Some Oscillation Criteria For First Order Delay Dynamic Equations

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Abstract: We present an oscillation criterion for first order delay dynamic equations on time scales, which contains well-known criteria for delay differential equations and delay difference equations as special cases. We illustrate our results by applying them to various kinds of time scales.

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1. Introduction

As is well known (cf. [3, Theorem 2.3.1]), a first order delay differential equation of the form

$$y'(t) + p(t)y(t-\tau) = 0$$

(where $t \in \mathbb{R}$, p is continuous and positive, and $\tau > 0$) is oscillatory provided

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)ds > \frac{1}{e}$$

holds. It is also well known (cf. [3, Theorem 7.5.1]) that a first order delay difference equation of the form

$$\Delta y_n + p_n y_{n-k} = 0$$

(where $n \in \mathbb{Z}$, $p_n > 0$, $k \in \mathbb{N}$, $\Delta y_n = y_{n+1} - y_n$) is oscillatory if

$$\liminf_{n\to\infty} \left\{ \frac{1}{k} \sum_{i=n-k}^{n-1} p_i \right\} > \frac{k^k}{(k+1)^{k+1}}$$

holds. In this paper we present a **generalization** and **extension** of these two results for first order delay dynamic equations (see also [4]) of the form

$$y^\Delta(t) + p(t)y(\tau(t)) = 0, \tag{1}$$

where $t \in \mathbb{T}$, $\mathbb{T}$ is a time scale (i.e., any nonempty closed subset of the reals) that is unbounded above, p is rd-continuous and positive, the *delay* function $\tau : \mathbb{T} \to \mathbb{T}$ satisfies $\tau(t) < t$ for all $t \in \mathbb{T}$ and $\lim_{t\to\infty} \tau(t) = \infty$, and $y^\Delta(t)$ is the delta derivative of $y : \mathbb{T} \to \mathbb{R}$ at $t \in \mathbb{T}$. If $\mathbb{T} = \mathbb{R}$, then $y^\Delta = y'$, while if $\mathbb{T} = \mathbb{Z}$, then $y^\Delta = \Delta y$. For further details concerning the time scales calculus we refer to [1, 2].

A function $p : \mathbb{T} \to \mathbb{R}$ is called *positively regressive* (we write $p \in \mathcal{R}^+$) if it is rd-continuous and satisfies $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}$, where $\mu : \mathbb{T} \to [0, \infty)$ is the *graininess* of the time scale defined by $\mu(t) = \sigma(t) - t$ with the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ defined by $\sigma(t) = \inf\{s \in \mathbb{T} : \; s > t\}$. It is well known that for any $p \in \mathcal{R}^+$ there exists a positive function y satisfying the initial value problem

$$y^\Delta = p(t)y, \quad y(t_0) = 1$$

(where $t_0 \in \mathbb{T}$), and we denote this $y(t)$ by $e_p(t, t_0)$ and call it the *exponential function.* Now let us assume that (1) possesses a positive solution y. Then

$$y^{\Delta}(t) = -p(t)y(\tau(t)) < 0$$

so that y is decreasing and therefore

$$\begin{aligned} 0 &= -\mu(t)\left[y^{\Delta}(t) + p(t)y(\tau(t))\right] = y(t) - y(\sigma(t)) - \mu(t)p(t)y(\tau(t)) \\ &< y(t) - \mu(t)p(t)y(t) = [1 - \mu(t)p(t)]\, y(t). \end{aligned}$$

Hence $1 - \mu(t)p(t) > 0$, which implies that $-p \in \mathcal{R}^+$, and thus there exists $\lambda > 0$ such that $-\lambda p \in \mathcal{R}^+$. The quantity

$$\alpha := \limsup_{\substack{t\to\infty \\ t\in\mathbb{T}}} \sup_{\substack{\lambda>0 \\ -\lambda p\in\mathcal{R}^+}} \{\lambda e_{-\lambda p}(t, \tau(t))\} \tag{2}$$

is therefore well defined. Now we can formulate our main result.

Theorem 1. *If (1) has an eventually positive solution, then α defined by (2) satisfies $\alpha \geq 1$.*

In the next section we present some auxiliary results, while we prove Theorem 1 in Section 3. Section 4 contains applications of Theorem 1 to various kinds of time scales. In the final section we present a result for a different dynamic equation and give more examples and applications.

2. Some Auxiliary Results

The following two easy lemmas are needed in the proof of Theorem 1.

Lemma 1. *Suppose $-p \in \mathcal{R}^+$ and $s \in \mathbb{T}$. If*

$$y^{\Delta}(t) + p(t)y(t) \leq 0 \quad \textit{for all} \quad t \geq s,$$

then

$$y(t) \leq e_{-p}(t, s)y(s) \quad \textit{for all} \quad t \geq s.$$

Proof. We put $f := y^{\Delta} + py$ and use [1, Theorem 2.77] to solve

$$y^{\Delta} = -p(t)y + f(t), \quad y(s) \text{ given}.$$

Thus for $t \geq s$,

$$y(t) = e_{-p}(t, s)y(s) + \int_s^t e_{-p}(t, \sigma(u))f(u)\Delta u.$$

The integrand is nonpositive as $-p \in \mathcal{R}^+$ and $f \leq 0$, so our claim follows. □

Lemma 2. *For nonnegative p with $-p \in \mathcal{R}^+$ we have the inequalities*

$$1 - \int_s^t p(u)\Delta u \leq e_{-p}(t, s) \leq \exp\left\{-\int_s^t p(u)\Delta u\right\} \quad \textit{for all} \quad t \geq s.$$

Proof. Fix $s \in \mathbb{T}$, denote $y(t) = -\int_s^t p(u)\Delta u$, and observe that

$$y^{\Delta}(t) = -p(t) \leq -p(t) - p(t)y(t).$$

We put $f := y^{\Delta} + py + p$ and use [1, Theorem 2.77] to solve

$$y^{\Delta} = -p(t)y + f(t) - p(t), \quad y(s) = 0.$$

Thus for $t \geq s$,

$$\begin{aligned} y(t) &= e_{-p}(t,s)y(s) + \int_s^t e_{-p}(t,\sigma(u))\left[f(u) - p(u)\right]\Delta u \\ &\leq -\int_s^t e_{-p}(t,\sigma(u))p(u)\Delta u \\ &= e_{-p}(t,s) - 1, \end{aligned}$$

where we have used [1, Theorem 2.39] in the last step. This establishes the left part of the asserted inequality. For the right part we use the representation [1, (2.15)]

$$e_{-p}(t,s) = \exp\left\{\int_s^t \xi_{\mu(u)}(-p(u))\Delta u\right\},$$

where we have for any p with $-p \in \mathcal{R}^+$

$$\xi_{\mu(u)}(-p(u)) = -p(u)$$

if $\mu(u) = 0$, and if $\mu(u) > 0$,

$$\begin{aligned} \xi_{\mu(u)}(-p(u)) &= \frac{\operatorname{Log}(1-\mu(u)p(u))}{\mu(u)} = \frac{\log(1-\mu(u)p(u))}{\mu(u)} \\ &= -p(u) - \frac{f(-\mu(u)p(u))}{\mu(u)} \leq -p(u), \end{aligned}$$

where $f : (-1,\infty) \to \mathbb{R}$ is defined by $f(x) = x - \log(1+x)$ and hence satisfies $f(x) \geq 0$ for all $x > -1$. □

We conclude this section with some remarks.

Remark 1. Let $s \in \mathbb{T}$. If p is rd-continuous and nonnegative, a similar proof as in Lemma 1 can be used to show that if

$$x^{\Delta}(t) + p(t)x(\sigma(t)) \leq 0 \quad \text{for all} \quad t \geq s,$$

then

$$x(s) \geq e_p(t,s)x(t) \quad \text{for all} \quad t \geq s.$$

Remark 2. If p is rd-continuous and nonnegative, a similar proof as in Lemma 2 can be used to show

$$1 + \int_s^t p(u)\Delta u \leq e_p(t,s) \leq \exp\left\{\int_s^t p(u)\Delta u\right\} \quad \text{for all} \quad t \geq s.$$

Remark 3. Denote $P := \int_s^t p(u)\Delta u$ for $t \geq s$, where p is nonnegative with $-p \in \mathcal{R}^+$. Then by Lemma 2, $1 - P \leq e_{-p}(t,s) \leq e^P$. For all $\lambda \in (0,1]$ we have $-\lambda p \in \mathcal{R}^+$ and hence by Lemma 2, $1 - \lambda P \leq e_{-\lambda p}(t,s) \leq e^{\lambda P}$, so that

$$\lambda - \lambda^2 P \leq \lambda e_{-\lambda p}(t,s) \leq \lambda e^{\lambda P}$$

and therefore

$$\frac{1}{4P} \leq \sup_{\substack{\lambda>0 \\ -\lambda p \in \mathcal{R}^+}} \{\lambda e_{-\lambda p}(t,s)\} \leq \frac{1}{eP}.$$

Thus we always have

$$\frac{1}{4\liminf_{t\to\infty}\int_{\tau(t)}^t p(s)\Delta s} \leq \alpha \leq \frac{1}{e\liminf_{t\to\infty}\int_{\tau(t)}^t p(s)\Delta s}$$

and

$$\frac{1}{4\alpha} \leq \liminf_{t\to\infty}\int_{\tau(t)}^t p(s)\Delta s \leq \frac{1}{e\alpha}.$$

3. The Oscillation Criterion

Now we have all the tools needed to prove our main result.

Proof of Theorem 1. Throughout we assume that y solves (1) and is eventually positive and that $\alpha < 1$. We proceed in two parts showing

$$\liminf_{t\to\infty} \frac{y(\tau(t))}{y(t)} = \infty \tag{3}$$

and

$$\liminf_{t\to\infty} \frac{y(\tau(t))}{y(t)} < \infty. \tag{4}$$

This contradiction shows $\alpha \geq 1$ and hence finishes the proof. First we show (3). Let $\beta \in (1, 1/\alpha)$. Then there exists $T_0 \in \mathbb{T}$ such that

$$\frac{1}{\sup_{\substack{\lambda>0 \\ -\lambda p \in \mathcal{R}^+}} \{\lambda e_{-\lambda p}(t, \tau(t))\}} \geq \beta \quad \text{for all} \quad t \geq T_0. \tag{5}$$

As y is eventually positive, it is eventually decreasing and hence $y(\tau(t)) \geq y(t)$ eventually so that

$$0 = y^{\Delta}(t) + p(t)y(\tau(t)) \geq y^{\Delta}(t) + p(t)y(t)$$

implies by Lemma 1 that there exists $T_1 \geq T_0$ with

$$\frac{y(\tau(t))}{y(t)} \geq \frac{1}{e_{-p}(t, \tau(t))} \overset{(5)}{\geq} \beta \quad \text{for all} \quad t \geq T_1.$$

Thus

$$0 = y^{\Delta}(t) + p(t)y(\tau(t)) \geq y^{\Delta}(t) + \beta p(t)y(t)$$

implies again by Lemma 1 that there exists $T_2 \geq T_1$ with

$$\frac{y(\tau(t))}{y(t)} \geq \frac{1}{e_{-\beta p}(t, \tau(t))} = \frac{\beta}{\beta e_{-\beta p}(t, \tau(t))} \overset{(5)}{\geq} \beta^2 \quad \text{for all} \quad t \geq T_2.$$

Proceeding in this manner we obtain a sequence $\{T_n\} \subset \mathbb{T}$ with

$$\frac{y(\tau(t))}{y(t)} \geq \beta^n \quad \text{for all} \quad t \geq T_n.$$

This proves (3) as $\beta > 1$. Now we show (4). Let $M \in (1/4, 1/(4\alpha))$. By Lemma 2 (see Remark 3) there exists $T \in \mathbb{T}$ such that

$$\int_{\tau(t)}^{t} p(s)\Delta s \geq M \quad \text{for all} \quad t \geq T.$$

Now

$$\int_{\tau(t)}^{\sigma(t)} p(s)\Delta s \geq \int_{\tau(t)}^{t} p(s)\Delta s \geq M \quad \text{for all} \quad t \geq T.$$

Let $t \geq T$. We consider the function $f : \mathbb{T} \to \mathbb{R}$ defined by

$$f(u) = \int_{\tau(t)}^{u} p(s)\Delta s - \frac{M}{2}$$

and find $f(\tau(t)) < 0$ and $f(t) > 0$. By the intermediate value theore [1, Theorem 1.115] there exists $t^* \in [\tau(t), t)$ such that $f(t^*) = 0$, or $f(t^*) < 0$ and $f(\sigma(t^*)) > 0$. Hence

$$\int_{\tau(t)}^{\sigma(t^*)} p(s)\Delta s = \frac{M}{2} + f(\sigma(t^*)) \geq \frac{M}{2} \tag{6}$$

and

$$\int_{t^*}^{\sigma(t)} p(s)\Delta s = \int_{\tau(t)}^{\sigma(t)} p(s)\Delta s - \left[f(t^*) + \frac{M}{2} \right] \geq \frac{M}{2} - f(t^*) \geq \frac{M}{2}. \tag{7}$$

Now we can estimate

$$\begin{aligned} y(t^*) &\geq y(t^*) - y(\sigma(t)) \overset{(1)}{=} \int_{t^*}^{\sigma(t)} p(s)y(\tau(s))\Delta s \\ &\geq y(\tau(t)) \int_{t^*}^{\sigma(t)} p(s)\Delta s \overset{(7)}{\geq} \frac{M}{2} y(\tau(t)) \\ &\geq \frac{M}{2} \left[y(\tau(t)) - y(\sigma(t^*)) \right] \overset{(1)}{=} \frac{M}{2} \int_{\tau(t)}^{\sigma(t^*)} p(s)y(\tau(s))\Delta s \\ &\geq \frac{M}{2} y(\tau(t^*)) \int_{\tau(t)}^{\sigma(t^*)} p(s)\Delta s \overset{(6)}{\geq} \frac{M^2}{4} y(\tau(t^*)), \end{aligned}$$

which proves (4). □

4. Applications

Example 1. Clearly, if $\mathbb{T} = \mathbb{R}$, then we get

$$\sup_{\substack{\lambda > 0 \\ -\lambda p \in \mathcal{R}^+}} \{\lambda e_{-\lambda p}(t, \tau(t))\} = \sup_{\lambda > 0} \left\{ \lambda e^{-\lambda \int_{\tau(t)}^{t} p(s)ds} \right\} = \frac{1}{e \int_{\tau(t)}^{t} p(s)ds},$$

and hence Theorem 1 yields the well-known result cited in the introduction as

$$\alpha < 1 \quad \Longleftrightarrow \quad \liminf_{t \to \infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}.$$

We now consider a time scale of the form

$$\mathbb{T} = \{t_n : n \in \mathbb{Z}\}, \tag{8}$$

where $\{t_n\}$ is a strictly increasing sequence of real numbers such that $\mathbb{T}$ is closed. For such time scales we present the following results.

Corollary 1. *Consider a time scale as described in* (8). *If*

$$y^{\Delta}(t) + p(t)y(\rho(t)) = 0 \quad \textit{for} \quad t \in \mathbb{T}$$

has an eventually positive solution, then

$$\liminf_{t \to \infty} \{\mu(t)p(t)\} \leq \frac{1}{4}.$$

Proof. We let $\tau(t) = \rho(t)$, find

$$\lambda e_{-\lambda p}(t, \tau(t)) = \lambda - \lambda^2 \mu(\tau(t))p(\tau(t)),$$

maximize, and apply Theorem 1. □

Example 2. If

$$y(4t) = y(t) - \mu(t)p(t)y(t/4) \quad \text{for} \quad t \in \{4^n : n \in \mathbb{N}_0\}$$

has an eventually positive solution, then

$$\liminf_{n \to \infty} \{4^n p(4^n)\} \leq \frac{1}{12}.$$

Corollary 2. *Consider a time scale as described in* (8)*. If*

$$y^{\Delta}(t)+p(t)y(\rho(\rho(t)))=0 \quad \textit{for} \quad t\in\mathbb{T}$$

has an eventually positive solution, then

$$\liminf_{t\to\infty}\frac{[N(t)+M(t)]\,[N(\sigma(t))+M(t)]}{[N(t)+N(\sigma(t))+M(t)]^3}\geq 1,$$

where

$$N(t)=\mu(t)p(t) \quad \textit{and} \quad M(t)=\sqrt{(N(t))^2+(N(\sigma(t))^2-N(t)N(\sigma(t))}.$$

Proof. We let $\tau(t)=\rho(\rho(t))$, find

$$\lambda e_{-\lambda p}(t,\tau(t))=\lambda\left[1-\lambda N(\rho(\rho(t)))\right]\left[1-\lambda N(\rho(t))\right],$$

maximize, and apply Theorem 1. □

Example 3. Let $h>0$. If

$$y(t+h)=y(t)-hp(t)y(t-2h) \quad \text{for} \quad t\in\{hn:\ n\in\mathbb{Z}\}=:h\mathbb{Z}$$

has an eventually positive solution, then

$$\liminf_{\substack{t\to\infty\\ t\in h\mathbb{Z}}}\frac{\left[p(t)+\tilde{M}(t)\right]\left[p(t+h)+\tilde{M}(t)\right]}{\left[p(t)+p(t+h)+\tilde{M}(t)\right]^3}\geq h,$$

where

$$\tilde{M}(t)=\sqrt{(p(t))^2+(p(t+h))^2-p(t)p(t+h)}.$$

Theorem 2. *Consider a time scale as described in* (8)*. Let $k\in\mathbb{N}$ and $\tau(t_n)=t_{n-k}$ for all $n\in\mathbb{Z}$. If* (1) *has an eventually positive solution, then*

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t}p(s)\Delta s\leq\left(\frac{k}{k+1}\right)^{k+1}. \tag{9}$$

Proof. We assume that (9) does not hold and show $\alpha<1$, which is a contradiction with Theorem 1. Note now that

$$\begin{aligned}\lambda e_{-\lambda p}(t_n,\tau(t_n)) &= \lambda\prod_{i=n-k}^{n-1}\left[1-\lambda\mu(t_i)p(t_i)\right]\\ &\leq \lambda\left\{1-\lambda\frac{\int_{\tau(t_n)}^{t_n}p(s)\Delta s}{k}\right\}^k\\ &= \lambda(1-\lambda S)^k,\end{aligned}$$

where we used the arithmetic-geometric inequality and put

$$S=\frac{1}{k}\int_{t_{n-k}}^{t_n}p(s)\Delta s=\frac{\sum_{i=n-k}^{n-1}(t_{i+1}-t_i)p(t_i)}{k}.$$

Now $f(\lambda)=\lambda(1-\lambda S)^k$ satisfies

$$f'(\lambda)=(1-\lambda S)^k-k\lambda S(1-\lambda S)^{k-1}=(1-\lambda S)^{k-1}\left[1-(k+1)\lambda S\right]$$

so that

$$f(\lambda)\leq f\left(\frac{1}{(k+1)S}\right)=\frac{1}{(k+1)S}\left(1-\frac{1}{k+1}\right)^k=\frac{k^k}{S(k+1)^{k+1}}.$$

Hence $\alpha\leq\left(\frac{k}{k+1}\right)^{k+1}\frac{1}{\liminf_{t\to\infty}\int_{\tau(t)}^{t}p(s)\Delta s}<1.$ □

Example 4. If we let $\mathbb{T} = \mathbb{Z}$ in Theorem 2, then we get the following result: Let $k \in \mathbb{N}$. If

$$y(n+1) = y(n) - p(n)y(n-k) \quad \text{for} \quad n \in \mathbb{Z}$$

has an eventually positive solution, then

$$\liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) \le \left(\frac{k}{k+1}\right)^{k+1}.$$

Example 5. If we let $\mathbb{T} = \overline{q^{\mathbb{Z}}} := \left\{q^k : k \in \mathbb{Z}\right\} \cup \{0\}$ with $q > 1$ in Theorem 2, then we get the following result: Let $k \in \mathbb{N}$. If

$$y\left(q^{n+1}\right) = y\left(q^n\right) - (q-1)q^n p(q^n) y\left(q^{n-k}\right) \quad \text{for} \quad n \in \mathbb{Z}$$

has an eventually positive solution, then

$$\liminf_{n\to\infty} \sum_{i=n-k}^{n-1} q^i p(q^i) \le \frac{\left(\frac{k}{k+1}\right)^{k+1}}{q-1}.$$

5. Further Oscillation Criteria

In this last section we consider the equation

$$x^{\Delta}(t) + p(t)x(\tau(\sigma(t))) = 0, \tag{10}$$

where p and τ satisfy the same assumptions as before. Since $\lambda p \in \mathcal{R}^+$ for all $\lambda > 0$, clearly the quantity

$$\alpha^* := \liminf_{\substack{t\to\infty \\ t\in\mathbb{T}}} \inf_{\lambda>0} \left\{\frac{e_{\lambda p}(t, \tau(\sigma(t)))}{\lambda}\right\} \tag{11}$$

is well defined. Our main result about equation (10) reads as follows.

Theorem 3. *If* (10) *has an eventually positive solution, then* α^* *defined by* (11) *satisfies* $\alpha^* \le 1$.

Proof. Throughout we assume that x solves (10) and is eventually positive and that $\alpha^* > 1$. We proceed as in the proof of Theorem 1 and show that x satisfies

$$\liminf_{t\to\infty} \frac{x(\tau(\sigma(t)))}{x(t)} = \infty \tag{12}$$

and

$$\liminf_{t\to\infty} \frac{x(\tau(\sigma(t)))}{x(t)} < \infty. \tag{13}$$

This contradiction shows then $\alpha^* \le 1$ and finishes the proof. We first show (12). Let $\beta^* \in (1, \alpha^*)$. Then there exists $T_0 \in \mathbb{T}$ such that

$$\inf_{\lambda>0} \left\{\frac{e_{\lambda p}(t, \tau(t))}{\lambda}\right\} \ge \beta^* \quad \text{for all} \quad t \ge T_0. \tag{14}$$

As x is eventually positive, it is eventually decreasing, and hence we conclude that $x(\tau(\sigma(t))) \ge x(\sigma(t))$ eventually so that

$$0 = x^{\Delta}(t) + p(t)x(\tau(\sigma(t))) \ge x^{\Delta}(t) + p(t)x(\sigma(t))$$

implies by Remark 1 that there exists $T_1 \ge T_0$ with

$$\frac{x(\tau(\sigma(t)))}{x(\sigma(t))} \ge e_p(t, \tau(\sigma(t)))\frac{x(t)}{x(\sigma(t))} \overset{(14)}{\ge} \beta^* \frac{x(t)}{x(\sigma(t))} \ge \beta^* \quad \text{for all} \quad t \ge T_1.$$

Therefore

$$0 = x^{\Delta}(t) + p(t)x(\tau(\sigma(t))) \ge x^{\Delta}(t) + \beta^* p(t)x(t),$$

implies again by Remark 1 that there exists $T_2 \geq T_1$ with

$$\begin{aligned}\frac{x(\tau(\sigma(t)))}{x(\sigma(t))} &\geq e_{\beta^* p}(t,\tau(\sigma(t)))\frac{x(t)}{x(\sigma(t))} = \beta^* \frac{e_{\beta^* p}(t,\tau(\sigma(t)))}{\beta^*}\frac{x(t)}{x(\sigma(t))} \\ &\overset{(14)}{\geq} (\beta^*)^2 \frac{x(t)}{x(\sigma(t))} \geq (\beta^*)^2 \quad \text{for all} \quad t \geq T_2.\end{aligned}$$

Proceeding in a way similar as in the first part of the proof of Theorem 1, we obtain (12). Now we show (13). By Remark 2 (see also Remark 3) there exists $M > 0$ and $T \in \mathbb{T}$ such that

$$\int_{\tau(\sigma(t))}^{t} p(s)\Delta s \geq M \quad \text{for all} \quad t \geq T$$

so that

$$\int_{\tau(\sigma(t))}^{\sigma(t)} \geq \int_{\tau(\sigma(t))}^{t} p(s)\Delta s \geq M \quad \text{for all} \quad t \geq T$$

and hence

$$\int_{\tau(\sigma(t))}^{\sigma(t)} \geq M \quad \text{for all} \quad t \geq T \tag{15}$$

holds. As in the second part of the proof of Theorem 1 we may find $t^* \in [\tau(\sigma(t)), t)$ such that

$$\int_{\tau(\sigma(t))}^{\sigma(t^*)} p(s)\Delta s \geq \frac{M}{2} \quad \text{and} \quad \int_{t^*}^{\sigma(t)} p(s)\Delta s \geq \frac{M}{2}. \tag{16}$$

Now we can estimate

$$\begin{aligned} x(t^*) &\geq x(t^*) - x(\sigma(t)) \overset{(10)}{=} \int_{t^*}^{\sigma(t)} p(s)x(\tau(\sigma(s)))\Delta s \\ &\geq x(\tau(\sigma(t))) \int_{t^*}^{\sigma(t)} p(s)\Delta s \overset{(16)}{\geq} \frac{M}{2} x(\tau(\sigma(t))) \\ &\geq [x(\tau(\sigma(t))) - x(\sigma(t^*))] \overset{(10)}{=} \frac{M}{2} \int_{\tau(\sigma(t))}^{\sigma(t^*)} p(s)x(\tau(\sigma(s)))\Delta s \\ &\geq \frac{M}{2} x(\tau(\sigma(t^*))) \int_{\tau(\sigma(t))}^{\sigma(t^*)} p(s)\Delta s \overset{(16)}{\geq} \frac{M^2}{4} x(\tau(\sigma(t^*))), \end{aligned}$$

i.e.,

$$x(t^*) \geq \frac{M^2}{4} x(\tau(\sigma(t^*))). \tag{17}$$

Clearly (17) implies (13). □

We can improve the condition from Theorem 3 by imposing an additional assumption. Define now

$$\tilde{\alpha} := \liminf_{\substack{t\to\infty \\ t\in\mathbb{T}}} \inf_{\lambda>0} \left\{ \frac{e_{\lambda p}(t,\tau(t))}{\lambda} \right\} \tag{18}$$

Theorem 4. *Assume that there exists $K > 0$ such that*

$$\int_{\tau(\sigma(t))}^{t} p(s)\Delta s \geq K \quad \textit{for all large} \quad t \in \mathbb{T}. \tag{19}$$

If (10) has an eventually positive solution, then $\tilde{\alpha}$ defined by (18) satisfies $\tilde{\alpha} \leq 1$.

Proof. We assume that x solves (10) and is eventually positive and that $\tilde{\alpha} < 1$. We proceed as in the proofs of Theorems 1 and 3 in two parts to show

$$\liminf_{t\to\infty} \frac{x(\tau(t))}{x(t)} = \infty \tag{20}$$

and

$$\liminf_{t\to\infty} \frac{x(\tau(t))}{x(t)} < \infty. \tag{21}$$

With the same notation as in the proof of the first part of Theorem 3, we find

$$\frac{x(\tau(t))}{x(\sigma(t))} \geq e_p(t, \tau(\sigma(t))) \overset{(14)}{\geq} \beta^* \quad \text{for all} \quad t \geq T_1$$

and may proceed as in the proof of the first part of Theorem 3 to reach (20). To show (21), note that there exists $M > 0$ and $T \in \mathbb{T}$ such that (15) holds. Therefore we can proceed with the same calculation as in the second part of Theorem 3 to obtain (17). Observe now the estimate

$$\begin{aligned} x(\tau(\sigma(t))) &\geq x(\tau(\sigma(t))) - x(t) \overset{(10)}{=} \int_{\tau(\sigma(t))}^{t} p(s)x(\tau(\sigma(s)))\Delta s \\ &\geq x(\tau(t)) \int_{\tau(\sigma(t))}^{t} p(s)\Delta s \overset{(19)}{\geq} Kx(\tau(t)) \end{aligned}$$

for large $t \in \mathbb{T}$, which combined with (17) yields (21). □

Example 6. For $\mathbb{T} = \mathbb{Z}$ and $\tau(t) = t - 2$ for $t \in \mathbb{Z}$, we have

$$\frac{e_{\lambda p}(t, \tau(t))}{\lambda} = \frac{[1 + \lambda p(t-2)]\,[1 + \lambda p(t-1)]}{\lambda},$$

which is minimized for

$$\left(\sqrt{p(t-2)} + \sqrt{p(t-1)}\right)^2.$$

Hence by Theorem 4,

$$\liminf_{n\to\infty} \left(\sqrt{p(n)} + \sqrt{p(n+1)}\right)^2 > 1$$

and there exists $K > 0$ with $p(n) \geq K$ for all large $n \in \mathbb{N}$, then

$$x(n+1) = x(n) - p(n)x(n-1) \quad \text{for} \quad n \in \mathbb{Z} \tag{22}$$

is oscillatory.

Example 7. For a more specific example of the kind as discussed in Example 6, consider (22) with

$$p(n) = \begin{cases} \frac{1}{8} & \text{for } n \text{ even} \\ \frac{1}{2} & \text{for } n \text{ odd.} \end{cases}$$

Here,

$$\liminf_{n\to\infty} p(n) = \frac{1}{8} < \frac{1}{4},$$

so the oscillation criterion from Corollary 1, i.e., the one known in the literature for difference equations, does not apply. However,

$$\liminf_{n\to\infty} \left(\sqrt{p(n)} + \sqrt{p(n+1)}\right) = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{8}}\right) > 1,$$

so Example 6, i.e., Theorem 4 applies and shows that the equation (22) is oscillatory.

A similar discussion as in Example 6, applying Theorem 4 for $\mathbb{T} = \mathbb{Z}$, yields the statements given in the following last examples of this paper. Of course similar examples (e.g., of the forms as presented in Section 4) can also be given for other time scales.

Example 8. Consider a difference equation of the form

$$x(n+1) = x(n) - p(n)x(n-2) \quad \text{for} \quad n \in \mathbb{Z}, \tag{23}$$

where p is three-periodic and takes values as follows:

$$p(1) = a, \quad p(2) = a, \quad p(3) = b, \quad \ldots \quad \text{with} \quad a, b > 0.$$

Then (23) is oscillatory provided

$$\left(\frac{3a+M}{a+M}\right)^2 \frac{a+M+2b}{2} > 1, \quad \text{where} \quad M = \sqrt{a^2+8ab}.$$

Example 9. Consider equation (23), where p is three-periodic and takes values as follows:

$$p(1) = a, \quad p(2) = b, \quad p(3) = c, \quad \ldots \quad \text{with} \quad a, b, c > 0.$$

Let

$$m = abc\left(\frac{3}{ab+ac+bc}\right)^{3/2}.$$

By the arithmetic-geometric inequality it can be shown that $0 < m \le 1$. Now define $\varphi \in [0, \pi/2)$ such that

$$m = \cos\varphi, \quad \text{and also put} \quad k = 2\cos\frac{\varphi}{3}.$$

Another way to calculate k is to use the formula

$$k = \left(m + i\sqrt{1-m^2}\right)^{1/3} + \left(m + i\sqrt{1-m^2}\right)^{-1/3}.$$

Then (23) is oscillatory provided

$$a+b+c+\frac{\sqrt{3(ab+ac+bc)}}{2}\left[k+\frac{1}{k}\right] > 1.$$

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