STURMIAN AND SPECTRAL THEORY FOR DISCRETE SYMPLECTIC SYSTEMS

MARTIN BOHNER[1], ONDŘEJ DOŠLÝ[2], AND WERNER KRATZ[3]

Abstract. We consider $2n \times 2n$ symplectic difference systems together with associated discrete quadratic functionals and eigenvalue problems. We establish Sturmian type comparison theorems for the numbers of focal points of conjoined bases of a pair of symplectic systems. Then, using this comparison result, we show that the numbers of focal points of two conjoined bases of one symplectic system differ by at most n. In the last part of the paper we prove the Rayleigh principle for symplectic eigenvalue problems and we show that finite eigenvectors of such eigenvalue problems form a complete orthogonal basis in the space of admissible sequences.

1. Introduction and Main Results

In this paper we deal with oscillation properties of *symplectic difference systems*

$$z_{k+1} = \mathcal{S}_k z_k, \quad k \in \{0, \dots, N\} \tag{S}$$

where the matrices $\mathcal{S}_k$ are *symplectic*, i.e.,

$$\mathcal{S}^T \mathcal{J} \mathcal{S} = \mathcal{J}, \quad \text{where} \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

(Note that here and in the entire paper we use the convention that an equation written as $A = B$ means that $A_k = B_k$ for all $k \in \{0, \dots, N\}$.) The system (S) is a natural discrete counterpart of the *linear Hamiltonian differential system*

$$z' = \mathcal{H}(t) z, \quad \text{where} \quad \mathcal{J}\mathcal{H}(t) + \mathcal{H}^T(t)\mathcal{J} = 0, \quad t \in [a, b], \tag{H}$$

whose oscillation theory is deeply developed, see e.g., [17,21–23]. Discrete symplectic systems play a key rôle in the numerical methods for solving Hamiltonian systems, since they "... present a proper way, i.e., the Hamiltonian way, for computing the Hamiltonian dynamics" [14, page 18]. Also, these systems

Date: Submitted to TAMS on June 20, 2007.

1991 *Mathematics Subject Classification.* 39A12, 39A13, 34B24, 49K99.

Key words and phrases. Discrete symplectic system, discrete quadratic functional, Sturmian separation result, Sturmian comparison result, Rayleigh principle, extended Picone identity.

are closely related to discrete quadratic functionals which arise as the second-order nonlinear problems in the discrete calculus of variations and optimal control theory, see, e.g., [2, 13, 15, 16, 19, 24–26].

If in (S) and (H) we write

$$z = \begin{pmatrix} x \\ u \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}$$

with $x, u \in \mathbb{R}^n$ and $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, A, B, C \in \mathbb{R}^{n\times n}$, then these systems can be rewritten in the forms

$$x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \quad u_{k+1} = \mathcal{C}_k x_k + \mathcal{D}_k u_k, \quad k \in \{0, \dots, N\} \tag{S}$$

and

$$x' = A(t)x + B(t)u, \quad u' = C(t)x - A^T(t)u, \quad t \in [a, b], \tag{H}$$

respectively. A matrix solution $Z = (X, U)$ of (S), where X_k and U_k are $n \times n$-matrices, is said to be a *conjoined basis* of (S) if the matrices $X_k^T U_k$ are symmetric and $\operatorname{rank}\left(\begin{smallmatrix} X_k \\ U_k \end{smallmatrix}\right) = n$ for all $k \in \{0, \dots, N\}$. A conjoined basis of (H) is defined in a similar way. Let (X, U) and $(\tilde{X}, \tilde{U})$ be two conjoined bases of (H) and denote by m and $\tilde{m}$ the number of points in $[a, b]$ satisfying $\det X(t) = 0$ and $\det \tilde{X}(t) = 0$, respectively (the so-called *focal points* of (X, U) and $(\tilde{X}, \tilde{U})$). The basic statement of oscillation theory of (H) (the so-called *Sturmian separation theorem* for (H)) states that $|m - \tilde{m}| \le n$ [22, Chapter VII, Corollary 1 of Theorem 7.9 on page 366]. Of course, if (H) corresponds to a second-order Sturm–Liouville equation, e.g.,

$$(r(t)x')' + p(t)x = 0, \quad t \in [a, b], \tag{SL}$$

then this statement reduces to the classical Sturmian theorem about separation of zeros of linearly independent solutions of (SL).

The aim of this paper is, among others, to establish the analogue for conjoined bases of the discrete system (S). To formulate this result, we will use the following notation and concepts. For a real and symmetric matrix P we write $P \ge 0$ if P is nonnegative definite, and $\operatorname{ind} P$ denotes the *index* of P, i.e., the number of negative eigenvalues (including multiplicities) of P. By $\operatorname{Ker} M$, $\operatorname{Im} M$, $\operatorname{rank} M$, M^T, and M^{-1} we denote the *kernel*, *image*, *rank*, *transpose*, and *inverse* of a matrix M, respectively. The notion of the *Moore–Penrose inverse* is fundamental in understanding the idea of multiplicity of a focal point as explained below, and therefore we will spend a few words discussing its definition and basic properties: For an $m \times n$-matrix M, there exists a unique $n \times m$-matrix N satisfying $NMN = N$ and $MNM = M$ such that both NM and MN are symmetric (see [1, Section 2.8], [3, Theorem 1.5], [5, Appendix]). This matrix N is called the Moore–Penrose inverse of M and is denoted by $M^\dagger$. It can be explicitly given (see [1, Lemma 2.8.3], [17, Remark 3.3.2]) by

$$M^\dagger = \lim_{t\to 0^+} \left\{(M^T M + tI)^{-1} M^T\right\} = \lim_{t\to 0^+} \left\{M^T (M M^T + tI)^{-1}\right\},$$

and in particular, these limits always exist, and we have

$$(M^T)^\dagger = (M^\dagger)^T, \quad (M^\dagger)^\dagger = M, \quad \text{and} \quad \operatorname{Ker}(M^\dagger)^T = \operatorname{Ker} M.$$

The main "reason" for the appearance of Moore–Penrose inverses in our theory is that for two matrices V and W, the following equivalences hold (see [1, Lemma 2.8.6], [4, Lemma 4], [5, Lemma A5], [6, Remark 2 (ii) and (iii)]):

$$\operatorname{Ker} V \subset \operatorname{Ker} W \quad \Longleftrightarrow \quad W = W V^\dagger V \quad \Longleftrightarrow \quad W^\dagger = V^\dagger V W^\dagger.$$

For a conjoined basis (X, U) of (S), the following matrices were introduced in [18]:

$$\begin{cases} M_k = (I - X_{k+1} X_{k+1}^\dagger) \mathcal{B}_k, \\ T_k = I - M_k^\dagger M_k, \\ P_k = T_k^T X_k X_{k+1}^\dagger \mathcal{B}_k T_k \end{cases} \tag{1.1}$$

for $k \in \{0, \dots, N\}$. Then obviously $M_k T_k = 0$ and it can be shown (see, e.g., [18]) that the matrix P_k is symmetric.

We say that a conjoined basis (X, U) has no *focal point* [6,8] in the interval $(k, k+1]$ if

$$\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k \quad \text{and} \quad X_k X_{k+1}^\dagger \mathcal{B}_k \geq 0 \tag{1.2}$$

holds. Note that if the first condition in (1.2) holds, then the matrix $X_k X_{k+1}^\dagger \mathcal{B}_k$ is really symmetric (see [8]), and it equals the matrix P_k given by (1.1) since $T_k = I$ in this case (see [18]). The *multiplicity* [18] of a focal point in the interval $(k, k+1]$ is defined as the number

$$\operatorname{rank} M_k + \operatorname{ind} P_k.$$

Throughout this paper, focal points of any conjoined basis are counted including their multiplicities.

Now we can formulate the Sturmian type separation theorem for conjoined bases of (S), which is a discrete version of the above mentioned separation theorem for (H).

Theorem 1.1. *The difference of the numbers of focal points in* $(0, N+1]$ *of any two conjoined bases of* (S) *is at most* n.

A natural extension of the Sturmian separation theorem is the Sturmian comparison theorem which compares the number of zeros of solutions of two equations (SL) or systems (H). Together with (S), we consider the system

$$z_{k+1} = \underset{\sim}{\mathcal{S}}_k z_k, \quad k \in \{0, \dots, N\}, \quad \text{where} \quad \underset{\sim}{\mathcal{S}} = \begin{pmatrix} \underset{\sim}{\mathcal{A}} & \underset{\sim}{\mathcal{B}} \\ \underset{\sim}{\mathcal{C}} & \underset{\sim}{\mathcal{D}} \end{pmatrix} \tag{$\underset{\sim}{\mathrm{S}}$}$$

and recall that the conjoined basis $Z = (X, U)$ of (S) (and similarly for ($\underset{\sim}{\mathrm{S}}$)) given by the initial condition $X_0 = 0$, $U_0 = I$ is called the *principal solution* of (S) at 0.

Theorem 1.2. *Define the* $2n \times 2n$*-matrices*

$$\mathcal{G} := \begin{pmatrix} \mathcal{A}^T \mathcal{B}\mathcal{B}^\dagger \mathcal{D}\mathcal{B}^\dagger \mathcal{A} - \mathcal{A}^T \mathcal{C} & \mathcal{C}^T - \mathcal{A}^T \mathcal{B}\mathcal{B}^\dagger \mathcal{D}\mathcal{B}^\dagger \\ \mathcal{C} - (\mathcal{B}^\dagger)^T \mathcal{D}^T \mathcal{B}\mathcal{B}^\dagger \mathcal{A} & \mathcal{B}\mathcal{B}^\dagger \mathcal{D}\mathcal{B}^\dagger \end{pmatrix} \tag{1.3}$$

and

$$\underset{\sim}{\mathcal{G}} := \begin{pmatrix} \underset{\sim}{\mathcal{A}}^T \underset{\sim}{\mathcal{B}}\underset{\sim}{\mathcal{B}}^\dagger \underset{\sim}{\mathcal{D}}\underset{\sim}{\mathcal{B}}^\dagger \underset{\sim}{\mathcal{A}} - \underset{\sim}{\mathcal{A}}^T \underset{\sim}{\mathcal{C}} & \underset{\sim}{\mathcal{C}}^T - \underset{\sim}{\mathcal{A}}^T \underset{\sim}{\mathcal{B}}\underset{\sim}{\mathcal{B}}^\dagger \underset{\sim}{\mathcal{D}}\underset{\sim}{\mathcal{B}}^\dagger \\ \underset{\sim}{\mathcal{C}} - (\underset{\sim}{\mathcal{B}}^\dagger)^T \underset{\sim}{\mathcal{D}}^T \underset{\sim}{\mathcal{B}}\underset{\sim}{\mathcal{B}}^\dagger \underset{\sim}{\mathcal{A}} & \underset{\sim}{\mathcal{B}}\underset{\sim}{\mathcal{B}}^\dagger \underset{\sim}{\mathcal{D}}\underset{\sim}{\mathcal{B}}^\dagger \end{pmatrix}$$

and suppose

$$\mathcal{G} \geq \underset{\sim}{\mathcal{G}} \quad \textit{and} \quad \operatorname{Im}(\mathcal{A} - \underset{\sim}{\mathcal{A}} \quad \mathcal{B}) \subset \operatorname{Im} \underset{\sim}{\mathcal{B}}. \tag{1.4}$$

If the principal solution of ($\underset{\sim}{\mathrm{S}}$) *has* m *focal points in* $(0, N+1]$, *then any conjoined basis of* (S) *has at most* $m + n$ *focal points in* $(0, N+1]$.

Theorem 1.3. *Suppose that* (1.4) *holds. If the principal solution of* (S) *has* m *focal points in* $(0, N+1]$, *then any conjoined basis of* ($\underset{\sim}{\mathrm{S}}$) *has at least* m *focal points in* $(0, N+1]$.

The proofs of these theorems are given in Section 2 and 3. These proofs are based on various results concerning eigenvalue problems associated with (S) in the form

$$\begin{cases} x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \quad u_{k+1} = \mathcal{C}_k x_k + \mathcal{D}_k u_k - \lambda x_{k+1}, \quad k \in \{0, \dots, N\} \\ x_0 = x_{N+1} = 0. \end{cases} \tag{E}$$

The eigenvalue problem (E) (including some preparatory material) is treated in the last Section 4 of this paper. In particular, attention is focussed on the *Rayleigh principle* for the discrete quadratic functional associated with (E) and on the *completeness* of finite eigenvectors in the space of the so-called admissible sequences. Section 4 also contains some technical statements used in the proofs of the results of our paper.

2. Sturmian Comparison Results

An important rôle in the proof of Theorems 1.2 and 1.3 is played by the associated discrete *quadratic functional*

$$\mathcal{F}(z) = \sum_{k=0}^{N} \left\{ x_k^T \mathcal{A}_k^T \mathcal{C}_k x_k + 2x_k^T \mathcal{C}_k^T \mathcal{B}_k u_k + u_k^T \mathcal{B}_k^T \mathcal{D}_k u_k \right\} \qquad \text{for} \qquad z = \begin{pmatrix} x \\ u \end{pmatrix}.$$

A pair of n-dimensional sequences $z = \{z_k\}_{k=0}^{N+1} = \{(x_k, u_k)\}_{k=0}^{N+1}$ is said to be *admissible* for $\mathcal{F}$ provided it satisfies the first equation in (S), i.e., the so-called *equation of motion* $x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k$ for all $k \in \{0, \dots, N\}$.

Now we recall some concepts and statements associated with the eigenvalue problem (E). This eigenvalue problem is treated in detail in Section 4, but some results we need to present already now. A number $\lambda \in \mathbb{R}$ is called a *finite eigenvalue* of (E) if there exists a solution $z = (x, u)$ of (E) such that $x_0 = 0 = x_{N+1}$ and $x = \{x_k\}_{k=0}^{N+1} \neq 0$, and then z is called a *finite eigenvector* corresponding to λ. Let $(\tilde{X}, \tilde{U}) = (\tilde{X}(\lambda), \tilde{U}(\lambda))$ be the principal solution of the symplectic system in (E) and denote by $\tilde{n}_1(\lambda)$ the number of focal points of $(\tilde{X}, \tilde{U})$ in $(0, N+1]$. Then by (4.8) given in Section 4,

$$\tilde{n}_1(\lambda) = n_2(\lambda), \tag{2.1}$$

where $n_2(\lambda)$ denotes the number of finite eigenvalues of (E) (counting multiplicities, see Definition 4.1 for details) which are less than or equal to λ.

The results in this section extend [16, Theorem 7.1], where the statements are proved (using a different technique than here, namely the Riccati technique) in the special case $m = 0$ (see also [7, Theorem 2]). Here we also use the following lemma from [16, Formula (7.1)] (see also [20, Lemma 1.32]), which can be proved by a direct computation.

Lemma 2.1. *If $z = (x, u)$ is admissible and $\mathcal{G}$ is given by* (1.3)*, then*

$$\mathcal{F}(z) = \sum_{k=0}^{N} \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix}^T \mathcal{G}_k \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix}.$$

Proof of Theorem 1.2. Let $Z = (X, U)$ be any conjoined basis of (S). Suppose that Z has p focal points in $(0, N+1]$. Then, corresponding to each focal point, we can construct $z^{[\nu]} = (x^{[\nu]}, u^{[\nu]})$, $1 \leq \nu \leq p$, as in [11, (12) and (13) on page 338] such that

$$x^{[\nu]} \quad \text{is admissible and} \quad x_{N+1}^{[\nu]} = 0 \quad \text{for all} \quad 1 \leq \nu \leq p.$$

Furthermore, since the principal solution $\underset{\sim}{Z} = (\underset{\sim}{X}, \underset{\sim}{U})$ of $(\underset{\sim}{\mathrm{S}})$ has m focal points in $(0, N+1]$, by (2.1) with $\lambda = 0$, applied to the eigenvalue problem

$$\begin{cases} x_{k+1} = \underset{\sim}{A}_k x_k + \underset{\sim}{B}_k u_k, \quad u_{k+1} = \underset{\sim}{C}_k x_k + \underset{\sim}{D}_k u_k - \lambda x_{k+1}, \quad k \in \{0, \dots, N\} \\ x_0 = x_{N+1} = 0, \end{cases} \tag{2.2}$$

this eigenvalue problem has m nonpositive finite eigenvalues λ_μ, $1 \le \mu \le m$, with corresponding orthonormal finite eigenvectors $\underset{\sim}{z}^{(\mu)} = (\underset{\sim}{x}^{(\mu)}, \underset{\sim}{u}^{(\mu)})$, $1 \le \mu \le m$. Moreover, by Theorem 4.6 given in Section 4, $\mathcal{F}(z) \ge \underset{\sim}{\mathcal{F}}(z) > 0$ for $z = (x, u)$ satisfying

$$z \perp \underset{\sim}{z}^{(\mu)}, \quad \text{i.e.,} \quad \left\langle z, \underset{\sim}{z}^{(\mu)} \right\rangle := \sum_{k=0}^{N} x_{k+1}^T \underset{\sim}{x}_{k+1}^{(\mu)} = 0, \ 1 \le \mu \le m, \quad x \ne 0.$$

Now suppose that $p > m + n$. Then there exists a nontrivial linear combination

$$\sum_{\nu=1}^{p} c_\nu \begin{pmatrix} \left\langle z^{[\nu]}, \underset{\sim}{z}^{(1)} \right\rangle \\ \left\langle z^{[\nu]}, \underset{\sim}{z}^{(2)} \right\rangle \\ \vdots \\ \left\langle z^{[\nu]}, \underset{\sim}{z}^{(m)} \right\rangle \\ x_0^{[\nu]} \end{pmatrix} = 0.$$

Define

$$z = (x, u) = \sum_{\nu=1}^{p} c_\nu z^{[\nu]}.$$

By construction, $x_{N+1} = 0$ and x is admissible as it is of the same form as in [11, (15) on page 339]. Moreover, $\sum_{\nu=1}^{p} c_\nu x_0^{[\nu]} = 0$ implies $x_0 = 0$ and we also have

$$0 = \sum_{\nu=1}^{p} c_\nu \left\langle z^{[\nu]}, \underset{\sim}{z}^{(\mu)} \right\rangle = \left\langle z, \underset{\sim}{z}^{(\mu)} \right\rangle \quad \text{for all} \quad 1 \le \mu \le m,$$

so $z \perp \tilde{z}^{(\mu)}$ for all $1 \le \mu \le m$. As in [11, Proof of Theorem 1 on page 339] we have $x \ne 0$ (the x there was of the same form and the only property that was used there was that not all $c_\nu = 0$, which is guaranteed in our current setting) and $\mathcal{F}(z) \le 0$. From the second condition in (1.4) it follows that there exists $\underset{\sim}{u} = \{\underset{\sim}{u}_k\}_{k=0}^N$ such that $x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k \underset{\sim}{u}_k$ for all $k \in \{0, \dots, N\}$, and hence $\underset{\sim}{z} = (x, \underset{\sim}{u})$ is admissible for $\underset{\sim}{\mathcal{F}}$ and by applying Lemma 2.1 twice and coupled with the first condition in (1.4), we find

$$\underset{\sim}{\mathcal{F}}(z) = \sum_{k=0}^{N} \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix}^T \underset{\sim}{G}_k \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix} \le \sum_{k=0}^{N} \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix}^T \mathcal{G}_k \begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix} = \mathcal{F}(z) \le 0.$$

Hence we have found an admissible $z=(x,u)$ with $x\neq 0$, $x_0=x_{N+1}=0$, $z\perp \underset{\sim}{z}^{(\mu)}$ for all $1\leq\mu\leq m$, and $\underset{\sim}{\mathcal{F}}(z)\leq 0$, contradicting the Rayleigh principle, Theorem 4.6 as stated and proved in Section 4, by which $\underset{\sim}{\mathcal{F}}(z)>0$ for all admissible z with $x_0=x_{N+1}=0$, $x\neq 0$, and $z\perp \underset{\sim}{z}^{(\mu)}$ for all $1\leq\mu\leq m$. $\square$

Proof of Theorem 1.3. We consider (E). Let $(X,U)=(X(\lambda),U(\lambda))$ be the principal solution of the symplectic system in (E) and let $(\underset{\sim}{X},\underset{\sim}{U})=(\underset{\sim}{X}(\lambda),\underset{\sim}{U}(\lambda))$ be any conjoined basis of the symplectic system in (2.2) such that $\underset{\sim}{X}_0(\lambda)\equiv\underset{\sim}{X}_0$ and $\underset{\sim}{U}_0(\lambda)\equiv\underset{\sim}{U}_0$ are constant. Let $m=n_1(\lambda)$ and $p=p(\lambda)$ be the numbers of focal points of (X,U) and $(\underset{\sim}{X},\underset{\sim}{U})$ in $(0,N+1]$, respectively. Then the assertion of Theorem 1.3 says

$$n_1(0)\leq p(0).$$

We show that $n_1(\lambda)\leq p(\lambda)$ for all $\lambda\in\mathbb{R}$. To do so, let (as in Section 4) λ_μ denote the finite eigenvalues of (E) with corresponding orthonormal eigenvectors $z^{(\mu)}$ for $1\leq\mu\leq r$ such that $\lambda_1\leq\ldots\leq\lambda_r$. Now, given $\lambda\in\mathbb{R}$, we have that

$$\lambda_m\leq\lambda<\lambda_{m+1}\quad\text{ for some }\quad m\in\{0,\ldots,r\},$$

where we put $\lambda_0=-\infty$ and $\lambda_{r+1}=\infty$. By (2.1), this means that $m=n_1(\lambda)$. First suppose that λ is not a finite eigenvalue of (E) so that $\lambda_m<\lambda<\lambda_{m+1}$. Put $\tilde z=\sum_{\mu=1}^m\beta_\mu z^{(\mu)}$, where the constants $\beta_1,\beta_2,\ldots,\beta_m$ are chosen in such a way that $\tilde z=(\tilde x,\tilde u)$ satisfies $\tilde p$ linear homogeneous conditions

$$\begin{cases}\underset{\sim}{M}_k^T\tilde x_{k+1}=0, & k\in\{0,\ldots,N-1\},\\ \tilde s_k\perp\{\alpha\in\mathbb{R}^n: & \alpha\text{ is an eigenvector corresponding}\\ & \text{to a negative eigenvalue of }\underset{\sim}{P}_k\},\quad k\in\{0,\ldots,N\},\end{cases}\tag{2.3}$$

where

$$\tilde s_k=\tilde u_k-Q_k\tilde x_k=\sum_{\mu=1}^m\beta_\mu(\tilde u_k^{(\mu)}-Q_k\tilde x_k^{(\mu)})$$

with the matrix Q satisfying $Q\underset{\sim}{X}=\underset{\sim}{U}\underset{\sim}{X}^\dagger\underset{\sim}{X}$ and the matrices $\underset{\sim}{M}$, $\underset{\sim}{P}$ given by (1.1) with $(X,U)=(\underset{\sim}{X},\underset{\sim}{U})$, and where $\tilde p$ equals to the number of focal points of $(\underset{\sim}{X},\underset{\sim}{U})$ in the open interval $(0,N+1)$ so that $\tilde p\leq p$. The sequence $\tilde z$ is admissible for $\mathcal{F}$ and by the second condition in (1.4) there exists $\underset{\sim}{u}$ such that $(\tilde x,\underset{\sim}{u})$ is admissible for $\underset{\sim}{\mathcal{F}}$. Since the value of the quadratic functional does not depend on the second component of an admissible sequence $z=(x,u)$ (see Lemma 2.1), we write also $\tilde z=(\tilde x,\underset{\sim}{u})$. Then by the extended Picone identity, Theorem 4.2 from Section 4, we have

$$\underset{\sim}{\mathcal{F}}_\lambda(\tilde z):=\underset{\sim}{\mathcal{F}}(\tilde z)-\lambda\langle\tilde z,\tilde z\rangle=\sum_{k=0}^N\tilde s_k^T\underset{\sim}{P}_k\tilde s_k\geq 0.$$

Note that the first condition in (2.3) implies that $\tilde{x}_k \in \operatorname{Im} \underset{\sim}{X}_k$ for all $k \in \{0, \dots, N+1\}$ by Proposition 4.4 from Section 4, and hence Theorem 4.2 can be applied. At the same time, by a direct computation using orthonormality of $z^{(1)}, z^{(2)}, \dots, z^{(m)}$ (see also Section 4), we have

$$\mathcal{F}_\lambda(\tilde{z}) := \mathcal{F}(\tilde{z}) - \lambda \langle \tilde{z}, \tilde{z} \rangle = \sum_{\mu=1}^{m} (\lambda_\mu - \lambda)\beta_\mu^2.$$

Now by Lemma 2.1 and the first condition in (1.4), we have

$$\sum_{\mu=1}^{m} (\lambda_\mu - \lambda)\beta_\mu^2 \geq \sum_{k=0}^{N} \tilde{s}_k^T P_k \tilde{s}_k \geq 0$$

since $\lambda > \lambda_\mu$ for all $1 \leq \mu \leq m$, this is possible only if $\beta_\mu = 0$ for all $1 \leq \mu \leq m$. This means that the system of $\tilde{p}$ linear homogeneous conditions (2.3) has only the trivial solution, and hence the number of conditions $\tilde{p}$ is greater than or equal to the number of parameters β_μ which is m. This proves the statement when λ is not a finite eigenvalue of (E). Since the functions $n_1(\lambda)$ and $p(\lambda)$ are continuous from the right by (4.8), by letting $\lambda \to \lambda_m^+$ we obtain the statement also in the case when $\lambda = \lambda_m$ is a finite eigenvalue of (E). $\square$

3. Sturmian Separation Results

Combining Theorems 1.2 and 1.3, we obtain the following statement. This result extends [11, Theorem 1], where the statement is proved in the special case $m = 0$ (see also [7, Theorem 1]).

Theorem 3.1. *If the principal solution of* (S) *has* m *focal points in* $(0, N+1]$*, then any conjoined basis of* (S) *has at least* m *and at most* $m+n$ *focal points in* $(0, N+1]$*.*

Proof. We apply Theorems 1.2 and 1.3 with $\underset{\sim}{\mathcal{S}} = \mathcal{S}$ and note that the assumptions in both theorems are satisfied since $\underset{\sim}{\mathcal{G}} = \mathcal{G}$ and $\operatorname{Im}(0 \quad \mathcal{B}) \subset \operatorname{Im} \mathcal{B}$ imply that (1.4) holds. $\square$

Now we prove Theorem 1.1 as stated in the introduction of this paper.

Proof of Theorem 1.1. Suppose that Z and $\tilde{Z}$ are two conjoined bases of (S) with p and $\tilde{p}$ focal points in $(0, N+1]$, respectively. Theorem 3.1 yields that

$$m \leq p \leq m+n \quad \text{ and } \quad m \leq \tilde{p} \leq m+n,$$

where m is the number of focal points of the principal solution in $(0, N+1]$. This yields the assertion. $\square$

Next we show that using the construction introduced in [9, page 1256] we can derive a more precise estimate for the difference of the numbers of focal points of the principal solution of (S) and of any other conjoined basis of (S). This statement can be regarded as a discrete version of [22, Problems VII.7, problem 2, page 367].

Theorem 3.2. *Let* $(\tilde{X}, \tilde{U})$ *be the principal solution of* (S) *and let* (X, U) *be any conjoined basis of this system. Let* m *and* p *denote the number of focal points of* $(\tilde{X}, \tilde{U})$ *and of* (X, U) *in* $(0, N+1]$*, respectively. Then*

$$m \leq p \leq m + \operatorname{rank} X_0.$$

Proof. We extend the eigenvalue problem (E) to a problem of the same kind on the interval $[-1, N+1]$. Define

$$W_{-1} := I \quad \text{and} \quad \mathcal{S}_{-1} := \begin{pmatrix} \mathcal{A}_{-1} & \mathcal{B}_{-1} \\ \mathcal{C}_{-1} & \mathcal{D}_{-1} \end{pmatrix} := \begin{pmatrix} U_0 K & X_0 \\ -X_0 K & U_0 \end{pmatrix}, \tag{3.1}$$

where $K := (X_0^T X_0 + U_0^T U_0)^{-1}$. Then by a direct computation we see that the matrix S_{-1} is symplectic and hence

$$\begin{cases} x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \quad u_{k+1} = \mathcal{C}_k x_k + \mathcal{D}_k u_k - \lambda x_{k+1}, \quad k \in \{-1, 0, 1, \ldots, N\} \\ x_{-1} = x_{N+1} = 0 \end{cases} \tag{3.2}$$

is an eigenvalue problem for a symplectic system. Now we extend also $(\tilde{X}, \tilde{U})$ and (X, U) to $[-1, N+1]$ by

$$\begin{pmatrix} \tilde{X}_{-1} \\ \tilde{U}_{-1} \end{pmatrix} := S_{-1}^{-1} \begin{pmatrix} \tilde{X}_0 \\ \tilde{U}_0 \end{pmatrix} = \begin{pmatrix} U_0^T & -X_0^T \\ K X_0^T & K U_0^T \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} = \begin{pmatrix} -X_0^T \\ K U_0^T \end{pmatrix}$$

and

$$\begin{pmatrix} X_{-1} \\ U_{-1} \end{pmatrix} := S_{-1}^{-1} \begin{pmatrix} X_0 \\ U_0 \end{pmatrix} = \begin{pmatrix} U_0^T & -X_0^T \\ K X_0^T & K U_0^T \end{pmatrix} \begin{pmatrix} X_0 \\ U_0 \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix}.$$

Hence

$$\tilde{M}_{-1} = (I - \tilde{X}_0 \tilde{X}_0^\dagger) \mathcal{B}_{-1} = X_0, \;\; \tilde{T}_{-1} = I - X_0^\dagger X_0, \;\; \tilde{P}_{-1} = \tilde{T}_{-1}^T \tilde{X}_{-1} \tilde{X}_0^\dagger \mathcal{B}_{-1} \tilde{T}_{-1} = 0,$$

which means that the extended $(\tilde{X}, \tilde{U})$ has $\operatorname{rank} X_0$ focal points in $(-1, 0]$ and, since $X_{-1} = 0$, the extended conjoined basis (X, U) has no additional focal point in $(-1, 0]$. Denote by $\hat{m}$ and $\hat{p}$ the number of focal points of the extended $(\tilde{X}, \tilde{U})$ and (X, U) in $(-1, N+1]$, respectively. Then $\hat{m} = m + \operatorname{rank} X_0$,

$\hat{p} = p$, and by Theorem 1.3 and $\underset{\sim}{\mathcal{S}} = \mathcal{S}$ applied once to (S) on $[0, N+1]$ and once on $[-1, N+1]$ (with S_{-1} defined by (3.1)), we obtain

$$m \le p = \hat{p} \le \hat{m} = m + \operatorname{rank} X_0,$$

which completes the proof. □

4. Spectral Theory

We consider the (slightly more general than in Section 1) symplectic eigenvalue problem associated with (S)

$$(4.1) \qquad \begin{cases} x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \quad u_{k+1} = \mathcal{C}_k x_k + \mathcal{D}_k u_k - \lambda \mathcal{W}_k x_{k+1}, \quad k \in \{0, \ldots, N\} \\ x_0 = x_{N+1} = 0, \end{cases}$$

where we assume that

$$\mathcal{W}_k \ge 0.$$

The following definition is as in [12, Definitions 2 and 3, Proposition 2 (v)].

Definition 4.1. A number $\lambda \in \mathbb{R}$ is called a *finite eigenvalue* of (4.1) provided it possesses a corresponding *finite eigenvector* $z = (x, u)$, i.e., z solves (4.1) such that $\{\mathcal{W}_k x_{k+1}\}_{k=1}^{N-1} \not\equiv 0$, and then

$$\dim \left\{ \{\mathcal{W}_k x_{k+1}\}_{k=0}^{N-1} : \ z = (x, u) \text{ solves } (4.1) \right\}$$

is called its *multiplicity*.

By [12, Proposition 2], finite eigenvectors corresponding to different finite eigenvalues are orthogonal with respect to the bilinear form

$$\langle z, \tilde{z} \rangle_{\mathcal{W}} := \langle z, \tilde{z} \rangle := \sum_{k=0}^{N} x_{k+1}^T \mathcal{W}_k \tilde{x}_{k+1} \quad \text{for} \quad z = (x, u) \text{ and } \tilde{z} = (\tilde{x}, \tilde{u}).$$

Moreover, we consider the discrete bilinear form $\mathcal{F}_\lambda$ associated with (4.1)

$$(4.2) \qquad \mathcal{F}_\lambda(z, \tilde{z}) = \sum_{k=0}^{N} \left\{ x_k^T \mathcal{A}_k^T \mathcal{C}_k \tilde{x}_k + x_k^T \mathcal{C}_k^T \mathcal{B}_k \tilde{u}_k + u_k^T \mathcal{B}_k^T \mathcal{C}_k \tilde{x}_k + u_k^T \mathcal{B}_k^T \mathcal{D}_k \tilde{u}_k \right\} - \lambda \langle z, \tilde{z} \rangle_{\mathcal{W}}$$

for admissible $z = (x, u)$ and $\tilde{z} = (\tilde{x}, \tilde{u})$, and the quadratic functional $\mathcal{F}_\lambda(z) = \mathcal{F}_\lambda(z, z)$, so that $\mathcal{F}_0(z) = \mathcal{F}(z)$ with the notation of Section 2 above.

4.1. **Picone's Identity.** We extend the generalized Picone identity from [10, Proposition 2.1] as follows. For the continuous version of this identity, see [17, Theorem 2.2.3].

Theorem 4.2 (Extended Picone Identity). *Suppose that $Z = (X, U)$ is a conjoined basis of the symplectic system of (4.1) for a fixed $\lambda \in \mathbb{R}$, let Q be symmetric with $QX = UX^{\dagger}X$, and define M, T, and P by (1.1). Let $\lambda_1, \ldots, \lambda_m$ be finite eigenvalues with corresponding orthonormal finite eigenfunctions $z^{(\mu)} = (x^{(\mu)}, u^{(\mu)})$, $1 \le \mu \le m$, with respect to $\langle \cdot, \cdot \rangle_{\mathcal{W}}$, and let $\beta_1, \ldots, \beta_m \in \mathbb{R}$ and put $\hat{z} := \sum_{\mu=1}^{m} \beta_\mu z^{(\mu)}$. Finally, suppose that $z = (x, u)$ is admissible, put $\tilde{z} := z + \hat{z}$, $\tilde{s} := \tilde{u} - Q\tilde{x}$, and assume that*

$$z \perp z^{(\mu)}, \quad \text{i.e.,} \quad \left\langle z, z^{(\mu)} \right\rangle_{\mathcal{W}} = 0 \text{ for } 1 \le \mu \le m \tag{4.3}$$

and that

$$\tilde{x}_k \in \operatorname{Im} X_k \quad \text{for all} \quad k \in \{0, \ldots, N+1\}. \tag{4.4}$$

Then we have that

$$\mathcal{F}_\lambda(z) - x_k^T u_k\Big|_{k=0}^{N+1} = \sum_{k=0}^{N} \tilde{s}_k^T P_k \tilde{s}_k + \sum_{\mu=1}^{m} (\lambda - \lambda_\mu)|\beta_\mu|^2 + \tilde{x}_k^T Q_k \tilde{x}_k\Big|_{k=0}^{N+1} - \tilde{x}_k^T \tilde{u}_k\Big|_{k=0}^{N+1}. \tag{4.5}$$

Proof. First, we obtain from [10, Proposition 2.1 (iv)] (observe that assumption (1.7) is not needed there) that

$$\begin{aligned} \mathcal{F}_\lambda(\tilde{z}) &= \sum_{k=0}^{N} \left\{ \tilde{x}_{k+1}^T \tilde{Q}_k \tilde{x}_{k+1} - \tilde{x}_k^T \tilde{Q}_k \tilde{x}_k + \tilde{s}_k^T P_k \tilde{s}_k \right\} \\ &= \tilde{x}_k^T Q_k \tilde{x}_k\Big|_{k=0}^{N+1} + \sum_{k=0}^{N} \tilde{s}_k^T P_k \tilde{s}_k \end{aligned}$$

because $\tilde{z}$ is admissible and (4.4) holds. Next, from [8, page 711] or [11, Lemma 1], from the recursion of (4.1) for $\lambda = \lambda_\mu$, $1 \le \mu \le m$, and from orthonormality we conclude that

$$\begin{aligned} \mathcal{F}_0(\hat{z}) &= \hat{x}_k^T \hat{u}_k\Big|_{k=0}^{N+1} + \sum_{k=0}^{N} \hat{x}_{k+1}^T \left\{ \mathcal{C}_k \hat{x}_k + \mathcal{D}_k \hat{u}_k - \hat{u}_{k+1} \right\} \\ &= \hat{x}_k^T \hat{u}_k\Big|_{k=0}^{N+1} + \sum_{\mu=1}^{m} \sum_{k=0}^{N} \hat{x}_{k+1}^T \beta_\mu \lambda_\mu x_{k+1}^{(\mu)} \\ &= \hat{x}_k^T \hat{u}_k\Big|_{k=0}^{N+1} + \sum_{\mu=1}^{m} \lambda_\mu |\beta_\mu|^2, \end{aligned}$$

and using (4.3),

$$\mathcal{F}_0(z, \hat{z}) = x_k^T \hat{u}_k\Big|_{k=0}^{N+1} + \sum_{\mu=1}^{m} \sum_{k=0}^{N} x_{k+1}^T \beta_\mu \lambda_\mu x_{k+1}^{(\mu)} = x_k^T \hat{u}_k\Big|_{k=0}^{N+1}$$

and

$$\mathcal{F}_0(\hat{z}, z) = \hat{x}_k^T u_k \big|_{k=0}^{N+1}.$$

Altogether, we can conclude that

$$\begin{aligned}
\mathcal{F}_\lambda(z) - x_k^T u_k \big|_{k=0}^{N+1} &= \mathcal{F}_0(z) - \lambda \langle z, z \rangle_{\mathcal{W}} - x_k^T u_k \big|_{k=0}^{N+1} \\
&= \mathcal{F}_0(\tilde{z}) - \mathcal{F}_0(\hat{z}) - \mathcal{F}_0(z, \hat{z}) - \mathcal{F}_0(\hat{z}, z) - \lambda \langle z, z \rangle_{\mathcal{W}} - x_k^T u_k \big|_{k=0}^{N+1} \\
&= \mathcal{F}_\lambda(\tilde{z}) + \lambda \langle z + \hat{z}, z + \hat{z} \rangle_{\mathcal{W}} - \mathcal{F}_0(\hat{z}) - (x_k^T \hat{u}_k + \hat{x}_k^T u_k + x_k^T u_k) \big|_{k=0}^{N+1} \\
&= \sum_{k=0}^{N} \tilde{s}_k^T P_k \tilde{s}_k + \tilde{x}_k^T Q_k \tilde{x}_k \big|_{k=0}^{N+1} - \tilde{x}_k^T \tilde{u}_k \big|_{k=0}^{N+1} - \sum_{\mu=1}^{m} \lambda_\mu |\beta_\mu|^2 + \lambda \langle \hat{z}, \hat{z} \rangle_{\mathcal{W}}
\end{aligned}$$

using that $z \perp \hat{z}$ by (4.3), and orthonormality yields our assertion (4.5). $\square$

Note first that the existence of a matrix Q with the requirements as in Theorem 4.2 is established in [8]. Next we analyze the crucial assumption (4.4) using notation (1.1).

Lemma 4.3. *Suppose that $Z = (X, U)$ is a conjoined basis of the symplectic system* (S) *and let $0 \le k \le N$. Then we have*

(i) $x_{k+1} \in \operatorname{Im} X_{k+1}$ *implies that* $M_k^T x_{k+1} = 0$*;*
(ii) $x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k$, $M_k^T x_{k+1} = 0$, *and* $x_k \in \operatorname{Im} X_k$ *imply that* $x_{k+1} \in \operatorname{Im} X_{k+1}$.

Proof. First, $x_{k+1} = X_{k+1} c \in \operatorname{Im} X_{k+1}$ implies by (1.1) that

$$M_k^T x_{k+1} = \mathcal{B}_k^T (I - X_{k+1} X_{k+1}^\dagger) X_{k+1} c = 0.$$

Hence, (i) is true. Next, $x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k$, $x_k = X_k c \in \operatorname{Im} X_k$, and $M_k^T x_{k+1} = 0$ imply that

$$\begin{aligned}
0 &= \mathcal{B}_k^T (I - X_{k+1} X_{k+1}^\dagger)(\mathcal{A}_k X_k c + \mathcal{B}_k u_k) \\
&= \mathcal{B}_k^T (I - X_{k+1} X_{k+1}^\dagger)(X_{k+1} c + \mathcal{B}_k (u_k - U_k c)) \\
&= \mathcal{B}_k^T (I - X_{k+1} X_{k+1}^\dagger) \mathcal{B}_k (u_k - U_k c) \\
&= \mathcal{B}_k^T (I - X_{k+1} X_{k+1}^\dagger)^T (I - X_{k+1} X_{k+1}^\dagger) \mathcal{B}_k (u_k - U_k c)
\end{aligned}$$

so that

$$0 = (I - X_{k+1} X_{k+1}^\dagger) \mathcal{B}_k (u_k - U_k c)$$

and therefore

$$x_{k+1} = X_{k+1} c + \mathcal{B}_k (u_k - U_k c) = X_{k+1} c + X_{k+1} X_{k+1}^\dagger \mathcal{B}_k (u_k - U_k c) \in \operatorname{Im} X_{k+1}$$

holds. Hence, (ii) is true. $\square$

Proposition 4.4. *Suppose that $Z = (X, U)$ is a conjoined basis of the symplectic system* (S) *and let $z = (x, u)$ be admissible with $x_0 = x_{N+1} = 0$. Then* (4.4) *holds for $\tilde{x} = x$ if and only if*

$$M_k^T x_{k+1} = 0 \quad \textit{for all} \quad 0 \le k \le N-1. \tag{4.6}$$

Proof. First, (4.4) implies (4.6) by Lemma 4.3 (i). Next, suppose that $z = (x, u)$ is admissible, $x_0 = x_{N+1} = 0$, and (4.6) holds. Then $x_0 = 0 \in \operatorname{Im} X_0$, and inductively $x_{k+1} \in \operatorname{Im} X_{k+1}$ for all $0 \le k \le N-1$. Moreover, $x_{N+1} = 0 \in \operatorname{Im} X_{N+1}$, so that (4.4) holds for $\tilde{x} = x$. □

4.2. **Rayleigh's Principle.** In this subsection we put $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathcal{W}}$ and assume throughout that

$$\mathcal{W}_k = I \quad \text{for all} \quad 0 \le k \le N \tag{4.7}$$

(as in Sections 2 and 3) and that $Z(\lambda) = (X(\lambda), U(\lambda))$ is the principal solution of the symplectic system in (E), i.e., that

$$X_0 = X_0(\lambda) \equiv 0 \quad \text{and} \quad U_0 = U_0(\lambda) \equiv I.$$

We need the following lemma.

Lemma 4.5. *There exists $\lambda_0 \in \mathbb{R}$ such that $\lambda \le \lambda_0$ implies $\mathcal{F}_\lambda(z) > 0$ for all admissible $z = (x, u)$ with $x_0 = x_{N+1} = 0$ and $x = \{x_k\}_{k=1}^N \not\equiv 0$.*

Proof. Let z be admissible with $x_0 = x_{N+1} = 0$. Then, by (4.2) and (4.7),

$$\begin{aligned} \mathcal{F}_\lambda(z) &= \sum_{k=0}^{N} \Big\{ x_k^T \mathcal{A}_k^T \mathcal{C}_k x_k + 2 x_k^T \mathcal{C}_k^T (x_{k+1} - \mathcal{A}_k x_k) \\ &\qquad + (x_{k+1} - \mathcal{A}_k x_k)^T \mathcal{B}_k \mathcal{B}_k^\dagger \mathcal{D}_k \mathcal{B}_k^\dagger (x_{k+1} - \mathcal{A}_k x_k) \; - \lambda x_{k+1}^T x_{k+1} \Big\} \\ &= \sum_{k=1}^{N} \left\{ x_k^T \mathcal{F}_k x_k - \lambda |x_k|^2 \right\} \end{aligned}$$

for certain matrices $\mathcal{F}_k$. This yields the assertion if λ_0 is sufficiently small. □

Now, using Lemma 4.5, we obtain from [12, Theorem 2]

$$n_1(\lambda) = n_2(\lambda), \; n_1(\lambda+) = n_1(\lambda), \; n_2(\lambda+) = n_2(\lambda) \quad \text{for all} \quad \lambda \in \mathbb{R}, \tag{4.8}$$

where, including multiplicities

$n_1(\lambda)$ denotes the number of focal points of $Z = Z(\lambda)$ in the interval $(0, N+1]$;
$n_2(\lambda)$ denotes the number of finite eigenvalues of (E) which are less than or equal to λ.

We let (see [12, Proposition 2])

$$\lambda_1 \leq \ldots \leq \lambda_r \quad \text{denote the finite eigenvalues of (E)}$$

including multiplicities with corresponding orthonormal finite eigenfunctions $z^{(\mu)}$, $1 \leq \mu \leq r$, and we put $\lambda_0 := -\infty$ and $\lambda_{r+1} := \infty$. Then, by [12, Proposition 2], $r \leq nN < \infty$.

Theorem 4.6 (Rayleigh Principle)**.** *With the above notation and assumptions we have for $0 \leq m \leq r$ that*

$$\lambda_{m+1} = \min \left\{ \tfrac{\mathcal{F}_0(z)}{\langle z,z \rangle} : \quad z = (x,u) \textit{ is admissible with } x_0 = x_{N+1} = 0, \right.$$
$$\left. z \perp z^{(\mu)} \textit{ for all } 1 \leq \mu \leq m, \textit{ and } x = \{x_k\}_{k=1}^N \not\equiv 0 \right\}.$$

Note that we include the cases $m = 0$, where the orthogonality condition on z becomes empty, and $m = r$, where $\lambda_{m+1} = \infty$.

Proof. Let $0 \leq m \leq r$ and $\lambda \in (\lambda_m, \lambda_{m+1})$. Then, by (4.8),

$$m = n_1(\lambda) = n_2(\lambda),$$

so the principal solution possesses exactly m focal points in the open interval $(0, N+1)$, because $N+1$ is not a focal point. The fact that $N+1$ is not a focal point follows from Lemma 4.5, which implies that $\operatorname{rank} M_k(\tilde{\lambda}) = 0$ for all $\tilde{\lambda} \leq \lambda_0$ so that by [9, Remark 3 (ii)]

$$\operatorname{rank} M_k(\lambda+) = \operatorname{rank} M_k(\lambda) = 0 \quad \text{for all} \quad 0 \leq k \leq N, \ \lambda \in (\lambda_m, \lambda_{m+1}),$$

in particular $\operatorname{rank} M_N(\lambda) = 0$.

First, we apply the extended Picone identity, Theorem 4.2, to $z = 0$ so that

$$\tilde{z} = \hat{z} = \sum_{\mu=1}^{m} \beta_\mu z^{(\mu)} \quad \text{with} \quad \tilde{x}_0 = \tilde{x}_{N+1} = x_0 = x_{N+1} = 0.$$

We use an argument similar to that applied in the proof of Theorem 1.3. Suppose that $\beta_1, \ldots, \beta_m$ satisfy the m linear and homogeneous equations

$$(4.9) \qquad \begin{cases} M_k^T \tilde{x}_{k+1} = 0, \quad k \in \{0, \ldots, N-1\}, \\ \tilde{s}_k \perp \{\alpha \in \mathbb{R}^n : \quad \alpha \text{ is an eigenvector corresponding} \\ \qquad\qquad\qquad\qquad \text{to a negative eigenvalue of } P_k\}, \quad k \in \{0, \ldots, N\}, \end{cases}$$

where

$$\tilde{s}_k = \tilde{u}_k - Q_k \tilde{x}_k.$$

Note that the number of these equations is just the number of focal points in $(0, N+1)$ by definition. Then, by Proposition 4.4, the assumption (4.4) holds, and we obtain from Theorem 4.2 that

$$0 = \mathcal{F}_\lambda(z) - x_k^T u_k\big|_{k=0}^{N+1} = \sum_{k=0}^{N} \tilde{s}_k^T P_k \tilde{s}_k + \sum_{\mu=1}^{m} (\lambda - \lambda_\mu)|\beta_\mu|^2,$$

where $\tilde{s}_k^T P_k \tilde{s}_k \geq 0$ for $0 \leq k \leq N$ by (4.9), and $\lambda - \lambda_\mu \geq \lambda - \lambda_m > 0$ for $1 \leq \mu \leq m$. Hence $\beta_1 = \cdots = \beta_m = 0$ so that (4.9) possesses only the trivial solution. Thus we have shown that

(4.10) the coefficient matrix corresponding to (4.9) is nonsingular.

Now suppose that z is admissible with $x_0 = x_{N+1} = 0$ and $z \perp z^{(\mu)}$ for $1 \leq \mu \leq m$. Then we apply Theorem 4.2 to $\tilde{z} = z + \hat{z} = z + \sum_{\mu=1}^{m} \beta_\mu z^{(\mu)}$, where we choose $\beta_1, \ldots, \beta_m$ such that (4.9) holds. This is possible because the m linear and inhomogeneous equations possess a unique solution by (4.10). Theorem 4.2 implies that

$$\begin{aligned} \mathcal{F}_\lambda(z) &= \mathcal{F}_\lambda(z) - x_k^T u_k\big|_{k=0}^{N+1} \\ &= \sum_{k=0}^{N} \tilde{s}_k^T P_k \tilde{s}_k + \sum_{\mu=1}^{m} (\lambda - \lambda_\mu)|\beta_\mu|^2 \\ &\geq \sum_{\mu=1}^{m} (\lambda - \lambda_\mu)|\beta_\mu|^2 \;\geq\; 0 \end{aligned}$$

so that

$$\mathcal{F}_0(z) \geq \lambda \langle z, z \rangle \qquad \text{for all} \qquad \lambda \in (\lambda_m, \lambda_{m+1}).$$

Hence $\mathcal{F}_0(z) \geq \lambda_{m+1} \langle z, z \rangle$, and $\mathcal{F}_0(z) = \lambda_{m+1} \langle z, z \rangle$ for $z = z^{(m+1)}$. For multiple finite eigenvalues use the fact that $\mathcal{F}_\lambda(z) = \mathcal{F}_\lambda(z + \hat{z})$ if z is admissible with $x_0 = x_{N+1} = 0$ and if $\hat{z}$ is a finite eigenvector corresponding to λ. Hence the assertion follows. ☐

Rayleigh's principle yields the following result.

Theorem 4.7 (Expansion Theorem). *Suppose that $z = (x, u)$ is admissible with $x_0 = x_{N+1} = 0$. Then*

$$x = \sum_{\mu=1}^{r} c_\mu x^{(\mu)}, \qquad \text{where} \qquad c_\mu = \left\langle z^{(\mu)}, z \right\rangle. \tag{4.11}$$

Proof. First, suppose that z is admissible with $x_0 = x_{N+1} = 0$ and $z \perp z^{(\mu)}$ for all $1 \leq \mu \leq r$. Then, by Theorem 4.6 with $m = r$, $\mathcal{F}_0(z) \geq \lambda \langle z, z \rangle$ for all $\lambda \in \mathbb{R}$. Hence

$$\langle z, z \rangle = \sum_{k=0}^{N} x_{k+1}^T x_{k+1} = 0 \qquad \text{so that} \qquad x = 0.$$

Now if z is admissible with $x_0 = x_{N+1} = 0$, then $z - \sum_{\mu=1}^{r} \left\langle z^{(\mu)}, z \right\rangle z^{(\mu)} \perp z^{(m)}$ for all $1 \leq m \leq r$, and (4.11) follows from what we have shown before. ☐

In our final result we prove that "extremal vectors of the functional $\mathcal{F}_0$ satisfy necessarily the corresponding Euler equations". The meaning of this becomes clear from the formulation of the following theorem.

Theorem 4.8. *Let be given any eigenvalue problem* (E) *with* (4.7) *and with corresponding functional* $\mathcal{F}_\lambda$*, and let* $m \in \{0, \ldots, r\}$*. Suppose that*

$$\mathcal{F}_0(\hat{z}) = \lambda_{m+1} \langle \hat{z}, \hat{z} \rangle \qquad \text{for some admissible } \hat{z} \text{ with } \hat{x}_0 = \hat{x}_{N+1} = 0 \quad \text{and } \hat{z} \perp z^{(\mu)} \text{ for all } 1 \le \mu \le m. \tag{4.12}$$

Then $\hat{z}$ *satisfies the* Euler equation*, i.e.,*

$$\begin{cases} \tilde{u}_{k+1} = \mathcal{C}_k \hat{x}_k + \mathcal{D}_k \tilde{u}_k - \lambda_{m+1} \hat{x}_{k+1} \text{ for suitable vectors } \tilde{u}_k \text{ with} \\ \mathcal{B}_k \tilde{u}_k = \mathcal{B}_k \hat{u}_k \text{ for } 0 \le k \le N. \end{cases} \tag{4.13}$$

Proof. Let $\hat{z}$ be admissible with $\hat{x}_0 = \hat{x}_{N+1} = 0$, $\hat{z} \perp z^{(\mu)}$ for all $1 \le \mu \le m$, and $\mathcal{F}_0(\hat{z}) = \lambda_{m+1} \langle \hat{z}, \hat{z} \rangle$. Then, by the expansion theorem, Theorem 4.7, $\hat{x} = \sum_{\mu=1}^r c_\mu x^{(\mu)}$, $c_\mu = \langle z^{(\mu)}, \hat{z} \rangle$, where $c_\mu = 0$ for $1 \le \mu \le m$, by using the notation above. It follows that (use e.g., Theorem 4.2)

$$\lambda_{m+1} \langle \hat{z}, \hat{z} \rangle = \mathcal{F}_0(\hat{z}) = \sum_{\mu=1}^r \lambda_\mu |c_\mu|^2 \ge \lambda_{m+1} \sum_{\mu=m+1}^r |c_\mu|^2 = \lambda_{m+1} \langle \hat{z}, \hat{z} \rangle$$

so that $c_\mu = 0$ for all $\mu \ge \ell + 1$, where $\lambda_{m+1} = \ldots = \lambda_\ell < \lambda_{\ell+1}$. Hence $\tilde{z} := \sum_{\mu=m+1}^{\ell} c_\mu z^{(\mu)}$ is an eigenfunction corresponding to λ_{m+1} with $\tilde{x} = \hat{x}$. Then (4.13) holds because

$$\tilde{u}_{k+1} = \mathcal{C}_k \tilde{x}_k + \mathcal{D}_k \tilde{u}_k - \lambda_{m+1} \tilde{x}_{k+1} = \mathcal{C}_k \hat{x}_k + \mathcal{D}_k \tilde{u}_k - \lambda_{m+1} \hat{x}_{k+1}$$

and

$$\mathcal{B}_k(\tilde{u}_k - \hat{u}_k) = (\tilde{x}_{k+1} - \hat{x}_{k+1}) - \mathcal{A}_k(\tilde{x}_k - \hat{x}_k) = 0.$$

The proof is complete. □

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[1]University of Missouri–Rolla, Department of Mathematics and Statistics and Department of Economics and Finance, Rolla, MO 65401, USA, [2]Masaryk University, Department of Mathematics, CZ-66295 Brno, Czech Republic, [3]Universität Ulm, Institut Angewandte Analysis, D-89069 Ulm, Germany