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Sturm-Liouville eigenvalue problems on time scales

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Abstract

For Sturm–Liouville eigenvalue problems on time scales with separated boundary conditions we give an oscillation theorem and establish Rayleigh's principle. Our results not only unifiy the corresponding theories for differential and difference equations, but are also new in the discrete case. © 1999 Published by Elsevier Science Inc. All rights reserved.

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1. Introduction

A time scale $\mathbf{T}$ is a closed subset of $\mathbb{R}$, and for a function $f : \mathbf{T} \to \mathbb{R}$ it is possible to define a derivative f^Δ in such a manner that

$$f^\Delta(t) = f'(t) \quad \text{for all } t \in \mathbb{R} \qquad \text{if } \mathbf{T} = \mathbb{R}$$

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and

$$f^\Delta(t) = \Delta f(t) = f(t+1) - f(t) \quad \text{for all } t \in \mathbb{Z} \text{ if } \mathbf{T} = \mathbb{Z}.$$

Moreover, jump operators $\sigma, \rho : \mathbf{T} \to \mathbf{T}$ can be introduced so that

$$\sigma(t) = \rho(t) = t \quad \text{for all } t \in \mathbb{R} \qquad \text{if } \mathbf{T} = \mathbb{R}$$

and

$$\sigma(t) = t + 1, \qquad \rho(t) = t - 1 \quad \text{for all } t \in \mathbb{Z} \text{ if } \mathbf{T} = \mathbb{Z}.$$

Further details of time scales will be provided in Section 2; but for reading this introduction it is sufficient to know the particular cases $\mathbb{R}$ and $\mathbb{Z}$ of $\mathbf{T}$.

In this paper we shall consider a Sturm–Liouville eigenvalue problem with separated boundary conditions on an arbitrary time scale $\mathbf{T}$ of the form

$$\text{(E)} \qquad L(y) + \lambda y^\sigma = 0, \quad \mathcal{R}_a(y) = \mathcal{R}_b(y) = 0,$$

where (we put $y^\sigma = y \circ \sigma$ and $y^{\Delta\Delta} = (y^\Delta)^\Delta$)

$$L(y) = y^{\Delta\Delta} + q y^\sigma$$

with $q : \mathbf{T} \to \mathbb{R}$ continuous, $a, b \in \mathbf{T}$ with $a < b$, and

$$\mathcal{R}_a(y) = \alpha y(\rho(a)) + \beta y^\Delta(\rho(a)), \quad \mathcal{R}_b(y) = \gamma y(b) + \delta y^\Delta(b)$$

with $(\alpha^2 + \beta^2)(\gamma^2 + \delta^2) \neq 0$. Eigenvalues and eigenfunctions of (E) are defined in the usual manner. Our main result is an *oscillation* theorem for the eigenvalue problem (E), and to state this theorem, we need the concept of *generalized zeros* of solutions of $L(y) + \lambda y^\sigma = 0$. For this, a solution y of $L(y) + \lambda y^\sigma = 0$ is said to have a *zero* at $t \in \mathbf{T}$ if $y(t) = 0$, and it has a *node* at $(t + \sigma(t))/2$ if $y(t)y(\sigma(t)) < 0$ (observe that there are no nodes in the case $\mathbf{T} = \mathbb{R}$). A generalized zero of y is then defined as its zero or its node. Our main result reads as follows:

Theorem 1 (Oscillation Theorem). *The eigenvalues of (E) may be arranged as* $-\infty < \lambda_1 < \lambda_2 < \lambda_3 < \ldots$, *and an eigenfunction corresponding to* λ_{k+1} *has exactly* k *generalized zeros in the open interval* (a, b).

By introducing a scalar product and the corresponding norm

$$\langle x, y \rangle = \int_{\rho(a)}^{b} x(t)y(t)\Delta t \quad \text{and} \quad \|x\| = \sqrt{\langle x, x \rangle}$$

on the set of so-called rd-continuous functions (the integral will be defined in the next section; however, for the moment we note that it is the "usual" integral from a to b if $\mathbf{T} = \mathbb{R}$ and the sum from $a - 1$ to $b - 1$ if $\mathbf{T} = \mathbb{Z}$), we shall also consider the *Rayleigh quotient*

$$R(x) = -\frac{\langle L(x), x^\sigma \rangle}{\|x^\sigma\|^2}, \quad x \neq 0.$$

Our main result concerning this Rayleigh quotient then reads as the following theorem.

Theorem 2 (Rayleigh's Principle). *We have*

$$\lambda_{k+1} = \min_{\substack{\mathscr{R}_a(x)=\mathscr{R}_b(x)=0 \\ \langle x^\sigma, y_\nu^\sigma\rangle = 0,\ 1\leqslant \nu\leqslant k}} R(x),$$

where y_ν are the normalized eigenfunctions corresponding to λ_ν, $1 \leqslant \nu \leqslant k$.

The plan of this paper is as follows: Section 2 gives an introduction of the concept of time scales and provides some preliminaries of the eigenvalue problem (E) such as a characterization of eigenvalues and Lagrange's identity. In Section 3 we state and prove the *extended Picone's identity* on time scales. This identity is the *essential* tool for proving most of the results in this paper. The proofs of Theorems 1 and 2 are contained in the final Section 4. Here, we shall also provide estimates for the Rayleigh quotient in terms of eigenvalues and eigenfunctions of

$$(\mathrm{E}^*) \qquad L(y) + \lambda y^\sigma = 0, \quad \mathscr{R}_a(y) = y(b) = 0.$$

A *comparison* result between eigenvalues of (E) and (E^*) is also given.

Finally, we review the literature on this subject. The concept of time scales has been introduced by Hilger [1] in order to unify continuous and discrete calculus (see also [2]). An up-to-date information of the subject is available in the recently published monograph [3]. Section 4.7 deals with Sturm–Liouville equations on time scales (see also [4]), and we will use some of their results. Further results that we need are contained in [5,6]. The methods we apply are rather similar to those in the recent monograph "Quadratic Functionals in Variational Analysis and Control Theory" [7] ch. 0. However, a great deal of additional effort has to be made which is due to the existence of nodes in the case of general time scales (of course there are no nodes in the case of $\mathbf{T} = \mathbb{R}$). We also wish to add that many of our results are new even for the case $\mathbf{T} = \mathbb{Z}$. However, for the discrete case, Theorem 1 can be found (with $\beta = \delta = 0$) in [8], Theorem 7.6 while [8] Theorem 7.7 gives a result related to Theorem 2 (with $\beta = \delta = k = 0$). Further references for the case $\mathbf{T} = \mathbb{Z}$ are contained in [9–12].

2. Preliminaries on time scales and eigenvalue problems

We let $\mathbf{T} \subset \mathbb{R}$ be closed so that the *jump operators*

$$\sigma(t) = \inf\{s \in \mathbf{T}:\ s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbf{T}:\ s < t\}$$

(supplemented by $\inf\emptyset := \sup \mathbf{T}$ and $\sup\emptyset := \inf \mathbf{T}$) are well-defined and map $\mathbf{T}$ into $\mathbf{T}$. We call a point $t \in \mathbf{T}$ *left-dense, left-scattered, right-dense,* *right-scattered* if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. The *graininess* $\mu : \mathbf{T} \to \mathbb{R}_0^+$ is defined by $\mu(t) = \sigma(t) - t$. We put $\mathbf{T}^\kappa = \mathbf{T}$ if $\mathbf{T}$ is unbounded above and $\mathbf{T}^\kappa = \mathbf{T}\backslash(\rho(\max \mathbf{T}), \max \mathbf{T}])$ otherwise. A function $f : \mathbf{T} \to \mathbb{R}$ is called *differentiable* at $t \in \mathbf{T}^\kappa$ if

$$f^\Delta(t) := \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s},$$

where

$$s \to t, \quad s \in \mathbf{T}\backslash\{\sigma(t)\}$$

exists, and it is called differentiable on $\mathbf{T}$ provided it is differentiable at each $t \in \mathbf{T}^\kappa$. A function $F : \mathbf{T} \to \mathbb{R}$ with $F^\Delta(t) = f(t)$ for all $t \in \mathbf{T}^\kappa$ is called an *antiderivative* of f on $\mathbf{T}$, and in this case we define

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s) \quad \text{for all } s, t \in \mathbf{T}.$$

Moreover, f is called *rd-continuous* on $\mathbf{T}$ if it is continuous at all right-dense points and has left-sided limits at all left-dense points $t \in \mathbf{T}$. The following results from [3], ch. 1 (see also [5], Lemma 1) will be needed.

Lemma 1. *Let $f, g : \mathbf{T} \to \mathbb{R}$ and $t \in \mathbf{T}^\kappa$. Then the following holds:*
(i) If $f^\Delta(t)$ exists, then f is continuous at t;
(ii) if t is right-scattered and f is continuous at t, then $f^\Delta(t) = (f(\sigma(t)) - f(t))/\mu(t)$;
(iii) if $f^\Delta(t)$ exists, then $f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t)$;
(iv) if $f^\Delta(t), g^\Delta(t)$ exist and $(fg)(t)$ is defined, then $(fg)^\Delta(t) = f(\sigma(t))g^\Delta(t) + f^\Delta(t)g(t)$;
(v) if f^Δ exists on $\mathbf{T}^\kappa$ and f is invertible on $\mathbf{T}$, then $(f^{-1})^\Delta = -(f^\sigma)^{-1} f^\Delta f^{-1}$ on $\mathbf{T}^\kappa$;
(vi) if f is rd-continuous on $\mathbf{T}$, then it has an antiderivative on $\mathbf{T}$.

Now we proceed to give some preliminaries on Sturm–Liouville eigenvalue problems. For $x, y \in \mathrm{C}^2_{\mathrm{rd}}(\mathbf{T})$, the set of twice differentiable functions with rd-continuous second derivative, we define the *Wronskian*

$$w(x, y) = xy^\Delta - yx^\Delta.$$

Lemma 2 (Lagrange's Identity, Green's Formula). *For $x, y \in \mathrm{C}^2_{\mathrm{rd}}(\mathbf{T})$ we have*
(i) $L(y)x^\sigma - L(x)y^\sigma = w^\Delta(x, y)$ on $[\rho(a), \rho(b)] \cap \mathbf{T}$;
(ii) $\langle L(y), x^\sigma\rangle - \langle L(x), y^\sigma\rangle = w(x, y)(b) - w(x, y)(\rho(a))$.

Proof. We refer to [4], Lemma 3.9. □

It is easy to see that $w(x,y)(\rho(a))=0$ if $\mathscr{R}_a(x)=\mathscr{R}_a(y)=0$ and $w(x,y)(b)=0$ if $\mathscr{R}_b(x)=\mathscr{R}_b(y)=0$. Hence the problem (E) is self-adjoint and all eigenvalues are real. In what follows we shall always assume without loss of generality that

$$\begin{aligned}&\beta>\alpha\mu(\rho(a)) \text{ if } \beta\neq\alpha\mu(\rho(a)), \quad \alpha=-1 \text{ if } \beta=\alpha\mu(\rho(a)),\\ &\quad\delta>0 \text{ if } \delta\neq 0, \quad \gamma=1 \text{ if } \delta=0.\end{aligned} \tag{1}$$

We will also assume throughout that

$$\begin{aligned}&\text{if } a \text{ is right-scattered}, \quad \text{then it is also left-scattered, and}\\ &\text{if } b \text{ is left-scattered}, \quad \text{then it is also right-scattered.}\end{aligned} \tag{2}$$

Now we shall provide a characterization for the eigenvalues of (E). For this, we denote the unique solutions (see [4]) of the initial value problems

$$L(y)+\lambda y^\sigma=0, \quad y(\rho(a))=\beta, \quad y^\Delta(\rho(a))=-\alpha$$

by $y(\cdot,\lambda)$, where $\lambda\in\mathbb{R}$, and we put $\Lambda(\lambda)=\mathscr{R}_b(y(\cdot,\lambda))$. With this notation, we have the following lemmas.

Lemma 3. *λ is an eigenvalue of (E) if and only if $\Lambda(\lambda)=0$.*

Proof. If $\mathscr{R}_b(y(\cdot,\lambda))=0$, then $y=y(\cdot,\lambda)$ satisfies

$$L(y)+\lambda y^\sigma=0, \quad \mathscr{R}_a(y)=\mathscr{R}_b(y)=0,$$

i.e., λ is an eigenvalue of (E). Conversely, let $\lambda\in\mathbb{R}$ be an eigenvalue of (E) with corresponding eigenfunction y. Then because of the unique solvability of initial value problems (observe $\mathscr{R}_a(y)=0$),

$$y=cy(\cdot,\lambda) \quad \text{with} \quad c=\frac{\beta y(\rho(a))-\alpha y^\Delta(\rho(a))}{\alpha^2+\beta^2}.$$

Hence $\mathscr{R}_b(y(\cdot,\lambda))=0$. □

The proof of Lemma 3 also shows that all eigenvalues of (E) are *simple*. Furthermore, in view of Lemma 2, eigenfunctions x,y corresponding to different eigenvalues are always orthogonal in the sense

$$x^\sigma\perp y^\sigma, \quad \text{i.e.,} \quad \langle x^\sigma,y^\sigma\rangle=0.$$

Lemma 4. *We have for all $t\in[\rho(a),b]\cap\mathbf{T}$ and $\lambda\in\mathbb{R}$*

$$y(t,\lambda)\frac{\partial}{\partial\lambda}y^\Delta(t,\lambda)-y^\Delta(t,\lambda)\frac{\partial}{\partial\lambda}y(t,\lambda)=-\int_{\rho(a)}^t\{y^\sigma(\tau,\lambda)\}^2\Delta\tau.$$

Proof. Let $v, \lambda \in \mathbb{R}$ with $v \neq \lambda$. Then

$$\begin{aligned}&\left[y(t,v)\{y^{\Delta}(t,\lambda)-y^{\Delta}(t,v)\}-y^{\Delta}(t,v)\{y(t,\lambda)-y(t,v)\}\right]^{\Delta}\\&\quad=\left[y(t,v)y^{\Delta}(t,\lambda)-y^{\Delta}(t,v)y(t,\lambda)\right]^{\Delta}\\&\quad=y^{\sigma}(t,v)y^{\Delta\Delta}(t,\lambda)-y^{\Delta\Delta}(t,v)y^{\sigma}(t,\lambda)=(v-\lambda)y^{\sigma}(t,v)y^{\sigma}(t,\lambda).\end{aligned}$$

We divide both sides by $\lambda - v$, let $v \to \lambda$, and integrate from $\rho(a)$ to t (observe that $(\partial/\partial\lambda)y(\rho(a),\lambda) = (\partial/\partial\lambda)y^{\Delta}(\rho(a),\lambda) = 0$) to obtain our desired equation. □

From Lemma 4 it follows that $(y^{\Delta}(t,\lambda))/(y(t,\lambda))$ is for each $t \in (a,b] \cap \mathbf{T}$ strictly decreasing in λ whenever $y(t,\lambda) \neq 0$. Note that Lemma 4 implies $(\partial/\partial\lambda)y(b,\lambda) \neq 0$ if $y(b,\lambda) = 0$, and it also implies $(\partial/\partial\lambda)\{\gamma + \delta(y^{\Delta}(b,\lambda))/(y(b,\lambda))\} \neq 0$. This together with Lemma 3 (observe $\Lambda(\lambda) = y(b,\lambda) \times \{\gamma + \delta(y^{\Delta}(b,\lambda))/(y(b,\lambda))\}$) shows that all eigenvalues of (E) are isolated.

Our main concern in this paper is to give the so-called *oscillation number* for any eigenfunction of (E). We define $n(\lambda)$ to be the number of generalized zeros of $y(\cdot,\lambda)$ in the open interval (a,b). The following result is needed later, and since its proof is analogous to the proof for the continuous case (in view of Lemma 1 (i), $y(t,\lambda)$ is continuous in t, whereas its continuity with respect to λ follows from [3], Section 2.6), it is omitted here.

Lemma 5. $\limsup_{v\to\lambda} n(v) \leqslant n(\lambda) + 1$ *for all* $\lambda \in \mathbb{R}$; *and if equality holds, then* $y(b,\lambda) = 0$.

Lemma 6. *Put* $\lambda^* = -\{1 + \max_{t\in[\rho(a),b]\cap\mathbf{T}} q(t)\}$. *Let* $\lambda \leqslant \lambda^*$ *and* y *be a nontrivial solution of* $L(y) + \lambda y^{\sigma} = 0$. *Then we have the following:*

(i) If y *has for some* $t_1 \in [\rho(a),b] \cap \mathbf{T}$ *a generalized zero in* $\mathcal{I} = \{t_1\} \cup (t_1, \sigma(t_1))$, *then* y^{Δ} *has no generalized zero in* $\mathcal{I}$;

(ii) if y *has a generalized zero in* $\mathcal{I}$, *then* yy^{Δ} *has a generalized zero in* $\mathcal{I}$ *also;*

(iii) yy^{Δ} *strictly increases on* $[\rho(a),b] \cap \mathbf{T}$;

(iv) if $(yy^{\Delta})(t_2) \geqslant 0$ *for some* $t_2 \in [\rho(a),b] \cap \mathbf{T}$, *then* y *has no generalized zeros in* $(t_2,b]$;

(v) $[\rho(a),b]$ *contains at most one generalized zero of* y;

(vi) if $n(\lambda) = 0$, *then* $\lim_{v\to-\infty} n(v) = 0$.

Proof. Assertion (i) follows from (use Lemma 1 (iii))

$$\begin{aligned}y^{\Delta}(t_1)y^{\Delta}(\sigma(t_1)) &= y^{\Delta}(t_1)\{y^{\Delta}(t_1) + \mu(t_1)y^{\Delta\Delta}(t_1)\}\\&= (y^{\Delta}(t_1))^2 - (\lambda + q(t_1))\mu(t_1)y(\sigma(t_1))y^{\Delta}(t_1)\\&= (y^{\Delta}(t_1))^2 - (\lambda + q(t_1))y(\sigma(t_1))\{y(\sigma(t_1)) - y(t_1)\}\\&= (y^{\Delta}(t_1))^2 - (\lambda + q(t_1))(y(\sigma(t_1)))^2 + (\lambda + q(t_1))y(t_1)y(\sigma(t_1))\\&\geqslant (y^{\Delta}(t_1))^2 + (y(\sigma(t_1)))^2 - y(t_1)y(\sigma(t_1)) > 0.\end{aligned}$$

For (ii), $(yy^\sigma)(t_1) \leqslant 0$ implies $(y^\Delta y^{\Delta^\sigma})(t_1) > 0$ by (i), and these two inequalities of course force $(yy^\Delta(yy^\Delta)^\sigma)(t_1) \leqslant 0$. Next,

$$\begin{aligned}(yy^\Delta)^\Delta(t) &= \left(y^\Delta(t)\right)^2 + y^{\Delta\Delta}(t)y^\sigma(t)\\ &= \left(y^\Delta(t)\right)^2 - (\lambda + q(t))(y^\sigma(t))^2\\ &\geqslant \left(y^\Delta(t)\right)^2 + (y^\sigma(t))^2 > 0\end{aligned}$$

proves (iii). The identity (use Lemma 1 (iii))

$$y(t)y^\sigma(t) = y(t)\left\{y(t) + \mu(t)y^\Delta(t)\right\} = y^2(t) + \mu(t)y(t)y^\Delta(t)$$

takes care of (iv), while (v) follows from (ii), (iii) and (iv). Finally assume that $n(\lambda) = 0$ and put $\kappa = \min_{t\in[\rho(a),\sigma(b)]\cap\mathbf{T}} y^2(t,\lambda)$. Then $\kappa > 0$ and

$$\begin{aligned}y(t,\nu)y^\Delta(t,\nu) &= -\alpha\beta + \int_{\rho(a)}^t \left\{\left(y^\Delta(\tau,\nu)\right)^2 - (\nu + q(\tau))(y^\sigma(\tau,\nu))^2\right\}\Delta\tau\\ &\geqslant \; -\alpha\beta + (\lambda^* - \nu)\kappa(t - \rho(a)) > 0\end{aligned}$$

for all sufficiently small ν so that (vi) follows from (iv). □

Theorem 3. *We have* $\lim_{\lambda\to-\infty} n(\lambda) = 0$ *and* $\lim_{\lambda\to-\infty} \Lambda(\lambda) = \infty$.

Proof. Let λ^* be as in Lemma 6 and $\lambda \leqslant \lambda^*$. Then $n(\lambda^*) \in \{0, 1\}$ according to Lemma 6 (v). If $n(\lambda^*) = 0$, then $\lim_{\nu\to-\infty} n(\nu) = 0$ because of Lemma 6 (vi). Hence we assume $n(\lambda^*) = 1$. Thus $\alpha\beta > 0$ (observe $y(\rho(a),\lambda^*)y^\Delta(\rho(a),\lambda^*) = -\alpha\beta$ and use Lemma 6 (iv)), and $[\rho(a), a]$ does not contain a generalized zero of $y(\cdot,\lambda^*)$ (observe Lemma 6 (v)). Therefore there exists some $\underset{\sim}{t} \in (\rho(a), b] \cap \mathbf{T}$ such that $[\rho(a), \underset{\sim}{t}]$ contains no generalized zero of $y(\cdot,\lambda^*)$. Put $\mathscr{T} = (\rho(a), \underset{\sim}{t}] \cap \mathbf{T}$. Hence $\mathscr{T} \neq \emptyset$. Therefore

$$\frac{y^\Delta(t,\lambda^*)}{y(t,\lambda^*)} < 0 \quad \text{for all} \in \mathscr{T}$$

because of Lemma 6 (iv).

We now assume that

$$\frac{y^\Delta(t,\lambda)}{y(t,\lambda)} \in \left(\frac{y^\Delta(t,\lambda^*)}{y(t,\lambda^*)}, 0\right) \quad \text{for all} \quad t \in \mathscr{T} \text{ and all } \lambda < \lambda^*.$$

This implies for all $t \in \mathscr{T}$ and all $\lambda < \lambda^*$ (use Lemma 1 (iii))

$$1 + \mu(t)\frac{y^\Delta(t,\lambda)}{y(t,\lambda)} \geqslant 1 + \mu(t)\frac{y^\Delta(t,\lambda^*)}{y(t,\lambda^*)} = \frac{y^\sigma(t,\lambda^*)}{y(t,\lambda^*)} > 0$$

and hence for all $\lambda < \lambda^*$

$$\begin{aligned}\frac{y^\Delta(\underline{t},\lambda)}{y(\underline{t},\lambda)} &= -\frac{\beta}{\alpha} + \int_{\rho(a)}^{\underline{t}} \left\{ -(\lambda+q(t)) - \left(\frac{y^\Delta(t,\lambda)}{y(t,\lambda)}\right)^2 \frac{1}{1+\mu(t)\frac{y^\Delta(t,\lambda)}{y(t,\lambda)}} \right\}\Delta t \\ &\geqslant -\frac{\beta}{\alpha} + \int_{\rho(a)}^{\underline{t}} \left\{ (\lambda^*-\lambda) - (\lambda^*+q(t)) - \left(\frac{y^\Delta(t,\lambda^*)}{y(t,\lambda^*)}\right)^2 \frac{1}{1+\mu(t)\frac{y^\Delta(t,\lambda^*)}{y(t,\lambda^*)}} \right\}\Delta t \\ &= \frac{y^\Delta(\underline{t},\lambda^*)}{y(\underline{t},\lambda^*)} + (\lambda^*-\lambda)\left(\underline{t}-\rho(a)\right)\end{aligned}$$

which tends to infinity as $\lambda \to -\infty$, a contradiction.

Hence and because $(y^\Delta/y)(t,\cdot)$ is strictly decreasing according to Lemma 4, there must exist an $s \in \mathcal{T}$ and $\underline{\lambda} < \lambda^*$ such that $y^\Delta(s,\underline{\lambda}) = 0$. Then $y(s,\underline{\lambda}) \neq 0$ and $y(\cdot,\underline{\lambda})$ has no generalized zero in (s,b) according to Lemma 6 (iv). If $y(\cdot,\underline{\lambda})$ had a generalized zero in $[\rho(a),s)$, then $(yy^\Delta)(t_1,\underline{\lambda}) \geqslant 0$ for some $t_1 \in [\rho(a),s) \cap \mathbf{T}$ because of Lemma 6 (ii), and this contradicts Lemma 6 (iii). Hence $n(\underline{\lambda}) = 0$ and $\lim_{\nu\to-\infty} n(\nu) = 0$ due to Lemma 6 (vi).

Finally, using what we have shown so far (note also $-y^\Delta(b,\lambda) = \alpha + \int_{\rho(a)}^b (\lambda + q(t))y^\sigma(t)\Delta t$) and from the proof of Lemma 6 (vi) it follows that

$$\lim_{\lambda\to-\infty} y(b,\lambda) = \lim_{\lambda\to-\infty} y^\Delta(b,\lambda) = \lim_{\lambda\to-\infty} \frac{y^\Delta(b,\lambda)}{y(b,\lambda)} = \infty.$$

Hence

$$\lim_{\lambda\to-\infty} \Lambda(\lambda) = \lim_{\lambda\to-\infty} \left\{ y(b,\lambda)\left(\gamma + \delta\frac{y^\Delta(b,\lambda)}{y(b,\lambda)}\right)\right\} = \infty$$

according to our assumption (1). □

From the results obtained so far, we know that the eigenvalues of (E) may be arranged as $-\infty < \lambda_1 < \lambda_2 < \cdots$ We shall write $\lambda_p = \infty$ for all $p > k$ if only k eigenvalues exist.

3. The extended Picone's identity

Besides elementary results in Section 2, the proof of our main results requires appropriate applications of the extended Picone's identity on time scales, which we shall derive in this section. Let us introduce some notation which is used below. By Ker M we denote the kernel of the indicated matrix M, and we put def M = dim Ker M. Moreover, we let $c^\dagger = 1/c$ if $c \neq 0$ and $0^\dagger = 0$. Finally, for a set S we abbreviate the number of elements of S by $|S|$.

Theorem 4 (Picone's Identity). *Let $t \in [\rho(a), \rho(b)] \cap \mathbf{T}$. Suppose*

$$\operatorname{Ker} y(t) \subset \operatorname{Ker} x(t) \quad \text{and} \quad \operatorname{Ker} y^\sigma(t) \subset \operatorname{Ker} x^\sigma(t),$$

where $x \in C^2_{\mathrm{rd}}(\mathbf{T})$ and y solves $L(y) + \lambda y^\sigma = 0$. Then at t,

$$-\{L(x) + \lambda x^\sigma\}x^\sigma = \left[xy^\dagger w(x,y)\right]^\Delta + (yy^\sigma)^\dagger w^2(x,y).$$

Proof. First suppose that $(yy^\sigma)(t) \neq 0$. Then Lemmas 1 (iv), (v) and 2 provide at t,

$$\begin{aligned}\left[xy^\dagger w(x,y)\right]^\Delta &= \frac{x^\sigma}{y^\sigma} w^\Delta(x,y) + \left(\frac{x^\Delta}{y^\Delta} - \frac{xy^\Delta}{yy^\sigma}\right) w(x,y) \\ &= \frac{x^\sigma}{y^\sigma}\{L(y)x^\sigma - L(x)y^\sigma\} - \frac{w^2(x,y)}{yy^\sigma} \\ &= -\{L(x) + \lambda x^\sigma\}x^\sigma - (yy^\sigma)^\dagger w^2(x,y).\end{aligned}$$

Next, suppose that t is right-scattered with $(yy^\sigma)(t) = 0$. Then we have $(xx^\sigma)(t) = 0$ and hence at t,

$$w(x,y) = xy^\Delta - yx^\Delta = x\frac{y^\sigma - y}{\mu} - y\frac{x^\sigma - x}{\mu} = \frac{xy^\sigma - yx^\sigma}{\mu} = 0.$$

If t is left-scattered, then $xy^\dagger w(x,y)$ is obviously continuous at t, and if t is left-dense, then it is also continuous at t, since according to l'Hopital rule (see [5], Theorem 3),

$$\lim_{s \to t^-}\left[xy^\dagger w(x,y)\right](s) = \frac{x^\Delta(t)}{y^\Delta(t)} w(x,y)(t) = 0 = \left[xy^\dagger w(x,y)\right](t).$$

Hence we may apply Lemmas 1 (ii) and 2 (i) to obtain at t,

$$\begin{aligned}\left[xy^\dagger w(x,y)\right]^\Delta &= \frac{[xy^\dagger w(x,y)]^\sigma - xy^\dagger w(x,y)}{\mu} = \frac{[xy^\dagger w(x,y)]^\sigma}{\mu} \\ &= x^\sigma (y^\sigma)^\dagger w^\Delta(x,y) = x^\sigma (y^\sigma)^\dagger \{L(y)x^\sigma - L(x)y^\sigma\} \\ &= x^\sigma (y^\sigma)^\dagger y^\sigma\{-\lambda x^\sigma - L(x)\} = -x^\sigma\{L(x) + \lambda x^\sigma\} \\ &= -x^\sigma\{L(x) + \lambda x^\sigma\} - (yy^\sigma)^\dagger w^2(x,y).\end{aligned}$$

Finally, if t is right-dense with $y(t) = y(\sigma(t)) = 0$, then we have $x(t) = x(\sigma(t)) = 0$, and l'Hopital rule yields

$$\begin{aligned}\left[xy^\dagger w(x,y)\right]^\Delta &= \lim_{s \to t}\frac{[xy^\dagger w(x,y)](\sigma(t)) - [xy^\dagger w(x,y)](t)}{\sigma(t) - s} \\ &= \lim_{s \to t}\frac{[xy^\dagger w(x,y)](s)}{s - t} = \lim_{s \to t}\left[xy^\dagger\right](s) \lim_{s \to t}\frac{w(x,y)(s) - w(x,y)(t)}{s - t} \\ &= \frac{x^\Delta(t)}{y^\Delta(t)} w^\Delta(x,y)(t) = 0 = -\{L(x) + \lambda x^\sigma\}(t) - (yy^\sigma)^\dagger w^2(x,y)(t).\end{aligned}$$

This proves our result. □

For a special case of Theorem 4 we refer to [6], Theorem 3. With the aid of Theorem 2 it is now possible to prove the following *key result*:

Theorem 5 (Extended Picone's Identity). *Let y be a solution of $L(y) + \lambda y^\sigma = 0$ and $k \in \mathbb{N}$. Let x_ν be solutions of $L(x_\nu) + \rho_\nu x_\nu^\sigma = 0$ with $\rho_\nu \in \mathbb{R}$, $1 \leqslant \nu \leqslant k$. Moreover, let $x \in \mathrm{C}^2_{\mathrm{rd}}(\mathbf{T})$ satisfy $x^\sigma \perp x_\nu^\sigma$, $1 \leqslant \nu \leqslant k$, and suppose $x_\nu^\sigma \perp x_\mu^\sigma$, $1 \leqslant \nu < \mu \leqslant k$, $\|x_\nu^\sigma\| = 1$, $1 \leqslant \nu \leqslant k$. Put $\tilde{x}_1 = \sum_{\nu=1}^k \beta_\nu x_\nu$ with $\beta_\nu \in \mathbb{R}$, $1 \leqslant \nu \leqslant k$, and $\tilde{x} = x + \tilde{x}_1$. Suppose $\mathrm{Ker}\, y(t) \subset \mathrm{Ker}\, x(t)$ holds for all $t \in [\rho(a), b] \cap \mathbf{T}$. Then*

$$-\langle L(x) + \lambda x^\sigma, x^\sigma\rangle = \int_{\rho(a)}^b \left[(yy^\sigma)^\dagger w^2(\tilde{x}, y)\right](t)\Delta t + \sum_{\nu=1}^k \beta_\nu^2(\lambda - \rho_\nu) + T(b) - T(\rho(a)),$$

where

$$T = \tilde{x} y^\dagger w(\tilde{x}, y) + w(\tilde{x}_1, x).$$

Proof. We apply Theorem 2 and Lemma 2 (ii) to obtain

$$\begin{aligned}
&\left[\tilde{x}y^\dagger w(\tilde{x}, y)\right](\rho(a)) - \left[\tilde{x}y^\dagger w(\tilde{x}, y)\right](b) - \int_{\rho(a)}^b \left[(yy^\sigma)^\dagger w^2(\tilde{x}, y)\right](t)\Delta t \\
&= \langle L(\tilde{x}) + \lambda\tilde{x}^\sigma, \tilde{x}^\sigma\rangle = \left\langle L(x) + \sum_{\nu=1}^k \beta_\nu L(x_\nu) + \lambda x^\sigma + \lambda\tilde{x}_1^\sigma, x^\sigma + \tilde{x}_1^\sigma\right\rangle \\
&= \langle L(x) + \lambda x^\sigma, x^\sigma\rangle + \langle L(x), \tilde{x}_1^\sigma\rangle - \sum_{\nu=1}^k \beta_\nu\rho_\nu\langle x_\nu^\sigma, \tilde{x}_1^\sigma\rangle + \lambda\langle\tilde{x}_1^\sigma, \tilde{x}_1^\sigma\rangle \\
&= \langle L(x) + \lambda x^\sigma, x^\sigma\rangle + \langle L(\tilde{x}_1), x\rangle + w(\tilde{x}_1, x)(b) \\
&\quad - w(\tilde{x}_1, x)(\rho(a)) + \sum_{\nu=1}^k \beta_\nu^2(\lambda - \rho_\nu),
\end{aligned}$$

and hence our claim follows. □

4. Proof of the main results

Let $\lambda_1^* < \lambda_2^* < \cdots$ denote the eigenvalues of (E^*) with corresponding normalized eigenfunctions y_ν^* (i.e., $\|y_\nu^*\| = 1$). In all the proofs of the results from this section we use the notation from Theorem 5.

Theorem 6. *If* $\lambda \in (\lambda_k^*, \lambda_{k+1}^*] \cap \mathbb{R}$, *then*

$$n(\lambda) = k.$$

Moreover, define a $k \times k$*-matrix* $Z^* = (z_{\mu\nu}^*)$ *by*

$$z_{\mu\nu}^* = \begin{cases} y_\nu^*(t_\mu) & \text{if } t_\mu \text{ is a zero of } y(\cdot,\lambda) \\ w(y, y_\nu^*)(t_\mu) & \text{if } t_\mu \text{ is a node of } y(\cdot,\lambda), \end{cases}$$

where $t_1 < t_2 < \ldots < t_k$ *are the generalized zeros of* $y(\cdot,\lambda)$. *Then* Z^* *is invertible.*

Proof. Let $x = 0$, $\rho_\nu = \lambda_\nu^*$, $x_\nu = y_\nu^*$, $y = y(\cdot,\lambda)$. Then $T(\rho(a)) = T(b) = 0$. Suppose that y has no generalized zeros in (a, b). Then by Theorem 3 (observe $\tilde{x}(b) = 0$ and $\tilde{x}(\rho(a)) = 0$ if $y(\rho(a)) = 0$),

$$0 < \sum_{\nu=1}^{k} (\lambda - \lambda_\nu^*) = -\int_{\rho(a)}^{b} \left[(yy^\sigma)^\dagger w^2(\tilde{x}, y)\right](t)\Delta t \leqslant 0,$$

a contradiction. Hence y has at least one generalized zero in (a, b), say $t_1 < t_2 < \cdots < t_p$ are all of them. Define a $p \times k$-matrix Z^* with $z_{\mu\nu}^*$ as above. Let $(\beta_1, \beta_2, \ldots, \beta_k)^{\mathrm{T}} \in \operatorname{Ker} Z^*$ and $\tilde{x} = \sum_{\nu=1}^{k} \beta_\nu y_\nu^*$. Hence $\tilde{x}(t_\mu) = 0$ for all zeros t_μ of y and $w(\tilde{x}, y)(t_\mu) = 0$ for all nodes t_μ of y. Therefore, by Theorem 5,

$$0 \leqslant \sum_{\nu=1}^{k} \beta_\nu^2 (\lambda - \lambda_\nu^*) = -\int_{\rho(a)}^{b} \left[(yy^\sigma)^\dagger w^2(\tilde{x}, y)\right](t)\Delta t \leqslant 0$$

so that $\beta_1 = \beta_2 = \cdots = \beta_k = 0$ and hence $\operatorname{Ker} Z^* = \{0\}$. Thus $p \geqslant k$. Further, by Lemma 5 and Theorem 3 we have $p \leqslant k$. Therefore $p = k$ and Z^* is invertible. □

According to Theorem 6, we have $y(t, \lambda_1^*) > 0$ for all $t \in (a, b) \cap \mathbf{T}$. Hence $y^\Delta(b, \lambda_1^*) < 0$ (note that if b is left-scattered, then it is also right-scattered due to (2), and observe also Lemma 6 (i)) and

$$\Lambda(\lambda_1^*) = \delta y^\Delta(b, \lambda_1^*) \leqslant 0, \quad \text{hence } \lambda_1 \leqslant \lambda_1^*$$

(observe (1) and Theorem 3). Because of Theorem 6 if $\delta \neq 0$, we also have

$$\operatorname{sgn} \Lambda(\lambda_{k+1}^*) = \operatorname{sgn} \delta y^\Delta(b, \lambda_{k+1}^*) = \operatorname{sgn} y^\Delta(b, \lambda_{k+1}^*) = (-1)^{k+1}$$

and hence $\lambda_{k+1} \leqslant \lambda_{k+1}^*$.

Theorem 7. *If* $\lambda_k^* < \infty$, *then*

$$\lambda_{k+1}^* \leqslant \inf_{\substack{x(\rho(a)) = x(b) = 0 \\ x^\sigma \perp y_\nu^{*\sigma},\ 1 \leqslant \nu \leqslant k}} R(x) \quad \textit{and} \quad \lambda_{k+1}^* \leqslant \inf_{\substack{\mathcal{R}_a(x) = \mathcal{R}_b(x) = 0 \\ x^\sigma \perp y_\nu^{*\sigma},\ 1 \leqslant \nu \leqslant k}} R(x).$$

Proof. Let $x \in C^2_{\mathrm{rd}}(\mathbf{T})$ with $x(\rho(a))\mathscr{R}_a(x) = x(b)\mathscr{R}_b(x) = 0$, $x \neq 0$, $x_\nu = y^*_\nu$, $\rho_\nu = \lambda^*_\nu$, $\lambda \in (\lambda^*_k, \lambda^*_{k+1}] \cap \mathbb{R}$, $y = y(\cdot,\lambda)$. Then $T(\rho(a)) = T(b) = 0$. By Theorem 6 we may choose $\beta_1, \ldots, \beta_k \in \mathbb{R}$ such that $\tilde{x}(t_\mu) = 0$ for all zeros t_μ of y and $w(\tilde{x}, y)(t_\mu) = 0$ for all nodes t_μ of y (observe that Z^* is invertible). Then, by Theorem 5,

$$-\langle L(x) + \lambda x^\sigma, x^\sigma\rangle = \int_a^b \left[(yy^\sigma)^\dagger w^2(\tilde{x}, y)\right](t)\Delta t + \sum_{\nu=1}^{k} \beta_\nu^2(\lambda - \lambda^*_\nu) \geqslant 0$$

so that $-\langle L(x), x^\sigma\rangle \geqslant \lambda\langle x^\sigma, x^\sigma\rangle$. □

Theorem 8. *The number of eigenvalues of (E) is* $|[a,b] \cap \mathbf{T}| - \operatorname{def}(\beta - \alpha\mu(\rho(a))) - \operatorname{def}\delta$.

Proof. First note that, if $|[a,b] \cap \mathbf{T}| = \infty$ and $\lambda^*_k < \infty$, there exists an $x \in C^2_{\mathrm{rd}}(\mathbf{T})$ with $x(\rho(a)) = x(b) = 0$, $x \neq 0$ which is orthogonal to all y^*_ν, $1 \leqslant \nu \leqslant k$. Hence Theorem 7 implies that $\lambda^*_{k+1} < \infty$. Therefore (E*) has infinitely many eigenvalues.

Next, suppose $|[a,b] \cap \mathbf{T}| = b - a + 1 < \infty$. Then all points of $[a,b] \cap \mathbf{T}$ are right-scattered (observe that b is right-scattered since it is left-scattered due to Eq. (2)). Hence

$$y(a) = \beta - \alpha\mu(\rho(a)),$$

and the recursion formula

$$y(\sigma^2(t)) = \{a(t) + \lambda b(t)\}y(\sigma(t)) + c(t)y(t), \quad t \in [\rho(a), \rho(b)] \cap \mathbf{T}$$

with

$$a = 1 + \frac{\mu^\sigma}{\mu} - q\mu\mu^\sigma, \quad b = -\mu\mu^\sigma, \quad c = -\frac{\mu^\sigma}{\mu}$$

can be easily verified so that our claim follows from Lemma 3. □

Proof of Theorem 1. We let $x = 0$, $x_\nu = y_\nu$, $\rho_\nu = \lambda_\nu$, $k \in \mathbb{N}_0$, $\lambda = \lambda_{k+1}$ if $\lambda_{k+1} < \infty$, and $y = y(\cdot,\lambda)$. Then $T(\rho(a)) = T(b) = 0$, and as in the proof of Theorem 6 we apply Theorem 5 to find $n(\lambda) \leqslant k$. Hence our claim follows as before. Moreover, we observe that the analogously defined $k \times k$-matrix Z is invertible. □

Theorem 7 (Comparison Theorem). *If* $k \in \mathbb{N}_0$ *with* $\lambda^*_k < \infty$, *then*

$$\lambda_k \leqslant \lambda^*_k < \lambda_{k+1}.$$

Proof. This is clear from Theorem 1 and the remark after Theorem 6. □

Proof of Theorem 2. We let $x(\rho(a))\mathscr{R}_a(x) = x(b)\mathscr{R}_b(x) = 0$, $x \neq 0$, $x_v = y_v$, $\rho_v = \lambda_v$, $k \in \mathbb{N}_0$, $\lambda = \lambda_{k+1}$ if $\lambda_{k+1} < \infty$, and $y = y(\cdot, \lambda)$. Then $T(\rho(a)) = T(b) = 0$ and as in the proof of Theorem 7 we apply Theorem 5 to find

$$\lambda^*_{k+1} \leqslant \inf_{\substack{x(\rho(a))\mathscr{R}_a(x)=x(b)\mathscr{R}_b(x)=0 \\ x^\sigma \perp y^\sigma_v,\ 1 \leqslant v \leqslant k}} R(x).$$

We note that $R(y_{k+1}) = \lambda_{k+1}$, and this implies our claim. □

We conclude this paper with specializing our results for the continuous and the discrete case.

Remark 1. If $\mathbf{T} = \mathbb{R}$, then our results are well-known (see e.g. [7], ch. 0), and they may be summarized as follows (we put $a = 0$ and $b = 1$): There exist infinitely many eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ of

$$\ddot{y} + (\lambda + q(t))y = 0, \quad \alpha y(0) + \beta \dot{y}(0) = \gamma y(1) + \delta \dot{y}(1) = 0,$$

and an eigenfunction corresponding to λ_{k+1} vanishes exactly k times in $(0, 1)$. We can compute λ_{k+1} by taking the minimum of

$$-\frac{\int_0^1 (\ddot{x}(t) + q(t)x(t))x(t)\,\mathrm{d}t}{\int_0^1 x^2(t)\,\mathrm{d}t}$$

over all C^2-functions x that satisfy the boundary conditions and $\int_0^1 x(t)y_v(t)\,\mathrm{d}t = 0$, $1 \leqslant v \leqslant k$ (where y_v are the eigenfunctions corresponding to λ_v with $\int_0^1 y_v^2(t)\,\mathrm{d}t = 1$).

Remark 2. If $\mathbf{T} = \mathbf{Z}$, then only special cases of Theorem 1 (with $\beta = \delta = 0$) and Theorem 2 (with $\beta = \delta = k = 0$) are known (see e.g. [8], ch. 7). Our results may be summarized as follows (we put $a = 1$ and $b = N \in \mathbb{N}$): there exist $N - \operatorname{def}(\beta - \alpha) - \operatorname{def} \delta$ eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ of

$$\Delta^2 y + (\lambda + q(i))(\Delta y + y) = 0, \quad \alpha y(0) + \beta \Delta y(0) = \gamma y(N) + \delta \Delta y(N) = 0,$$

and an eigenfunction corresponding to λ_{k+1} has exactly k zeros or changes of sign in $(0, 1)$ (where an interval $(i, i+1)$ for $i \in \mathbb{Z}$ is said to contain a change of sign of y whenever $y(i)y(i+1) < 0$). We can compute λ_{k+1} by taking the minimum of

$$-\frac{\sum_{i=1}^N (\Delta^2 x(i) + q(i-1)x(i))x(i)}{\sum_{i=1}^N x^2(i)}$$

over all sequences x that satisfy the boundary conditions and $\sum_{i=0}^N x(i)y_v(i) = 0$, $1 \leqslant v \leqslant k$ (where y_v are the eigenfunctions corresponding to λ_v with $\sum_{i=0}^N y_v^2(i) = 1$).

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References

[1] S. Hilger, Analysis on measure chains – A unified approach to continuous and discrete calculus, Res. Math. 18 (1990) 19–56.
[2] B. Aulbach, S. Hilger, Linear dynamic processes with inhomogeneous time scale, Nonlinear Dynamics and Quantum Dynamical Systems, Akademie Verlag, Berlin, 1990.
[3] V. Lakshmikantham, S. Sivasundaram, B. Kaymakçalan. Dynamic Systems on Measure Chains, Kluwer Academic Publishers, Dordrecht, 1996.
[4] L. Erbe, S. Hilger, Sturmian theory on measure chains, Differential Equations Dynamical Systems 1 (3) (1993) 223–246.
[5] R.P. Agarwal, M. Bohner, Basic calculus on time scales and some of its applications (submitted).
[6] R.P. Agarwal, M. Bohner, Quadratic functionals for second order matrix equations on time scales, Nonlinear Anal. (to appear).
[7] W. Kratz, Quadratic Functionals in Variational Analysis and Control Theory, Akademie Verlag, Berlin, 1995.
[8] W.G. Kelley, A.C. Peterson, Difference Equations: An Introduction with Applications, Academic Press, New York, 1991.
[9] R.P. Agarwal, Difference Equations and Inequalities, Marcel Dekker, New York, 1992.
[10] C.D. Ahlbrandt, A.C. Peterson, Discrete Hamiltonian Systems: Difference Equations, Continued Fractions, and Riccati Equations, Kluwer Academic Publishers, Dordrecht, 1996.
[11] M. Bohner, Linear Hamiltonian difference systems: Disconjugacy and Jacobi-type conditions, J. Math. Anal. Appl. 199 (1996) 804–826.
[12] M. Bohner, On disconjugacy for Sturm–Liouville difference equations, J. Difference Equations 2 (2) (1996) 227–237.