\left(\mathrm{i}\frac{\ln t}{\ln q}z\right)}{\sqrt{\mu(t)}} \quad \text{and} \quad \widetilde{e}^{\,-}(t,z) = \frac{\exp\left(-\mathrm{i}\frac{\ln t}{\ln q}z\right)}{\sqrt{\mu(t)}}$$

are solutions of $(l_1 y)(t) = \lambda y(t)$ for $\lambda = 2\sqrt{q}\cos z$ and $z \in \overline{\mathbb{C}}_{+}$. The Wronskian of these two solutions is obtained as

$$W\left[\left\{\widetilde{e}^{\,+}(t,z)\right\}_{t\in q^{\mathbb{Z}}}, \left\{\widetilde{e}^{\,-}(t,z)\right\}_{t\in q^{\mathbb{Z}}}\right] = -\frac{1}{\sqrt{q}}2\mathrm{i}\sin z, \quad z \in \overline{\mathbb{C}}_{+}.$$

For all $z \in \overline{\mathbb{C}}_{+} \setminus \{k\pi : k \in \mathbb{Z}\}$, we define the Green function of L_1 by

$$G_{t,r}(z) := \frac{1}{2\mathrm{i}\sqrt{q}\sin z}\begin{cases}\widetilde{e}^{\,+}(t,z)\widetilde{e}^{\,-}(r,z) & \text{if } r = \dfrac{t}{q^k},\ k \in \mathbb{N} \\ \widetilde{e}^{\,+}(r,z)\widetilde{e}^{\,-}(t,z) & \text{if } r = tq^k,\ k \in \mathbb{N}_0,\end{cases} \quad t \in q^{\mathbb{Z}}. \tag{3.2}$$

It is easy to see that

$$R_\lambda(L_1)\psi(t) = \int_{q^{\mathbb{Z}}} G_{t,r}(z)\psi(r)\Delta r \quad \text{for } \psi \in \ell^2\left(q^{\mathbb{Z}}\right) \tag{3.3}$$

is the resolvent of L_1.

Lemma 1. *For every $\delta > 0$, there is a number c_δ such that*

$$\|R_\lambda(L_1)\|_q \geq \frac{c_\delta}{|\sin z|\sqrt{1 - \exp(-2\operatorname{Im} z)}} \tag{3.4}$$

for all $z \in \mathbb{C}_{+}$ and $|\operatorname{Im} z| < \delta$.

Proof. Let us consider the function

$$g(r,z) := \begin{cases}\dfrac{\exp\left(-\mathrm{i}\frac{\ln r}{\ln q}z\right)}{\sqrt{\mu(r)}} & \text{if } r = \dfrac{t}{q^k},\ k \in \mathbb{N} \\ 0 & \text{if } r = tq^k,\ k \in \mathbb{N}_0.\end{cases}$$

Thus we obtain

$$\|g(\cdot,z)\|_q^2=\int_{q^{\mathbb{Z}}}|g(r,z)|^2\Delta r=\sum_{r\in\left(0,\frac{t}{q}\right]\cap q^{\mathbb{Z}}}\left|\exp\left(-\mathrm{i}\frac{\ln r}{\ln q}z\right)\right|^2<\infty.$$

Then $g(\cdot,z)\in\ell^2(q^{\mathbb{Z}})$. Using (3.2) and (3.3), we obtain

$$R_\lambda(L_1)(g(\cdot,z))=\frac{\exp\left(\mathrm{i}\frac{\ln t}{\ln q}z\right)}{2\mathrm{i}\sqrt{q}\sin z\sqrt{\mu(t)}}\|g(\cdot,z)\|_q^2.$$

Since

$$\left|\exp\left(\mathrm{i}\frac{\ln t}{\ln q}z\right)\right|>\frac{1}{2}\exp\left(-\frac{\ln t}{\ln q}\operatorname{Im}z\right),$$

we get that

$$\begin{aligned}\|R_\lambda(L_1)(g(\cdot,z))\|_q^2&=\frac{\|g(\cdot,z)\|_q^4}{4q|\sin z|^2}\int_{q^{\mathbb{Z}}}\frac{1}{\mu(t)}\left|\exp\left(\mathrm{i}\frac{\ln t}{\ln q}z\right)\right|^2\Delta t\\&=\frac{\|g(\cdot,z)\|_q^4}{4q|\sin z|^2}\sum_{t\in q^{\mathbb{Z}}}\left|\exp\left(\mathrm{i}\frac{\ln t}{\ln q}z\right)\right|^2\\&\ge\frac{\|g(\cdot,z)\|_q^4}{16q|\sin z|^2}\sum_{t\in q^{\mathbb{Z}}}\exp\left(-2\operatorname{Im}z\right)^{\frac{\ln t}{\ln q}}\\&\ge\frac{\|g(\cdot,z)\|_q^4}{16q\|\sin z\|^2}\sum_{t\in[q,\infty)\cap q^{\mathbb{Z}}}\exp\left(-2\operatorname{Im}z\right)^{\frac{\ln t}{\ln q}}\\&=\frac{\|g(\cdot,z)\|_q^4}{16q|\sin z|^2}\frac{\exp\left(-2\operatorname{Im}z\right)}{1-\exp\left(-2\operatorname{Im}z\right)}.\end{aligned}$$

By choosing $c_\delta=\frac{\|g(\cdot,z)\|_q}{4\sqrt{q}\exp(\operatorname{Im}z)}$, the proof is complete. □

Theorem 2. $\sigma(L_1)=\sigma_c(L_1)=[-2\sqrt{q},2\sqrt{q}]$, *where* $\sigma(L_1)$ *and* $\sigma_c(L_1)$ *denote the spectrum and continuous spectrum of the operator* L_1*, respectively.*

Proof. It is easy to see that L_1 has no eigenvalues, so the spectrum of the operator L_1 consists only of its continuous spectrum. If $\lambda_0=2\sqrt{q}\cos z_0\in\sigma_c(L_1)$, then $\|R_{\lambda_0}(L_1)\|_q\to\infty$. Using (3.2) and (3.3), we have $|\sin z_0|\to 0$ and $\operatorname{Im}z_0\to 0$. If $\operatorname{Im}z_0\to 0$, then $\lambda_0=2\sqrt{q}\cos z_0\in[-2\sqrt{q},2\sqrt{q}]$.

It follows from (3.4) that $\|R_\lambda(L_1)\|_q\to\infty$ for $\lambda=2\sqrt{q}\cos z\in[-2\sqrt{q},2\sqrt{q}]$. Now, we have to show that the range $\mathcal{R}(L_1-\lambda I)$ of values of the operator $L_1-\lambda I$ is dense in the space $\ell^2(q^{\mathbb{Z}})$. It is obvious that the orthogonal complement of $\mathcal{R}(L_1-\lambda I)$ coincides with the space of solutions of the equation $l_1^*y=\lambda y$ such that $y\in\ell^2(q^{\mathbb{Z}})$. Since L_1 is a selfadjoint operator and has no eigenvalues, the orthogonal complement of the set $\mathcal{R}(L_1-\lambda I)$ consists only of the zero element. This completes the proof. □

Theorem 3. *If* (2.2) *holds, then* $\sigma_c(L)=[-2\sqrt{q},2\sqrt{q}]$.

Proof. Let L_2 denote the operator generated in $\ell^2(q^{\mathbb{Z}})$ by the difference expression

$$(l_2y)(t):=q(\gamma(t)-1)y(qt)+\beta(t)y(t)+(\gamma(t/q)-1)y(t/q)\quad\text{for } t\in q^{\mathbb{Z}}.$$

It is evident that $L=L_1+L_2$ and $L_1^*=L_1$. From (2.2), we find that L_2 is a compact operator in $\ell^2(q^{\mathbb{Z}})$. Using Theorem 2 and Weyl's theorem of a compact perturbation [20, p. 13], we obtain

$$\sigma_c(L)=\sigma(L_1)=\sigma_c(L_1)=\left[-2\sqrt{q},2\sqrt{q}\right].$$

The proof is complete. □

4. Eigenvalues and spectral singularities of L

It is well known that the function η has an analytic continuation from the real axis to $\mathbb{C}_+$ (see (2.8)). If we define

$$f(z) = \frac{2\mathrm{i}\eta(z)}{\sqrt{q}} \sin z \quad \text{for } z \in \overline{\mathbb{C}}_+,$$

then

$$f(z) = W\left[e^-(z), e^+(z)\right] \quad \text{for } z \in \overline{\mathbb{C}}_+ \tag{4.1}$$

by (2.8). So we get that f is analytic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}}_+$, and

$$f(z) = f(z + 2\pi).$$

Let us define the semi-strips

$$P_0 = \left\{ z : z \in \mathbb{C}_+,\ -\frac{\pi}{2} \le \operatorname{Re} z \le \frac{3\pi}{2} \right\} \quad \text{and} \quad P = P_0 \cup \left[-\frac{\pi}{2}, \frac{3\pi}{2} \right].$$

We will denote the set of all eigenvalues of L by $\sigma_d(L)$. From the definition of the eigenvalues of L, we have

$$\sigma_d(L) = \left\{ \lambda : \lambda = 2\sqrt{q} \cos z,\ z \in P_0,\ f(z) = 0 \right\}. \tag{4.2}$$

For all $z \in P$ with $f(z) \neq 0$, we define the Green function of L by

$$\mathcal{G}_{t,r}(z) := \begin{cases} \dfrac{e^+(t,z)e^-(r,z)}{qf(z)} & \text{if } r = \dfrac{t}{q^k},\ k \in \mathbb{N} \\[2ex] \dfrac{e^+(r,z)e^-(t,z)}{qf(z)} & \text{if } r = tq^k,\ k \in \mathbb{N}_0. \end{cases} \tag{4.3}$$

It is obvious that

$$R_\lambda(L)\phi(t) = \int_{q^{\mathbb{Z}}} \mathcal{G}_{t,r}(z)\phi(r)\Delta r \quad \text{for } \phi \in \ell^2\left(q^{\mathbb{Z}}\right) \tag{4.4}$$

is the resolvent of L.

If $f(z_0) = 0$ for some $z_0 \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$, then $\lambda_0 = 2\sqrt{q}\cos z_0 \in [-2\sqrt{q}, 2\sqrt{q}] = \sigma_c(L)$. So, from (4.2)–(4.4), we get that

$$\sigma_{ss}(L) = \left\{ \lambda : \lambda = 2\sqrt{q} \cos z,\ z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2} \right] \setminus \{0, \pi\},\ f(z) = 0 \right\}, \tag{4.5}$$

where $\sigma_{ss}(L)$ denotes the set of all spectral singularities of L [6, p. 306].

Hereafter, we investigate quantitative properties of the eigenvalues and the spectral singularities of L. From (2.3), (2.4) and (4.1), we get that

$$\begin{aligned} f(z) &= W\left[e^-(z), e^+(z)\right] \\ &= \frac{1}{\sqrt{q}} \left(\prod_{r \in q^{\mathbb{Z}}} \gamma(r) \right)^{-1} \\ &\quad \times \left\{ \gamma^2(1) \mathrm{e}^{\mathrm{i}z} \left(1 + \sum_{r \in q^{\mathbb{N}}} A^+(q,r) \exp\left(\mathrm{i} \frac{\ln r}{\ln q} z \right) \right) \right. \\ &\quad \times \left(1 + \sum_{r \in q^{-\mathbb{N}}} A^-(1,r) \exp\left(-\mathrm{i} \frac{\ln r}{\ln q} z \right) \right) \end{aligned}$$

$$
- \mathrm{e}^{-\mathrm{i}z}\left(1+\sum_{r\in q^{-\mathbb{N}}} A^{-}(q,r)\exp\left(-\mathrm{i}\frac{\ln r}{\ln q}z\right)\right)
\times\left(1+\sum_{r\in q^{\mathbb{N}}} A^{+}(1,r)\exp\left(\mathrm{i}\frac{\ln r}{\ln q}z\right)\right)\Bigg\},
$$

and, from (2.5), we have

$$
f(z)=\frac{1}{\sqrt{q}}\left(\prod_{r\in q^{\mathbb{Z}}}\gamma(r)\right)^{-1}\mathrm{e}^{-\mathrm{i}z}\left(1+\mathrm{o}(1)\right), \quad z\in P_0, \operatorname{Im} z\to\infty, \tag{4.6}
$$

which shows the boundedness of the zeros of f in P_0.

Theorem 4. *Under the condition (2.2),*

(i) *the set $\sigma_d(L)$ is bounded and countable, and its limit points lie only in the interval $[-2\sqrt{q}, 2\sqrt{q}]$;*
(ii) *$\sigma_{ss}(L)\subset[-2\sqrt{q}, 2\sqrt{q}]$, $\sigma_{ss}(L)=\overline{\sigma_{ss}(L)}$, and the linear Lebesgue measure of the set $\sigma_{ss}(L)$ is zero.*

Proof. The boundedness of the set of zeros of f in P_0 is clear from (4.6). Since f is a 2π-periodic function and is analytic in $\mathbb{C}_+$, we obtain that f has at most a countable number of zeros in P_0. By the uniqueness of analytic functions, we find that the limit points of zeros of f in P_0 can lie only in $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$. The closedness and the property of having a linear Lebesgue measure of the set of zeros lying in $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ can be obtained from the boundary uniqueness theorem of analytic functions [21]. □

Definition 3. The multiplicity of a zero of f in P is called the multiplicity of the corresponding eigenvalue or spectral singularity of L.

Theorem 5. *If, for some $\varepsilon>0$,*

$$
\sup_{t\in q^{\mathbb{Z}}}\left\{\exp\left(\varepsilon\left|\frac{\ln t}{\ln q}\right|\right)\left(|1-\gamma(t)|+|\beta(t)|\right)\right\}<\infty \tag{4.7}
$$

holds, then the operator L has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.

Proof. Under the condition (4.7), $A^{\pm}(t,r)$ satisfy

$$
\begin{cases}
|A^{+}(t,r)|\le c\exp\left(-\frac{\varepsilon}{4}\left|\frac{\ln r}{\ln q}\right|\right) & \text{for } t\in\{1,q\},\ r=q^{k},\ k\in\mathbb{N},\\
|A^{-}(t,r)|\le c\exp\left(-\frac{\varepsilon}{4}\left|\frac{\ln r}{\ln q}\right|\right) & \text{for } t\in\{1,q\},\ r=\frac{1}{q^{k}},\ k\in\mathbb{N}.
\end{cases} \tag{4.8}
$$

By (4.8), f has an analytic continuation to the half-plane $\operatorname{Im} z>-\frac{\varepsilon}{4}$. Since f is a 2π-periodic function, the limit points of its zeros in P cannot lie in $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Using Theorem 4, we get that the bounded sets $\sigma_d(L)$ and $\sigma_{ss}(L)$ have no limit points, i.e., the sets $\sigma_d(L)$ and $\sigma_{ss}(L)$ have a finite number of elements. From the analyticity of f in $\operatorname{Im} z>-\frac{\varepsilon}{4}$, we obtain that all zeros of f in P have a finite multiplicity. Consequently, all eigenvalues and spectral singularities have a finite multiplicity. □

Acknowledgement

This work was supported by the Scientific and Technical Research Council of Turkey.

References

[1] E. Bairamov, Ö. Çakar, A.O. Çelebi, Quadratic pencil of Schrödinger operators with spectral singularities: Discrete spectrum and principal functions, J. Math. Anal. Appl. 216 (1) (1997) 303–320.

[2] E. Bairamov, Ö. Cakar, C. Yanik, Spectral singularities of the Klein–Gordon s-wave equation, Indian J. Pure Appl. Math. 32 (6) (2001) 851–857.

[3] A.M. Krall, E. Bairamov, Ö. Çakar, Spectrum and spectral singularities of a quadratic pencil of a Schrödinger operator with a general boundary condition, J. Differential Equations 151 (2) (1999) 252–267.

[4] V.E. Lyance, A differential operator with spectral singularities I, II, Amer. Math. Soc. Transl. 60 (2) (1967) 185–225, 227–283.

[5] M.A. Naĭmark, Investigation of the spectrum and the expansion in eigenfunctions of a non-selfadjoint differential operator of the second order on a semi-axis, Amer. Math. Soc. Transl. Ser. 2 16 (1960) 103–193.

[6] M.A. Naĭmark, Linear Differential Operators. Part II: Linear Differential Operators in Hilbert Space (With additional material by the author, and a supplement by V. È. Ljance. Translated from the Russian by E. R. Dawson. English translation edited by W. N. Everitt), Frederick Ungar Publishing Co., New York, 1968.

[7] M. Adıvar, E. Bairamov, Spectral properties of non-selfadjoint difference operators, J. Math. Anal. Appl. 261 (2) (2001) 461–478.

[8] E. Bairamov, Ö. Çakar, A.M. Krall, Non-selfadjoint difference operators and Jacobi matrices with spectral singularities, Math. Nachr. 229 (2001) 5–14.

[9] E. Bairamov, A.O. Çelebi, Spectrum and spectral expansion for the non-selfadjoint discrete Dirac operators, Quart. J. Math. Oxford Ser. 2 50 (200) (1999) 371–384.

[10] A.M. Krall, E. Bairamov, Ö. Çakar, Spectral analysis of non-selfadjoint discrete Schrödinger operators with spectral singularities, Math. Nachr. 231 (2001) 89–104.

[11] R.P. Agarwal, Difference Equations and Inequalities, in: Monographs and Textbooks in Pure and Applied Mathematics, vol. 228, Marcel Dekker Inc., New York, 2000.

[12] R.P. Agarwal, P.J.Y. Wong, Advanced Topics in Difference Equations, in: Mathematics and its Applications, vol. 404, Kluwer Academic Publishers Group, Dordrecht, 1997.

[13] V. Kac, P. Cheung, Quantum calculus, in: Universitext, Springer-Verlag, New York, 2002.

[14] M. Bohner, A. Peterson, Dynamic Equations on Time Scales, Birkhäuser Boston Inc., Boston, MA, 2001.

[15] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser Boston Inc., Boston, MA, 2003.

[16] C. Berg, A. Ruffing, Generalized q-Hermite polynomials, Comm. Math. Phys. 223 (1) (2001) 29–46.

[17] J.-P. Bézivin, Sur les équations fonctionnelles aux q-différences, Aequationes Math. 43 (2–3) (1992) 159–176.

[18] W.J. Trijtzinsky, Analytic theory of linear q-difference equations, Acta Math. 61 (1933) 1–38.

[19] C. Zhang, Sur la sommabilité des séries entières solutions formelles d'une équation aux q-différences. I, C. R. Acad. Sci. Paris Sér. I Math. 327 (4) (1998) 349–352.

[20] I.M. Glazman, Direct methods of qualitative spectral analysis of singular differential operators, in: Israel Program for Scientific Translations, Jerusalem, 1965, 1966. Translated from the Russian by the IPST staff.

[21] E.P. Dolzhenko, Boundary value uniqueness theorems for analytic functions, Math. Notes 26 (1979) 437–442.