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Separated and State-Constrained Separated Linear Programming Problems on Time Scales

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ABSTRACT: Separated linear programming problems can be used to model a wide range of real-world applications such as in communications, manufacturing, transportation, and so on. In this paper, we investigate novel formulations for two classes of these problems using the methodology of time scales. As a special case, we obtain the classical separated continuous-time model and the state-constrained separated continuous-time model. We establish some of the fundamental theorems such as the weak duality theorem and the optimality condition on arbitrary time scales, while the strong duality theorem is presented for isolated time scales. Examples are given to demonstrate our new results.

Key Words: Time scales, Separated linear programming problem, State constrained, Weak duality theorem, Optimality condition, Strong duality theorem.

Contents

1. Introduction

It is well known that discrete-time linear programming problems have numerous applications in areas such as portfolio optimization, crew scheduling, manufacturing, transportation, telecommunication, agriculture, and so on. Continuous-time linear programming problems were first studied by Bellman [6] as a bottleneck process. He established the weak duality theorem and optimality conditions. A computational approach has been presented by Bellman and Dreyfus [7]. The

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strong duality theorem was studied by Tyndall [35,36] and Levinson [26]. Grinold [23] has established strong duality without discretizing the continuous problem. A numerical solution to continuous-time linear programming was considered by Buie and Abrham [22]. Jasiulek [25] has characterized the extreme points of the feasibility set of continuous-time linear programming problems. Wen et al. [43] have presented an approximation approach to solve continuous-time problems. A new class called separated continuous-time linear programming problems has been investigated by Anderson and Nash [2,3] and Pullan [30,31,32,33]. This class has many applications such as, for example, job-shop scheduling problems. A simplex-based algorithm for solving the separated type of problems has been considered by Weiss [42]. An approximation algorithm has been used by Wang et al. [41] to solve separated continuous-time linear programming problems. Luo and Bertsimas [27] have presented an extension of the separated model called state-constrained separated model. Xiaoqing [37] has studied duality theorems for separated continuous linear programming and its extensions. Separated continuous linear programs with an application for service operation have been studied by Wang [39]. An application of separated problems to emergency department staffing has been presented by Wang [38]. Wang [40] has investigated duality theorems and solution methods for stochastic separated continuous programming.

The theory of time scales, on the other hand, was first introduced by Stefan Hilger in 1988 in his PhD dissertation, see [24]. The purpose of this theory is to unify discrete and continuous analysis and to offer an extension to cases "in between". Many applications in mathematical modelling exist for this theory, e.g., to optimal control [5,18,19,20,21,29], population biology [9], calculus of variations [8,10,13], and economics [4,11,14,15,34].

In this paper, we demonstrate that separated problems can be efficiently formulated and solved using time scales techniques. The new formulation yields the separated continuous-time model and the state-constrained separated continuous-time model as special cases (i.e., by setting $\mathbb{T} = \mathbb{R}$). The paper is organized as follows: In Section 2, some examples related to time scales calculus are given. In Section 3, we recall some recent results by the authors [1] about linear primal and dual programs on time scales. In Section 4, the basic structures of the primal and dual separated linear programming models are formulated. For the separated model, the weak duality theorem and the optimality condition theorem are established for arbitrary time scales, and the strong duality theorem is presented for isolated time scales. In Section 5, examples are given to demonstrate the duality theorems for separated models. In Section 6, we present the state-constrained separated primal and dual models and establish the weak duality theorem and the optimality condition theorem for state-constrained separated models on arbitrary time scales, while the strong duality is stated and proved for isolated time scales. Examples are presented in Section 7 to demonstrate our new results for state-constrained model. In Section 8, some conclusions are given.

2. Time Scales Calculus

In this section, instead of introducing the basic definitions, derivative, and integral on time scales, we refer the reader to the monographs [12,16,17], in which comprehensive details and complete proofs are given. For readers not familiar with the time scales calculus, we give the following few examples. Throughout, $\mathbb{T}$ is the time scale, σ is the forward jump operator, μ is the graininess, $f : \mathbb{T} \to \mathbb{R}$ is a function, $f^\sigma = f \circ \sigma$ is the advance of f, f^Δ is the delta derivative of f, and $\int_a^b f(t)\Delta t$ is the time scales integral of f between $a, b \in \mathbb{T}$.

Example 2.1. *If $\mathbb{T} = \mathbb{R}$, then*

$$\sigma(t) = t, \quad \mu(t) \equiv 0, \quad f^\Delta(t) = f'(t) \quad \text{for} \quad t \in \mathbb{T},$$

and

$$\int_a^b f(t)\Delta t = \int_a^b f(t)\mathrm{d}t, \quad \text{where} \quad a, b \in \mathbb{T} \text{ with } a < b,$$

is the usual Riemann integral from calculus.

Example 2.2. *If $\mathbb{T} = \{t_k \in \mathbb{R} : \ k \in \mathbb{N}_0\}$ with $t_k < t_{k+1}$ for all $k \in \mathbb{N}_0$ consists only of isolated points (i.e., it is an isolated time scale), then*

$$\sigma(t_k) = t_{k+1}, \quad \mu(t_k) = t_{k+1} - t_k, \quad f^\Delta(t_k) = \frac{f(t_{k+1}) - f(t_k)}{t_{k+1} - t_k} \quad \text{for} \quad k \in \mathbb{N}_0,$$

and

$$\int_{t_m}^{t_n} f(t)\Delta t = \sum_{k=m}^{n-1} \mu(t_k) f(t_k), \quad \text{where} \quad m, n \in \mathbb{N}_0 \text{ with } m < n. \tag{2.1}$$

The examples in Sections 5 and 7 are specific cases of Example 2.2 as follows.

Example 2.3. *Let $h > 0$. If $\mathbb{T} = h\mathbb{Z} = \{hk : \ k \in \mathbb{Z}\}$, then*

$$\sigma(t) = t + h, \quad \mu(t) \equiv h, \quad f^\Delta(t) = \frac{f(t+h) - f(t)}{h} \quad \text{for} \quad t \in \mathbb{T},$$

and

$$\int_a^b f(t)\Delta t = h \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh), \quad \text{where} \quad a, b \in \mathbb{T} \text{ with } a < b.$$

Example 2.4. *Let $q > 1$. If $\mathbb{T} = q^{\mathbb{N}_0} = \{q^n : \ n \in \mathbb{N}_0\}$, then*

$$\sigma(t) = qt, \quad \mu(t) = (q-1)t, \quad f^\Delta(t) = \frac{f(qt) - f(t)}{(q-1)t} \quad \text{for} \quad t \in \mathbb{T},$$

and

$$\int_{q^m}^{q^n} f(t)\Delta t = (q-1) \sum_{k=m}^{n-1} q^k f(q^k), \quad \text{where} \quad m, n \in \mathbb{N}_0 \text{ with } m < n.$$

3. Linear Programming Problems

Throughout this paper, $\mathbb{T}$ stands for a time scale, we assume $0 \in \mathbb{T}$, we let $T \in \mathbb{T}$, and we use $\mathcal{I}$ to denote the time scales interval

$$\mathcal{I} = [0, T] \cap \mathbb{T}.$$

By E_k, we denote the space of all rd-continuous functions (i.e., functions that are continuous in points $t \in \mathbb{T}$ with $\sigma(t) = t$, and their left-sided limits exist in points $t \in \mathbb{T}$ with $\rho(t) = t$, where the backward jump ρ is defined analagously to the forward jump ρ) from $\mathcal{I}$ into $\mathbb{R}^k$. In [1], the authors have introduced the primal time scales programming problem as

$$\begin{cases} \text{Maximize} \quad U(x) = \displaystyle\int_0^{\sigma(T)} f^\top(t)x(t)\Delta t \\ \text{subject to} \quad B(t)x(t) \le g(t) + \displaystyle\int_0^t K(t,s)x(s)\Delta s, \quad t \in \mathcal{I} \\ \text{and} \quad x \in E_n, \quad x(t) \ge 0, \quad t \in \mathcal{I}, \end{cases} \tag{P}$$

where $f \in E_n$, $g \in E_m$, and B and K are rd-continuous $m \times n$ matrix-valued functions. Moreover, in [1], the dual time scales programming problem is introduced as

$$\begin{cases} \text{Minimize} \quad V(z) = \displaystyle\int_0^{\sigma(T)} g^\top(t)z(t)\Delta t \\ \text{subject to} \quad B^\top(t)z(t) \ge f(t) + \displaystyle\int_{\sigma(t)}^{\sigma(T)} K^\top(s,t)z(s)\Delta s, \quad t \in \mathcal{I} \\ \text{and} \quad z \in E_m, \quad z(t) \ge 0, \quad t \in \mathcal{I}. \end{cases} \tag{D}$$

A feasible solution of (P) (or (D)) is any one that satisfies the given constraints. An optimal solution to (P) (or (D)) is a feasible solution with the largest (or smallest) objective function value. In [1], the following results are established.

Theorem 3.1 (Weak Duality Theorem)**.** *If x and z are arbitrary feasible solutions of* (P) *and* (D)*, respectively, then $U(x) \le V(z)$.*

Theorem 3.2 (Optimality Condition)**.** *If there exist feasible solutions x^* and z^* of* (P) *and* (D)*, respectively, such that $U(x^*) = V(z^*)$, then x^* and z^* are optimal solutions of their respective problems.*

Theorem 3.3 (Strong Duality Theorem)**.** *Assume $\mathbb{T}$ is an isolated time scale. If* (P) *has an optimal solution x^*, then* (D) *has an optimal solution z^* such that $U(x^*) = V(z^*)$.*

4. Separated Problems

In this section, we formulate the primal and the dual models for separated linear programming problems on arbitrary time scales. This formulation is an extension of separated continuous-time linear programming problems that are presented in [2,3,30,31,32,33,38,39] by using the methodology of time scales introduced by the authors in [1]. The primal time scales separated linear programming model is formulated as

$$\left\{\begin{array}{l} \text{Maximize} \quad U(x) = \displaystyle\int_0^{\sigma(T)} f^\top(t)x(t)\Delta t \\ \text{subject to} \quad \begin{array}{l} \displaystyle\int_0^t G(t,s)x(s)\Delta s \le a(t), \quad t \in \mathfrak{I} \\ B(t)x(t) \le b(t), \quad t \in \mathfrak{I} \end{array} \\ \text{and} \quad x \in E_n, \quad x(t) \ge 0, \quad t \in \mathfrak{I}, \end{array}\right. \tag{SP}$$

where $f \in E_n$, $a \in E_{m_1}$, $b \in E_{m_2}$, and B and G are rd-continuous matrix-valued functions of size $m_2 \times n$ and $m_1 \times n$, respectively. Rewriting the two inequalities in (SP) as one inequality

$$\begin{pmatrix} 0 \\ B(t) \end{pmatrix} x(t) \le \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} + \int_0^t \begin{pmatrix} -G(t,s) \\ 0 \end{pmatrix} x(s)\Delta s,$$

we can put (SP) in the form (P), then find the dual (D), and then rewrite (D) as

$$\left\{\begin{array}{l} \text{Minimize} \quad V(y,z) = \displaystyle\int_0^{\sigma(T)} \left[a^\top(t)y(t) + b^\top(t)z(t)\right]\Delta t \\ \text{subject to} \quad \displaystyle\int_{\sigma(t)}^{\sigma(T)} G^\top(s,t)y(s)\Delta s + B^\top(t)z(t) \ge f(t), \quad t \in \mathfrak{I} \\ \text{and} \quad y \in E_{m_1}, \quad z \in E_{m_2}, \quad y(t), z(t) \ge 0, \quad t \in \mathfrak{I}. \end{array}\right. \tag{SD}$$

A feasible solution of (SP) (or (SD)) is any one that satisfies the given constraints. An optimal solution to (SP) (or (SD)) is a feasible solution with the largest (or smallest) objective function value. Theorems 3.1–3.3 can now be rewritten as follows.

Theorem 4.1 (Weak Duality Theorem). *If x and (y,z) are arbitrary feasible solutions of* (SP) *and* (SD)*, respectively, then $U(x) \le V(y,z)$.*

Theorem 4.2 (Optimality Condition). *If there exist feasible solutions x^* and (y^*, z^*) of* (SP) *and* (SD)*, respectively, such that $U(x^*) = V(y^*, z^*)$, then x^* and (y^*, z^*) are optimal solutions of their respective problems.*

Theorem 4.3 (Strong Duality Theorem). *Assume $\mathbb{T}$ is an isolated time scale. If* (SP) *has an optimal solution x^*, then* (SD) *has an optimal solution (y^*, z^*) such that $U(x^*) = V(y^*, z^*)$.*

5. Examples (Separated)

In this section, three examples are given in order to illustrate our duality theorems on isolated time scales.

Example 5.1. *Let $\mathbb{T} = \mathbb{Z}$ and $\mathcal{I} = \{0, 1, 2, 3\}$. Then, we consider the isolated time scales separated linear programming primal model*

$$\begin{cases} \textit{Maximize} \quad U(x) = \displaystyle\int_0^{\sigma(3)} tx(t)\Delta t = \sum_{t=0}^{3} tx(t) \\ \textit{subject to} \quad \displaystyle\int_0^t x(s)\Delta s = \sum_{s=0}^{t-1} x(s) \le t^2, \quad t \in \mathcal{I} \\ \qquad\qquad\quad 6x(t) \le t + 1, \quad t \in \mathcal{I} \\ \textit{and} \quad x(t) \ge 0, \quad t \in \mathcal{I}, \end{cases}$$

where we have used σ and the integral given in Example 2.3 with $h = 1$. Using MATLAB command linprog or LINDO solver, we have

$$\begin{aligned} &x^*(0) = 0.000000, \qquad x^*(1) = 0.333333, \qquad x^*(2) = 0.500000, \\ &x^*(3) = 0.666667, \qquad U(x^*) = 3.333333. \end{aligned}$$

On the other hand, the isolated time scales separated linear programming dual model is

$$\begin{cases} \textit{Minimize} \quad \begin{aligned} V(y,z) &= \int_0^{\sigma(3)} \left[t^2 y(t) + (t+1)z(t)\right] \Delta t \\ &= \sum_{t=0}^{3} \left[t^2 y(t) + (t+1)z(t)\right] \end{aligned} \\ \textit{subject to} \quad \displaystyle\int_{\sigma(t)}^{\sigma(3)} y(s)\Delta s = \sum_{s=t+1}^{3} y(s) \ge t - 6z(t), \quad t \in \mathcal{I} \\ \textit{and} \quad y(t), z(t) \ge 0, \quad t \in \mathcal{I}, \end{cases}$$

where we have used again Example 2.3 with $h = 1$. Using MATLAB command linprog or LINDO solver, we have

$$\begin{aligned} &z^*(0) = 0.000000, \quad z^*(1) = 0.166667, \quad z^*(2) = 0.333333, \quad z^*(3) = 0.500000, \\ &y^*(0) = 0.000000, \quad y^*(1) = 0.000000, \quad y^*(2) = 0.000000, \quad y^*(3) = 0.000000, \end{aligned}$$

and the optimal value is $V(y^*, z^*) = 3.333333$, confirming $U(x^*) = V(y^*, z^*)$.

Example 5.2. *Let $\mathbb{T} = 5\mathbb{Z}$ and $\mathcal{I} = \{0, 5, 10, 15, 20\}$. Then, we consider the isolated time scales separated linear programming primal model*

$$\begin{cases} \textit{Maximize} \quad U(x) = \displaystyle\int_0^{\sigma(20)} tx(t)\Delta t = 25 \sum_{k=0}^{4} kx(5k) \\ \textit{subject to} \quad \displaystyle\int_0^t x(s)\Delta s = 5 \sum_{k=0}^{\frac{t}{5}-1} x(5k) \le 2t, \quad t \in \mathcal{I} \\ \qquad\qquad\quad 8x(t) \le t + 1, \quad t \in \mathcal{I} \\ \textit{and} \quad x(t) \ge 0, \quad t \in \mathcal{I}, \end{cases}$$

where we have used σ *and the integral given in Example 2.3. Using MATLAB command linprog or LINDO solver, we have*

$$\begin{aligned} x^*(0) &= 0.000, & x^*(5) &= 0.750, & x^*(10) &= 1.375, \\ x^*(15) &= 2.000, & x^*(20) &= 2.625, & U(x^*) &= 500.000. \end{aligned}$$

On the other hand, the isolated time scales separated linear programming dual model is

$$\begin{cases} \text{Minimize} & V(y,z) = \displaystyle\int_0^{\sigma(20)} [2ty(t) + (t+1)z(t)]\,\Delta t \\ & \qquad = 5\displaystyle\sum_{k=0}^{4} [10ky(5k) + (5k+1)z(5k)] \\ \text{subject to} & \displaystyle\int_{\sigma(t)}^{\sigma(20)} y(s)\Delta s = 5\sum_{k=\frac{t}{5}+1}^{4} y(5k) \geq t - 8z(t), \quad t \in \mathcal{I} \\ \text{and} & y(t), z(t) \geq 0, \quad t \in \mathcal{I}, \end{cases}$$

where we have used once more Example 2.3. Using MATLAB command linprog or LINDO solver, we have

$$\begin{aligned} z^*(0) &= 0.000, & z^*(5) &= 0.625, & z^*(10) &= 1.250, \\ z^*(15) &= 1.875, & z^*(20) &= 2.500, & & \\ y^*(0) &= 0.000, & y^*(5) &= 0.000, & y^*(10) &= 0.000, \\ y^*(15) &= 0.000, & y^*(20) &= 0.000, & & \end{aligned}$$

and the optimal value is $V(y^*, z^*) = 500$*, confirming* $U(x^*) = V(y^*, z^*)$*.*

Example 5.3. *Let* $\mathbb{T} = 2^{\mathbb{N}_0}$ *and* $\mathcal{I} = \{1, 2, 4\}$*. Then, we consider the isolated time scales separated linear programming primal model*

$$\begin{cases} \text{Maximize} & U(x) = \displaystyle\int_1^{\sigma(2^2)} tx(t)\Delta t = \sum_{k=0}^{2} 4^k x(2^k) \\ \text{subject to} & \displaystyle\int_1^t x(s)\Delta s = \sum_{k=0}^{\log_2 t - 1} 2^k x(2^k) \leq t^2, \quad t \in \mathcal{I} \\ & 6x(t) \leq t + 1, \quad t \in \mathcal{I} \\ \text{and} & x(t) \geq 0, \quad t \in \mathcal{I}, \end{cases}$$

where we have used σ *and the integral given in Example 2.4. Using MATLAB command linprog or LINDO solver, we have*

$$\begin{aligned} x^*(1) &= 0.333333, & x^*(2) &= 0.500000, \\ x^*(4) &= 0.833333, & U(x^*) &= 15.66667. \end{aligned}$$

On the other hand, the isolated time scales separated linear programming dual model is

$$\begin{cases} \textit{Minimize} & V(y,z) & = \displaystyle\int_{\frac{1}{2}}^{\sigma(2^2)} \left[t^2 y(t) + (t+1)z(t)\right] \Delta t \\ & & = \displaystyle\sum_{k=0}^{2} 2^k \left[(2^k)^2 y(2^k) + (2^k+1) z(2^k)\right] \\ \textit{subject to} & \displaystyle\int_{\sigma(t)}^{\sigma(4)} y(s)\Delta s = & \displaystyle\sum_{k=1+\log_2 t}^{2} 2^k y(2^k) \geq t - 6z(t), \quad t \in \mathfrak{I} \\ \textit{and} \quad y(t), z(t) \geq 0, \quad t \in \mathfrak{I}, \end{cases}$$

where we have used again Example 2.4. Using MATLAB command linprog or LINDO solver, we have

$$\begin{aligned} &y^*(1) = 0.000000, \qquad && y^*(2) = 0.000000, \qquad && y^*(4) = 0.000000, \\ &z^*(1) = 0.166667, && z^*(2) = 0.333333, && z^*(4) = 0.666667, \end{aligned}$$

and the optimal value is $V(y^*, z^*) = 15.66667$, confirming $U(x^*) = V(y, z^*)$.

6. State-Constrained Separated Problems

In this section, we formulate primal and the dual models for state-constrained separated linear programming problems on arbitrary time scales $\mathbb{T}$. This formulation extends state-constrained separated continuous-time linear programming problems as presented in [28,39,43], using the methodology of time scales as introduced by Al-Salih and Bohner [1]. The primal time scales state-constrained separated linear programming model is formulated as

$$\begin{cases} \text{Maximize} & U(u,x) = \displaystyle\int_0^{\sigma(T)} \left[c^\top(t)u(t) + f^\top(t)x(t)\right] \Delta t \\ & \displaystyle\int_0^t G(t,s)u(s)\Delta s + B(t)x(t) \leq a(t), \quad t \in \mathfrak{I} \\ \text{subject to} & H(t)u(t) \leq b(t), \quad t \in \mathfrak{I} \\ & F(t)x(t) \leq h(t), \quad t \in \mathfrak{I} \\ \text{and} \quad u, x \in E_n, \quad x(t), u(t) \geq 0, \quad t \in \mathfrak{I}, \end{cases} \tag{SCP}$$

where $c, f \in E_n$, $a \in E_{m_1}$, $b \in E_{m_2}$, $h \in E_{m_3}$, B and G are rd-continuous matrix-valued functions of size $m_1 \times n$, and H and F are rd-continuous matrix-valued functions of size $m_2 \times n$ and $m_3 \times n$, respectively. Rewriting the three inequalities in (SCP) as one inequality

$$\begin{pmatrix} H(t) & 0 \\ 0 & B(t) \\ 0 & F(t) \end{pmatrix} \begin{pmatrix} u(t) \\ x(t) \end{pmatrix} \leq \begin{pmatrix} b(t) \\ a(t) \\ h(t) \end{pmatrix} \int_0^t \begin{pmatrix} 0 & 0 \\ -G(t,s) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u(s) \\ x(s) \end{pmatrix} \Delta s,$$

we can put (SCP) in the form (P), then find the dual (D), and then rewrite (D) as

$$\begin{cases} \text{Minimize} \quad V(y,w,z) = \displaystyle\int_0^{\sigma(T)} \left[a^\top(t)y(t) + b^\top(t)w(t) + h^\top(t)z(t)\right] \Delta t \\ \text{subject to} \quad \displaystyle\int_{\sigma(t)}^{\sigma(T)} G^\top(s,t)y(s)\Delta s + H^\top(t)w(t) \geq c(t), \quad t \in \mathfrak{I} \\ \qquad\qquad\quad B^\top(t)y(t) + F^\top(t)z(t) \geq f(t), \quad t \in \mathfrak{I} \\ \text{and} \quad y \in E_{m_1}, \quad w \in E_{m_2}, \quad z \in E_{m_3}, \quad y(t), w(t), z(t) \geq 0, \quad t \in \mathfrak{I}. \end{cases} \tag{SCD}$$

A feasible solution of (SCP) (or (SCD)) is any one that satisfies the given constraints. An optimal solution to (SCP) (or (SCD)) is a feasible solution with the largest (or smallest) objective function value. Theorems 3.1–3.3 can now be rewritten as follows.

Theorem 6.1 (Weak Duality Theorem). *If (u,x) and (y,w,z) are arbitrary feasible solutions of (SCP) and (SCD), respectively, then $U(u,x) \leq V(y,w,z)$.*

Theorem 6.2 (Optimality Condition). *If there are feasible solutions (u^*,x^*) and (y^*,w^*,z^*) of (SCP) and (SCD), respectively, with $U(u^*,x^*) = V(y^*,w^*,z^*)$, then x^* and z^* are optimal solutions of their respective problems.*

Theorem 6.3 (Strong Duality Theorem). *Assume $\mathbb{T}$ is an isolated time scale. If (SCP) has an optimal solution (u^*,x^*), then (SCD) has an optimal solution (y^*,w^*,z^*) such that $U(u^*,x^*) = V(y^*,w^*,z^*)$.*

7. Examples (State-Constrained Separated)

In this section, three examples are given in order to illustrate our duality theorems on isolated time scales.

Example 7.1. *Let $\mathbb{T} = \mathbb{Z}$ and $\mathfrak{I} = \{0,1,2,3\}$. Then, we consider the isolated time scales state-constrained separated linear programming primal model*

$$\begin{cases} \textit{Maximize} \quad U(x) = \displaystyle\int_0^{\sigma(3)} [tu(t) + 5tx(t)]\,\Delta t = \sum_{t=0}^{3} [tu(t) + 5tx(t)] \\ \textit{subject to} \quad \displaystyle\int_0^t x(s)\Delta s = \sum_{s=0}^{t-1} x(s) \leq t^2 + 3 - 10x(t), \quad t \in \mathfrak{I} \\ \qquad\qquad\quad 6u(t) \leq t+1, \quad t \in \mathfrak{I} \\ \qquad\qquad\quad 2x(t) \leq t+2, \quad t \in \mathfrak{I} \\ \textit{and} \quad x(t), u(t) \geq 0, \quad t \in \mathfrak{I}, \end{cases}$$

where we have used σ and the integral given in Example 2.3 with $h = 1$. Using MATLAB command linprog or LINDO solver, we have

$$\begin{aligned} x^*(0) &= 0.000000, & x^*(1) &= 0.400000, & x^*(2) &= 0.700000, \\ x^*(3) &= 1.150000, & u^*(0) &= 0.000000, & u^*(1) &= 0.000000, \\ u^*(2) &= 0.500000, & u^*(3) &= 0.666667, & U(u^*,x^*) &= 29.25000. \end{aligned}$$

On the other hand, the isolated time scales state-constrained separated linear programming dual model is

$$
\begin{cases}
\text{Minimize} & V(y,w,z) = \displaystyle\int_0^{\sigma(3)} \big[(t^2+3)y(t)+(t+1)w(t)+(t+2)z(t)\big]\,\Delta t \\
& \qquad\qquad = \displaystyle\sum_{t=0}^{3} \big[(t^2+3)y(t)+(t+1)w(t)+(t+2)z(t)\big] \\
\text{subject to} & \displaystyle\int_{\sigma(t)}^{\sigma(3)} y(s)\Delta s = \sum_{s=t+1}^{3} y(s) \ge t-6w(t), \quad t\in\mathfrak{I} \\
& 10y(t)+2z(t)\ge 5t, \quad t\in\mathfrak{I} \\
\text{and} & y(t),w(t),z(t)\ge 0, \quad t\in\mathfrak{I},
\end{cases}
$$

where we have used again Example 2.3 with $h=1$. Using MATLAB command linprog or LINDO solver, we have

$$
\begin{aligned}
&y^*(0)=0.00000, && y^*(1)=0.50000, && y^*(2)=1.00000, && y^*(3)=1.50000,\\
&w^*(0)=0.00000, && w^*(1)=0.16667, && w^*(2)=0.08333, && w^*(3)=0.50000,\\
&z^*(0)=0.00000, && z^*(1)=0.00000, && z^*(2)=0.00000, && z^*(3)=0.00000,
\end{aligned}
$$

and the optimal value is

$$V(y^*,w^*,z^*)=29.25000,$$

confirming $U(u^*,x^*)=V(y^*,w^*,z^*)$.

Example 7.2. *Let $\mathbb{T}=5\mathbb{Z}$ and $\mathfrak{I}=\{0,5,10,15,20\}$. Then, we consider the isolated time scales state-constrained separated linear programming primal model*

$$
\begin{cases}
\text{Maximize} & U(u,x) = \displaystyle\int_0^{\sigma(20)} \big[tu(t)+t^2x(t)\big]\,\Delta t \\
& \qquad\qquad = 5\displaystyle\sum_{k=0}^{4} \big[5ku(5k)+(5k)^2x(5k)\big] \\
\text{subject to} & \displaystyle\int_0^{t} u(s)\Delta s = 5\sum_{k=0}^{\frac{t}{5}-1} u(5k) \le t+1-10x(t), \quad t\in\mathfrak{I} \\
& 3u(t)\le t+1, \quad t\in\mathfrak{I} \\
& 2x(t)\le t^2+3, \quad t\in\mathfrak{I} \\
\text{and} & x(t),u(t)\ge 0, \quad t\in\mathfrak{I},
\end{cases}
$$

where we have used σ and the integral given in Example 2.3. Using MATLAB command linprog or LINDO solver, we have

$$
\begin{aligned}
&x^*(0)=0.000000, && x^*(5)=0.600000, && x^*(10)=1.100000,\\
&x^*(15)=1.600000, && x^*(20)=2.100000, && \\
&u^*(0)=0.000000, && u^*(5)=0.000000, && u^*(10)=0.000000,\\
&u^*(15)=0.000000, && u^*(20)=13.666667, && U(x^*)=7991.667000.
\end{aligned}
$$

On the other hand, the isolated time scales state-constrained separated linear programming dual model is

$$\begin{cases} \textit{Minimize} & V(y,w,z) = \displaystyle\int_0^{\sigma(20)} \left[(t+1)(y(t)+w(t))+(t^2+3)z(t)\right]\Delta t \\ & \qquad = 5\displaystyle\sum_{k=0}^{4}\left[(5k+1)(y(5k)+w(5k))+(25k^2+3)z(5k)\right] \\ \textit{subject to} & \displaystyle\int_{\sigma(t)}^{\sigma(20)} y(s)\Delta s = 5\sum_{k=\frac{t}{5}+1}^{4} y(5k) \geq t-3w(t), \quad t\in\mathfrak{I} \\ & 10y(t)+2z(t)\geq t^2, \quad t\in\mathfrak{I} \\ \textit{and} & y(t),w(t),z(t)\geq 0, \quad t\in\mathfrak{I}, \end{cases}$$

where we have used again Example 2.3. Using MATLAB command linprog or LINDO solver, we have

$$\begin{aligned} &y^*(0)=0.000000, && y^*(5)=2.500000, && y^*(10)=10.000000,\\ &y^*(15)=22.500000, && y^*(20)=40.000000, && \\ &w^*(0)=0.000000, && w^*(5)=0.000000, && w^*(10)=0.000000,\\ &w^*(15)=0.000000, && w^*(20)=6.666667, && \\ &z^*(0)=0.000000, && z^*(5)=0.000000, && z^*(10)=0.000000,\\ &z^*(15)=0.000000, && z^*(20)=0.000000, && \end{aligned}$$

and the optimal value is

$$V(y^*,w^*,z^*)=7991.667000,$$

confirming $U(u^*,x^*)=V(y^*,w^*,z^*)$.

Example 7.3. *Let* $\mathbb{T}=2^{\mathbb{N}_0}$ *and* $\mathfrak{I}=\{1,2,4\}$. *Then, we consider the isolated time scales state-constrained separated linear programming primal model*

$$\begin{cases} \textit{Maximize} & U(x) = \displaystyle\int_1^{\sigma(2^2)} \left[tu(t)+t^3x(t)\right]\Delta t \\ & \qquad = \displaystyle\sum_{k=0}^{2} 2^k\left[2^k u(2^k)+(2^k)^3 x(2^k)\right] \\ \textit{subject to} & \displaystyle\int_1^t 5u(s)\Delta s = 5\sum_{k=0}^{\log_2 t-1} 2^k u(2^k) \leq 2t+1-20x(t), \quad t\in\mathfrak{I} \\ & 10u(t)\leq t, \quad t\in\mathfrak{I} \\ & 4x(t)\leq t^2+3, \quad t\in\mathfrak{I} \\ \textit{and} & x(t),u(t)\geq 0, \quad t\in\mathfrak{I}, \end{cases}$$

where we have used σ and the integral given in Example 2.4. Using MATLAB command linprog or LINDO solver, we have

$$\begin{aligned} &x^*(1) = 0.10, && x^*(2) = 0.25, && x^*(4) = 0.45,\\ &u^*(1) = 0.00, && u^*(2) = 0.00, && u^*(4) = 0.40,\\ & && U(u^*, x^*) = 125.75. && \end{aligned}$$

On the other hand, the isolated time scales state-constrained separated linear programming dual model is

$$\begin{cases} \textit{Minimize} & V(y,w,z) = \displaystyle\int_1^{\sigma(2^2)} \left[(2t+1)y(t) + tw(t) + (t^2+3)z(t)\right] \Delta t \\ & \qquad = \displaystyle\sum_{k=0}^{2} 2^k \left[(2\cdot 2^k + 1)y(2^k) + 2^k w(2^k) + (4^k+3)z(2^k)\right] \\ \textit{subject to} & \displaystyle\int_{\sigma(t)}^{\sigma(4)} 5y(s)\Delta s = 5 \sum_{k=1+\log_2 t}^{2} 2^k y(2^k) \geq t - 10w(t), \quad t \in \mathfrak{I} \\ & 20y(t) + 4z(t) \geq t^3, \quad t \in \mathfrak{I} \\ \textit{and} \quad y(t), w(t), z(t) \geq 0, & t \in \mathfrak{I}, \end{cases}$$

where we have used again Example 2.4. Using MATLAB command linprog or LINDO solver, we have

$$\begin{aligned} &y^*(1) = 0.05, && y^*(2) = 0.40, && y^*(4) = 3.20,\\ &w^*(1) = 0.00, && w^*(2) = 0.00, && w^*(4) = 0.40,\\ &z^*(1) = 0.00, && z^*(2) = 0.00, && z^*(4) = 0.00,\\ & && V(y^*, w^*, z^*) = 125.75, && \end{aligned}$$

confirming $U(u^*, x^*) = V(y^*, w^*, z^*)$.

8. Conclusions

An efficient formulation and a computational approach have been successfully constructed in this paper to solve two classes of separated linear programming problems on arbitrary time scales. Discretization-based methods have been used recently to solve this class of problems, but unfortunately these methods can only obtain the approximate solutions in most cases. Our formulation has addressed this issue by finding the exact optimal solution of the problem using an isolated time scales approach. Another key issue for the discretization-based methods is to solve both primal and dual models at the same time to abstain the error bound of the solution, so another by-product of this paper is to obtain the optimal solution by either solving the primal or the dual problem only, which will reduce the large computational effort. Moreover, to ensure that our new formulation is a useful formulation, we have established some fundamental theorems such as the weak duality theorem and the optimality condition on arbitrary time scales, while the strong duality theorem is presented for isolated time scales.

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