SOME DISCRETE FRACTIONAL INEQUALITIES OF CHEBYSHEV TYPE

MARTIN BOHNER*
Missouri S&T, Department of Mathematics and Statistics,
Rolla, MO 65409-0020, USA

RUI A. C. FERREIRA†
Lusophone University of Humanities and Technologies, Department of Mathematics,
1749-024 Lisbon, Portugal

Abstract
Using the discrete fractional sum operator, we establish some inequalities of Chebyshev type.

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1 Introduction

In 1882, Chebyshev proved the following result [3]:

Let $f$ and $g$ be two integrable functions in $[0, 1]$. If both functions are simultaneously increasing or decreasing for the same values of $x$ in $[0, 1]$, then

$$\int_0^1 f(x)g(x)dx \geq \int_0^1 f(x)dx \int_0^1 g(x)dx.$$

If one function is increasing and the other decreasing for the same values of $x$ in $[0, 1]$, then

$$\int_0^1 f(x)g(x)dx \leq \int_0^1 f(x)dx \int_0^1 g(x)dx.$$

Since then, continuous and discrete generalizations and extensions of such inequalities have appeared in the literature (see [2, 8] and references therein). In 2009, Belarbi and Dahmani [1] proved that

$$(f^\alpha g)(t) \geq \frac{\Gamma(\alpha+1)}{t^\alpha} (f^\alpha f)(t)(f^\alpha g)(t), \quad t > 0, \quad \alpha > 0, \quad (1.1)$$

*E-mail address: bohner@mst.edu
†E-mail address: ruiacferreira@ulusofona.pt
where $I^\alpha$ is the Riemann–Liouville fractional integral operator of order $\alpha \geq 0$ [6], and $f$ and $g$ are two synchronous functions (cf. Definition 2.5 below). Moreover, much more recently, a $q$-analogue of inequality (1.1) has appeared in the literature [7].

It is our aim with this paper to establish a discrete version of inequality (1.1) as well as some other related results. We will do this by using the discrete fractional sum operator defined by Miller and Ross [5] in 1989.

This paper is organized as follows: in Section 2 we provide the reader fundamental concepts and results needed throughout the paper. In Section 3 we state and prove our main achievements.

## 2 Preliminaries on Discrete Fractional Calculus

In this section we introduce the reader to basic concepts and results about discrete fractional calculus.

The power function is defined by

$$x^{(y)} = \frac{\Gamma(x+1)}{\Gamma(x+1-y)}, \text{ for } x, x-y \in \mathbb{R}\backslash (\mathbb{Z}\backslash \mathbb{N}_0).$$

**Remark 2.1.** Using the properties of the Gamma function, it is easily seen that for $x \geq y \geq 0$, we get $x^{(y)} \geq 0$.

For $a \in \mathbb{R}$ and $0 < \alpha \leq 1$, we define the set $\mathbb{N}_a^0 = \{a + \alpha, a + \alpha + 1, a + \alpha + 2, \ldots\}$. Also, we use the notation $\sigma(s) = s + 1$ for the shift operator and $(\Delta f)(t) = f(t+1)-f(t)$ for the forward difference operator.

For a function $f : \mathbb{N}_a^0 \rightarrow \mathbb{R}$, the discrete fractional sum of order $\alpha \geq 0$ is defined as

$$(a\Delta^0 f)(t) = f(t), \quad t \in \mathbb{N}_a^0,$$

$$(a\Delta^{-\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-\sigma(s))^{(\alpha-1)} f(s), \quad t \in \mathbb{N}_a^0, \quad \alpha > 0.$$

**Remark 2.2.** Note that the operator $a\Delta^{-\alpha}$ with $\alpha > 0$ maps functions defined on $\mathbb{N}_a^0$ to functions defined on $\mathbb{N}_a^\alpha$. Also observe that if $\alpha = 1$, we get the summation operator

$$(a\Delta^{-1} f)(t) = \sum_{s=a}^{t-1} f(s).$$

The following result will be used in the sequel.

**Lemma 2.3** (See [4, Corollary 10]). If $a \in \mathbb{R}$ and $\mu, \mu + \nu \in \mathbb{R}\backslash \{\ldots, -2, -1\}$, then

$$\left(\Delta^{-\nu} (s-a+\mu)^{\mu}\right)(t) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} (t-a+\mu)^{\mu+\nu}, \quad t \in \mathbb{N}_a^\nu.$$

**Remark 2.4.** The function $t \rightarrow (t-a)^{(\alpha)}$ defined on $\mathbb{N}_a^\alpha$, $a \in \mathbb{R}$ and $\alpha \geq 0$, is increasing. Indeed, we have that $\Delta(t-a)^{(\alpha)} = \alpha(t-a)^{(\alpha-1)}$ and $(t-a)^{(\alpha-1)} \geq 0$.

**Definition 2.5.** Two functions $f$ and $g$ are called synchronous, respectively asynchronous, on $\mathbb{N}_a^0$ if for all $\tau, s \in \mathbb{N}_a^0$, we have $(f(\tau) - f(s))(g(\tau) - g(s)) \geq 0$, respectively $(f(\tau) - f(s))(g(\tau) - g(s)) \leq 0$. 


3 Discrete Fractional Inequalities

We start by proving the main result of this paper.

**Theorem 3.1.** If $\alpha > 0$ and $f, g$ are two synchronous functions on $\mathbb{N}_a^\alpha$, then

$$\left(\Delta^{-\alpha}fg\right)(t) \geq \frac{\Gamma(\alpha+1)}{(t-a)^{\alpha}} \left(\Delta^{-\alpha}f\right)(t) \left(\Delta^{-\alpha}g\right)(t), \quad t \in \mathbb{N}_a^\alpha. \quad (3.1)$$

**Proof.** Since the functions $f$ and $g$ are synchronous on $\mathbb{N}_a^\alpha$, then for all $\tau, s \in \mathbb{N}_a^\alpha$, we have

$$f(\tau)g(\tau) + f(s)g(s) \geq f(\tau)g(s) + f(s)g(\tau). \quad (3.2)$$

Now, multiplying both sides of (3.2) by $\frac{(t-\sigma(\tau))^{(\alpha-1)}}{\Gamma(\alpha)}$, $t \in \mathbb{N}_a^\alpha$ and $\tau \in \{a, a+1, \ldots, t-\alpha\}$, we get

$$\frac{(t-\sigma(\tau))^{(\alpha-1)}}{\Gamma(\alpha)} f(\tau)g(\tau) + \frac{(t-\sigma(\tau))^{(\alpha-1)}}{\Gamma(\alpha)} f(s)g(s) \geq \frac{(t-\sigma(\tau))^{(\alpha-1)}}{\Gamma(\alpha)} f(\tau)g(s) + \frac{(t-\sigma(\tau))^{(\alpha-1)}}{\Gamma(\alpha)} f(s)g(\tau). \quad (3.3)$$

Now, summing both sides of (3.3) for $\tau \in \{a, a+1, \ldots, t-\alpha\}$, we obtain

$$\left(\Delta^{-\alpha}fg\right)(t) + f(s)g(s) \left(\Delta^{-\alpha}1\right)(t) \geq g(s) \left(\Delta^{-\alpha}f\right)(t) + f(s) \left(\Delta^{-\alpha}g\right)(t). \quad (3.4)$$

Multiplying both sides of (3.4) by $\frac{(t-\sigma(s))^{(\alpha-1)}}{\Gamma(\alpha)}$, $t \in \mathbb{N}_a^\alpha$ and $s \in \{a, a+1, \ldots, t-\alpha\}$, we obtain

$$\frac{(t-\sigma(s))^{(\alpha-1)}}{\Gamma(\alpha)} \left(\Delta^{-\alpha}fg\right)(t) + \frac{(t-\sigma(s))^{(\alpha-1)}}{\Gamma(\alpha)} f(s)g(s) \left(\Delta^{-\alpha}1\right)(t) \geq \frac{(t-\sigma(s))^{(\alpha-1)}}{\Gamma(\alpha)} g(s) \left(\Delta^{-\alpha}f\right)(t) + \frac{(t-\sigma(s))^{(\alpha-1)}}{\Gamma(\alpha)} f(s) \left(\Delta^{-\alpha}g\right)(t), \quad (3.5)$$

and again, summing both sides of (3.5) for $s \in \{a, a+1, \ldots, t-\alpha\}$, we get

$$\left(\Delta^{-\alpha}1\right)(t) \left(\Delta^{-\alpha}fg\right)(t) + \left(\Delta^{-\alpha}fg\right)(t) \left(\Delta^{-\alpha}1\right)(t) \geq \left(\Delta^{-\alpha}g\right)(t) \left(\Delta^{-\alpha}f\right)(t) + \left(\Delta^{-\alpha}f\right)(t) \left(\Delta^{-\alpha}g\right)(t),$$

i.e.,

$$\left(\Delta^{-\alpha}f\right)(t) \left(\Delta^{-\alpha}g\right)(t) \leq \left(\Delta^{-\alpha}1\right)(t) \left(\Delta^{-\alpha}fg\right)(t) = \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} \left(\Delta^{-\alpha}fg\right)(t),$$

where we have used Lemma 2.3. This shows (3.1). \qed
Remark 3.2. The inequality sign in (3.1) is reversed if the functions are asynchronous on $\mathbb{N}_a^0$.

Example 3.3. Let $\beta \geq 0$ and consider the functions $f_\beta$ defined by

$$f_\beta(t) = (t + \beta)^{(\beta)}, \quad t \in \mathbb{N}_a^0.$$ 

By Remark 2.4, it follows that $f_\beta$ and $f_\gamma$ are synchronous functions for $\beta, \gamma \geq 0$. Therefore, by Lemma 2.3 and Theorem 3.1, the inequality

$$\left(\alpha \Delta^{-\alpha} f_\beta f_\gamma\right)(t) \geq \frac{\Gamma(\alpha + 1)}{t^{(\alpha)}} \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + \alpha + 1)} (t + \gamma)^{(\gamma + \alpha)} \frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)} (t + \beta)^{(\beta + \alpha)}$$

holds for all $t \in \mathbb{N}_a^0$.

Theorem 3.4. If $\alpha, \beta > 0$ and $f, g$ are two synchronous functions on $\mathbb{N}_a^0$, then

$$\frac{(t - a)^{(\alpha)}}{\Gamma(\alpha + 1)} \left(\alpha \Delta^{-\alpha} f g\right)(t) + \frac{(t - a)^{(\beta)}}{\Gamma(\beta + 1)} \left(\alpha \Delta^{-\alpha} f g\right)(t) \geq \left(\alpha \Delta^{-\alpha} f\right)(t) \left(\alpha \Delta^{-\alpha} g\right)(t), \quad t \in \mathbb{N}_a^0. \quad (3.6)$$

Proof. Proceeding as in the proof of Theorem 3.1 and using inequality (3.4), we can write

$$\frac{(t - \sigma(s))^{(\beta - 1)}}{\Gamma(\beta)} \left(\alpha \Delta^{-\alpha} f g\right)(t) + \frac{(t - \sigma(s))^{(\beta - 1)}}{\Gamma(\beta)} f(s)g(s) \left(\alpha \Delta^{-\alpha} 1\right)(t) \geq \frac{(t - \sigma(s))^{(\beta - 1)}}{\Gamma(\beta)} g(s) \left(\alpha \Delta^{-\alpha} f\right)(t) + \frac{(t - \sigma(s))^{(\beta - 1)}}{\Gamma(\beta)} f(s) \left(\alpha \Delta^{-\alpha} g\right)(t). \quad (3.7)$$

Now, summing both sides of (3.7) for $s \in \{a, a + 1, \ldots, t - \beta\}$, we obtain the desired inequality (3.6).

Remark 3.5. If we let $\alpha = \beta$ in Theorem 3.4, we obtain Theorem 3.1.

We end this manuscript with a generalization of Theorem 3.1.

Theorem 3.6. Assume that $f_i, \ 1 \leq i \leq n$, are $n \in \mathbb{N}$ functions on $\mathbb{N}_a^0$ satisfying

$$\prod_{i=1}^{k-1} f_i \text{ and } f_k \text{ are synchronous for all } k \in \{2, \ldots, n\}, \quad (3.8)$$

$$f_i \geq 0 \text{ for } 3 \leq i \leq n. \quad (3.9)$$

Suppose that $\alpha > 0$. Then, for all $t \in \mathbb{N}_a^\alpha$, we have

$$\left(\alpha \Delta^{-\alpha} \prod_{i=1}^{n} f_i\right)(t) \geq \left(\frac{\Gamma(\alpha + 1)}{(t - a)^{(\alpha)}}\right)^{n-1} \prod_{i=1}^{n} \left(\alpha \Delta^{-\alpha} f_i\right)(t). \quad (3.10)$$
Proof. In view of (3.8) and (3.9), we have

\[
(a^\Delta-n\prod_{i=1}^{n} f_i)(t) \geq \Gamma(\alpha+1)\left(\frac{t-a}{\alpha}\right)^{\alpha}\left(a^\Delta_{-\alpha}\prod_{i=1}^{n-1} f_i\right)(t) (a^\Delta_{-\alpha}f_n)(t) \\
\geq \left(\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)}\right)^{\alpha}\left(\frac{t-a}{\alpha}\right)^{\alpha}\left(a^\Delta_{-\alpha}\prod_{i=1}^{n-2} f_i\right)(t) \prod_{i=1}^{n-1} (a^\Delta_{-\alpha}f_k)(t) \\
\vdots \\
\geq \Gamma(\alpha+1)\left(\frac{t-a}{\alpha}\right)^{\alpha}\prod_{i=1}^{n-1} (a^\Delta_{-\alpha}f_i)(t),
\]

where we repeatedly applied Theorem 3.1. \qed

Remark 3.7. If the functions \( f_i, 1 \leq i \leq n \), in Theorem 3.6 are either all nonnegative increasing or nonnegative decreasing, then both (3.8) and (3.9) are satisfied.

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