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SOME OSCILLATION RESULTS FOR SECOND-ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. This article is concerned with the oscillation of second-order neutral differential equations. Four new oscillation criteria for such equations are given and illustrated by some examples.

1. Introduction

It is well known that neutral differential equations have many applications, e.g., in physics and biology. Hence, there is a permanent interest in obtaining sufficient conditions for the oscillation of solutions of various classes of second-order neutral differential equations, see e.g., [1–9]. Following this trend, this article is concerned with the oscillatory behavior of the second-order nonlinear neutral functional differential equation

$$\big(r(t)|Z'(t)|^{\alpha-1}Z'(t)\big)' + q(t)f(x(\sigma(t))) = 0, \quad t \ge t_0 > 0, \tag{1.1}$$

where

$$Z(t) := x(t) + p(t)x(\tau(t)) \quad \text{and} \quad \alpha > 0.$$

Furthermore, we assume that the following conditions hold.

(A_1) $r, p, q, \in C(I, \mathbb{R})$, $\tau, \sigma \in C^1(I, \mathbb{R})$, $r(t) > 0$, $p(t) \ge 1$, $p(t) \not\equiv 1$ eventually, τ, σ are strictly increasing on I,

$$\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty,$$

and $q(t) \ge 0$ for $t \in I$, $I = [t_0, \infty)$.

(A_2) $f \in C^1(\mathbb{R}, \mathbb{R})$ and there exists $k > 0$ such that

$$\frac{f(x)}{|x|^{\alpha-1}x} \ge k \quad \text{for all} \quad x \ne 0.$$

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By a solution of (1.1) we mean a function $x \in \mathrm{C}([T_x, \infty))$, $T_x \geq t_0$, which satisfies (1.1) on $[T_x, \infty)$. We consider only those solutions x of (1.1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$, and otherwise it is said to be nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Regarding the known oscillation criteria for equations of the form (1.1), Grammatikopoulos et al. [4] studied the neutral equation

$$\big(x(t) + p(t)x(t-\tau)\big)'' + q(t)x(t-\sigma) = 0 \quad \text{for} \quad 0 \leq p(t) \leq 1.$$

Ye and Xu [9] examined the neutral differential equation (1.1) for the case when $0 \leq p(t) \leq 1$. Agarwal et al. [2] considered (1.1) for the case when $p(t) > 1$, $\tau(t) > t$, and $\int_{t_0}^{\infty} \frac{\mathrm{d}s}{r^{\frac{1}{\alpha}}(s)} = \infty$. Later, Baculíková and Džurina [3] investigated the oscillatory nature of differential equations of the type

$$\big(r(t)(x(t) + p(t)x(\tau(t)))'\big)' + q(t)x(\sigma(t)) = 0, \tag{1.2}$$

under the assumptions

$$\int_{t_0}^{\infty} \frac{\mathrm{d}s}{r(s)} = \infty, \quad 0 \leq p(t) \leq p_0 < \infty,$$
$$\tau'(t) \geq \tau_0 > 0, \quad \text{and} \quad \tau \circ \sigma = \sigma \circ \tau.$$

Han et al. [5] studied the equation (1.2) in the case when

$$\int_{t_0}^{\infty} \frac{\mathrm{d}s}{r(s)} < \infty, \quad 0 \leq p(t) \leq p_0 < \infty,$$
$$\tau'(t) \geq \tau_0 > 0, \quad \text{and} \quad \tau \circ \sigma = \sigma \circ \tau.$$

Recently, Li et al. [7] considered equation (1.1) under the conditions

$$0 \leq p(t) \leq p_0 < \infty, \quad \tau'(t) \geq \tau_0 > 0,$$
$$\tau \circ \sigma = \sigma \circ \tau, \quad \tau(t) \geq t, \quad \text{and} \quad \sigma(t) \geq t.$$

Obviously, the assumptions given in [7] are quite restrictive. Now a problem is how to derive some new oscillation criteria for (1.1) which remove those restrictions. Motivated by an idea from [2], in this paper we give some new oscillation criteria for equation (1.1).

In this paper, we also use the following notation:

(A_3) τ^{-1} is the inverse function of τ and

$$p^*(t) := \frac{1}{p(\tau^{-1}(t))}\left(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\right) > 0,$$

$$p_*(t) := \frac{1}{p(\tau^{-1}(t))}\left(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\frac{m(\tau^{-1}(\tau^{-1}(t)))}{m(\tau^{-1}(t))}\right) > 0$$

(for all sufficiently large t, where m will be specified later),

$$Q(t) := q(t)(p^*(\sigma(t)))^\alpha, \quad Q_*(t) := q(t)(p_*(\sigma(t)))^\alpha,$$

$$\pi(t) := \int_t^\infty \frac{\mathrm{d}s}{r^{\frac{1}{\alpha}}(s)}, \quad \text{and} \quad (\rho'(t))_+ := \max\{0, \rho'(t)\}.$$

2. Oscillation Criteria

In this section, we establish four new oscillation criteria for (1.1). All occurring functional inequalities are assumed to hold eventually, i.e., they are satisfied for all t large enough.

Theorem 2.1. *Assume* (A_1)–(A_3). *Let* $\pi(t_0) < \infty$,

$$\tau(t) > t \quad \text{and} \quad \sigma(t) \le \tau(t) \quad \text{for all} \quad t \in I.$$

If there exist functions $\rho, m \in \mathrm{C}^1(I, (0,\infty))$ *such that*

$$\frac{m(t)}{r^{\frac{1}{\alpha}}(t)\pi(t)} + m'(t) \le 0, \tag{2.1}$$

$$\int_{t_0}^\infty \left[\rho(s)Q(s) - \frac{1}{k(\alpha+1)^{\alpha+1}}\frac{((\rho'(s))_+)^{\alpha+1} r(\tau^{-1}(\sigma(s)))}{\rho^\alpha(s)((\tau^{-1}\circ\sigma)'(s))^\alpha}\right]\mathrm{d}s = \infty, \tag{2.2}$$

and

$$\int_{t_0}^\infty \left[kq(s)\pi^\alpha(s)(p_*(\sigma(s)))^\alpha - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{\pi(s)r^{\frac{1}{\alpha}}(s)}\right]\mathrm{d}s = \infty \tag{2.3}$$

hold, then (1.1) *is oscillatory.*

Proof: Let x be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \ge t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \ge t_1$. Then it follows from (1.1) that

$$\left(r(t)|Z'(t)|^{\alpha-1}Z'(t)\right)' \le -kq(t)x^\alpha(\sigma(t)) \le 0 \quad \text{for all} \quad t \ge t_1. \tag{2.4}$$

Hence there exist two possible cases of the sign of $Z'(t)$. If $Z'(t) > 0$, then as in [2, (8.6)], we get

$$x(t) \ge p^*(t)Z(\tau^{-1}(t)). \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$\left(r(t)(Z'(t))^{\alpha}\right)' + kq(t)(p^*(\sigma(t)))^{\alpha}(Z(\tau^{-1}(\sigma(t))))^{\alpha} \le 0. \tag{2.6}$$

Define

$$u(t) := \rho(t)\frac{r(t)(Z'(t))^{\alpha}}{(Z(\tau^{-1}(\sigma(t))))^{\alpha}}.$$

Similar as in the proof of [9, Theorem 2.1], we can obtain a contradiction to (2.2). If $Z'(t) < 0$, then we define the function ω by

$$\omega(t) := \frac{r(t)(-Z'(t))^{\alpha-1}Z'(t)}{Z^{\alpha}(t)}, \quad t \ge t_1. \tag{2.7}$$

Then $\omega(t) < 0$ for $t \ge t_1$. Recalling (2.4), we have that $r|Z'|^{\alpha-1}Z'$ is nonincreasing. Thus, we get

$$(r(s))^{\frac{1}{\alpha}}Z'(s) \le (r(t))^{\frac{1}{\alpha}}Z'(t) \quad \text{for all} \quad s \ge t \ge t_1. \tag{2.8}$$

Dividing (2.8) by $(r(s))^{\frac{1}{\alpha}}$ and integrating the resulting inequality from t to ℓ, we get

$$Z(\ell) \le Z(t) + (r(t))^{\frac{1}{\alpha}}Z'(t)\int_t^{\ell}\frac{\mathrm{d}s}{(r(s))^{\frac{1}{\alpha}}} \quad \text{for all} \quad l \ge t \ge t_1. \tag{2.9}$$

Letting $\ell \to \infty$ in (2.9), we obtain

$$0 \le Z(t) + (r(t))^{\frac{1}{\alpha}}Z'(t)\pi(t) \quad \text{for all} \quad t \ge t_1,$$

i.e.,

$$(r(t))^{\frac{1}{\alpha}}\pi(t)\frac{Z'(t)}{Z(t)} \ge -1 \quad \text{for all} \quad t \ge t_1. \tag{2.10}$$

Hence, (2.10) together with (2.7) yields

$$-1 \le \omega(t)\pi^{\alpha}(t) \le 0 \quad \text{for all} \quad t \ge t_1. \tag{2.11}$$

On the other hand, (2.10) also gives

$$\frac{Z'(t)}{Z(t)} \ge -\frac{1}{r^{\frac{1}{\alpha}}(t)\pi(t)} \quad \text{for all} \quad t \ge t_1. \tag{2.12}$$

By (2.12) and (2.1), we have

$$\begin{aligned}\left(\frac{Z}{m}\right)'(t) &= \frac{Z'(t)m(t) - Z(t)m'(t)}{m^2(t)} \\ &\ge -\frac{Z(t)}{m^2(t)}\left[\frac{m(t)}{r^{\frac{1}{\alpha}}(t)\pi(t)} + m'(t)\right] \ge 0\end{aligned}$$

so that Z/m is nondecreasing on $[t_1,\infty)$. Since $\tau^{-1}(t)\ge\tau^{-1}(\tau^{-1}(t))$ by (A_1) and

$$\begin{aligned} x(t) &= \frac{1}{p(\tau^{-1}(t))}\left(Z(\tau^{-1}(t)) - x(\tau^{-1}(t))\right) \\ &\ge \frac{Z(\tau^{-1}(t))}{p(\tau^{-1}(t))} - \frac{Z(\tau^{-1}(\tau^{-1}(t)))}{p(\tau^{-1}(t))p(\tau^{-1}(\tau^{-1}(t)))} \end{aligned} \tag{2.13}$$

as in [2, (8.6)], we therefore get

$$\begin{aligned} x(t) &\ge \frac{1}{p(\tau^{-1}(t))}\left(1 - \frac{1}{p(\tau^{-1}(\tau^{-1}(t)))}\frac{m(\tau^{-1}(\tau^{-1}(t)))}{m(\tau^{-1}(t))}\right)Z(\tau^{-1}(t)) \\ &= p_*(t)Z(\tau^{-1}(t)). \end{aligned} \tag{2.14}$$

Now differentiating (2.7), we get

$$\omega'(t) = \frac{(r(t)(-Z'(t))^{\alpha-1}Z'(t))'}{Z^\alpha(t)} - \frac{\alpha r(t)(-Z'(t))^{\alpha-1}Z^{\alpha-1}(t)(Z'(t))^2}{Z^{2\alpha}(t)},$$

from where (2.4) and (2.14) imply

$$\omega'(t) \le -kq(t)(p_*(\sigma(t)))^\alpha - \frac{\alpha r(t)(-Z'(t))^{\alpha-1}Z^{\alpha-1}(t)(Z'(t))^2}{Z^{2\alpha}(t)} \tag{2.15}$$

on noting that we assume $\sigma(t)\le\tau(t)$. Thus, from (2.7) and (2.15), we have

$$\omega'(t) + kq(t)(p_*(\sigma(t)))^\alpha + \frac{\alpha}{r^{\frac{1}{\alpha}}(t)}(-\omega(t))^{\frac{\alpha+1}{\alpha}} \le 0,\ t\ge t_1. \tag{2.16}$$

Multiplying (2.16) by $\pi^\alpha(t)$, integrating the resulting inequality from t_1 to t, and using integration by parts implies

$$\begin{aligned} 0 &\ge \pi^\alpha(t)\omega(t) - \pi^\alpha(t_1)\omega(t_1) + \alpha\int_{t_1}^t r^{-\frac{1}{\alpha}}(s)\pi^{\alpha-1}(s)\omega(s)\mathrm{d}s \\ &+ k\int_{t_1}^t q(s)(p_*(\sigma(s)))^\alpha\pi^\alpha(s)\mathrm{d}s + \alpha\int_{t_1}^t \frac{\pi^\alpha(s)}{r^{\frac{1}{\alpha}}(s)}(-\omega(s))^{\frac{\alpha+1}{\alpha}}\mathrm{d}s. \end{aligned} \tag{2.17}$$

We now use Young's inequality

$$|ab| \le \frac{1}{p}|a|^p + \frac{1}{q}|b|^q \quad \text{for} \quad a,b\in\mathbb{R}, \quad p,q>1, \quad \frac{1}{p}+\frac{1}{q}=1$$

with $p=(\alpha+1)/\alpha$, $q=\alpha+1$, and

$$a = (\alpha+1)^{\frac{\alpha}{\alpha+1}}\pi^{\frac{\alpha^2}{\alpha+1}}(t)\omega(t), \quad b = \frac{\alpha}{(\alpha+1)^{\frac{\alpha}{\alpha+1}}}\pi^{-\frac{1}{\alpha+1}}(t)$$

to obtain

$$-\alpha\pi^{\alpha-1}(t)\omega(t) \le \alpha\pi^\alpha(t)(-\omega(t))^{\frac{\alpha+1}{\alpha}} + \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{\pi(t)},$$

i.e.,

$$-\alpha\frac{\pi^{\alpha-1}(t)\omega(t)}{r^{\frac{1}{\alpha}}(t)} \le \alpha\frac{\pi^\alpha(t)(-\omega(t))^{\frac{\alpha+1}{\alpha}}}{r^{\frac{1}{\alpha}}(t)} + \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{\pi(t)r^{\frac{1}{\alpha}}(t)},$$

from which we conclude by (2.17) that

$$0 \geq \pi^\alpha(t)\omega(t) - \pi^\alpha(t_1)\omega(t_1) + \int_{t_1}^{t} \left[kq(s)\pi^\alpha(s)(p_*(\sigma(s)))^\alpha - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{1}{\pi(s)r^{\frac{1}{\alpha}}(s)} \right] \mathrm{d}s$$

holds, a contradiction to (2.11) when using (2.3). This completes the proof.

Theorem 2.2. *Assume* (A_1)–(A_3). *Let* $\pi(t_0) < \infty$,

$$\tau(t) > t \quad \textit{and} \quad \sigma(t) \geq \tau(t) \quad \textit{for all} \quad t \in I.$$

If there exist functions $\rho, m \in \mathrm{C}^1(I, (0, \infty))$ *such that* (2.1),

$$\int_{t_0}^{\infty} \left[\rho(s)Q(s) - \frac{1}{k(\alpha+1)^{\alpha+1}} \frac{((\rho'(s))_+)^{\alpha+1} r(s)}{\rho^\alpha(s)} \right] \mathrm{d}s = \infty, \tag{2.18}$$

and

$$\int_{t_0}^{\infty} \left[kq(s)\pi^\alpha(s)(p_*(\sigma(s)))^\alpha \left(\frac{m(\tau^{-1}(\sigma(s)))}{m(s)} \right)^\alpha - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{1}{\pi(s)r^{\frac{1}{\alpha}}(s)} \right] \mathrm{d}s = \infty \tag{2.19}$$

hold, then (1.1) *is oscillatory.*

Proof: Let x be a nonoscillatory solution of (1.1). Proceeding as in the proof of Theorem 2.1, we have (2.4), and thus there exist two possible cases of the sign of $Z'(t)$. If $Z'(t) > 0$, as in [2, (8.6)], we get (2.5). It follows from (2.4) and (2.5) that (2.6) holds. Define

$$u(t) := \rho(t) \frac{r(t)(Z'(t))^\alpha}{(Z(t))^\alpha}.$$

Similar as in the proof of [9, Theorem 2.1], we can obtain a contradiction to (2.18). If $Z'(t) < 0$, then as in the proof of Theorem 2.1, Z/m is nondecreasing. Hence we get (2.14). Next, define the function ω by (2.7). Differentiating (2.7) and using the condition $\sigma(t) \geq \tau(t)$, we get

$$\omega'(t) \leq -kq(t)(p_*(\sigma(t)))^\alpha \left(\frac{m(\tau^{-1}(\sigma(t)))}{m(t)} \right)^\alpha - \frac{\alpha r(t)(-Z'(t))^{\alpha-1} Z^{\alpha-1}(t)(Z'(t))^2}{Z^{2\alpha}(t)}.$$

The rest of the proof is similar to the proof of Theorem 2.1 and so is omitted. This completes the proof.

We note that Theorem 2.1 and Theorem 2.2 focus on the oscillation of equation (1.1) for the case when $\tau(t) > t$. Now we will establish some oscillation results for (1.1) under the condition $\tau(t) < t$.

Theorem 2.3. *Assume* (A_1)–(A_3). *Let* $\pi(t_0) < \infty$,

$$\tau(t) < t \quad \textit{and} \quad \sigma(t) \leq \tau(t) \quad \textit{for all} \quad t \in I.$$

If there exist functions $\rho, m \in \mathrm{C}^1(I, (0,\infty))$ *such that*

$$\frac{m(t)}{r^{\frac{1}{\alpha}}(t) \int_{t_1}^{t} \frac{1}{r^{\frac{1}{\alpha}}(s)} \mathrm{d}s} - m'(t) \leq 0 \quad \textit{for all sufficiently large } t_1, \tag{2.20}$$

$$\int_{t_0}^{\infty} \left[\rho(s) Q_*(s) - \frac{1}{k(\alpha+1)^{\alpha+1}} \frac{((\rho'(s))_+)^{\alpha+1} r(\tau^{-1}(\sigma(s)))}{\rho^{\alpha}(s)((\tau^{-1} \circ \sigma)'(s))^{\alpha}} \right] \mathrm{d}s = \infty, \tag{2.21}$$

and

$$\int_{t_0}^{\infty} \left[kq(s) \pi^{\alpha}(s) (p^*(\sigma(s)))^{\alpha} - \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1} \frac{1}{\pi(s) r^{\frac{1}{\alpha}}(s)} \right] \mathrm{d}s = \infty \tag{2.22}$$

hold, then (1.1) *is oscillatory.*

Proof: Let x be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$. Then it follows from (1.1) that (2.4) holds. Hence there exist two possible cases of the sign of $Z'(t)$. If $Z'(t) > 0$, then as in [2, (8.6)], we get (2.13). On the other hand, we have

$$Z(t) = Z(t_1) + \int_{t_1}^{t} \frac{(r(s)(Z'(s))^{\alpha})^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}}(s)} \mathrm{d}s \geq \left(r^{\frac{1}{\alpha}}(t) \int_{t_1}^{t} \frac{1}{r^{\frac{1}{\alpha}}(s)} \mathrm{d}s \right) Z'(t).$$

Hence

$$\begin{aligned} \left(\frac{Z}{m} \right)'(t) &= \frac{Z'(t)m(t) - Z(t)m'(t)}{m^2(t)} \\ &\leq \frac{Z(t)}{m^2(t)} \left(\frac{m(t)}{r^{\frac{1}{\alpha}}(t) \int_{t_1}^{t} \frac{1}{r^{\frac{1}{\alpha}}(s)} \mathrm{d}s} - m'(t) \right) \leq 0 \end{aligned}$$

and thus Z/m is nonincreasing. From (2.13), we therefore obtain (2.14). It follows from (2.4) and (2.14) that

$$(r(t)(Z'(t))^{\alpha})' + kq(t)(p_*(\sigma(t)))^{\alpha}(Z(\tau^{-1}(\sigma(t))))^{\alpha} \leq 0. \tag{2.23}$$

Define

$$v(t) := \rho(t) \frac{r(t)(Z'(t))^{\alpha}}{(Z(\tau^{-1}(\sigma(t))))^{\alpha}}.$$

Similar to the proof of [9, Theorem 2.1], we can obtain a contradiction to (2.21). If $Z'(t) < 0$, then, using [2, (8.6)], we have (2.13). From $\tau^{-1}(t) \leq \tau^{-1}(\tau^{-1}(t))$ and (2.13), we get (2.5). The remainder of the proof is similar to the proof of Theorem 2.1 and hence is omitted. This completes the proof.

Theorem 2.4. *Assume* (A_1)–(A_3). *Let* $\pi(t_0) < \infty$,

$$\tau(t) < t \quad \text{and} \quad \sigma(t) \geq \tau(t) \quad \text{for all} \quad t \in I.$$

If there exist functions $\rho, m, h \in C^1(I, (0, \infty))$ *such that* (2.20),

$$\frac{h(t)}{r^{\frac{1}{\alpha}}(t)\pi(t)} + h'(t) \leq 0, \tag{2.24}$$

$$\int_{t_0}^{\infty} \left[\rho(s)Q_*(s) - \frac{1}{k(\alpha+1)^{\alpha+1}} \frac{((\rho'(s))_+)^{\alpha+1} r(s)}{\rho^\alpha(s)}\right] \mathrm{d}s = \infty, \tag{2.25}$$

and

$$\int_{t_0}^{\infty} \left[kq(s)\pi^\alpha(s)(p^*(\sigma(s)))^\alpha \left(\frac{h(\tau^{-1}(\sigma(s)))}{h(s)}\right)^\alpha - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{1}{\pi(s) r^{\frac{1}{\alpha}}(s)}\right] \mathrm{d}s = \infty \tag{2.26}$$

hold, then (1.1) *is oscillatory.*

Proof: Let x be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$. Then it follows from (1.1) that (2.4) holds. Therefore, there exist two possible cases of the sign of $Z'(t)$. If $Z'(t) > 0$, then as in [2, (8.6)], we get (2.13). On the other hand, we see that Z/m is nonincreasing due to Theorem 2.3. From (2.13), we have (2.14). It follows from (2.4) and (2.14) that (2.23) holds. Define

$$v(t) := \rho(t) \frac{r(t)(Z'(t))^\alpha}{(Z(t))^\alpha}.$$

Similar as in the proof of [9, Theorem 2.1], we can obtain a contradiction to (2.25). If $Z'(t) < 0$, then, similar as in the proof of Theorem 2.3, we get (2.5). On the other hand, by the proof of Theorem 2.1, we see that Z/h is nondecreasing. Thus, we have

$$\frac{Z(\tau^{-1}(\sigma(t)))}{Z(t)} \geq \frac{h(\tau^{-1}(\sigma(t)))}{h(t)}.$$

The rest of the proof is similar to the proof of Theorem 2.1 and so is omitted. The proof is complete.

3. Examples and Remarks

In this section, we give two examples to illustrate the main results.

Example 3.1. Consider the equation

$$\left(e^t\left(x(t)+2e^{2\pi}x(t+2\pi)\right)'\right)'+\sqrt{2}(1+2e^{2\pi})e^t x\left(t-\frac{\pi}{4}\right)=0,\ t\geq 1. \quad (3.1)$$

Let $\rho(t)=1$ and $m(t)=e^{-t}$. It follows from Theorem 2.1 that every solution of equation (3.1) is oscillatory. For example, $x(t)=\sin t$ is an oscillatory solution of (3.1).

Example 3.2. Consider the equation

$$\left(e^t\left(x(t)+2e^{2\pi}x(t-2\pi)\right)'\right)'+\sqrt{2}(1+2e^{2\pi})e^t x\left(t-\frac{\pi}{4}\right)=0,\ t\geq 1. \quad (3.2)$$

Let $\rho(t)=1$, $m(t)=e^t$ and $h(t)=e^{-t}$. It follows from Theorem 2.4 that every solution of equation (3.2) is oscillatory. For example, $x(t)=\sin t$ is an oscillatory solution of (3.2).

Remark 3.1. The obtained results improve some results in the literature, e.g., it is easy to see that criteria given in [5, 7] cannot be applied to equations (3.1) and (3.2).

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References

[1] Ravi P. Agarwal, Martin Bohner and Wan-Tong Li, *Nonoscillation and oscillation: theory for functional differential equations*, Volume 267 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 2004.

[2] Ravi P. Agarwal, Said R. Grace and Donal O'Regan, *The oscillation of certain higher-order functional differential equations*, Math. Comput. Modelling, 37(7-8), (2003), 705–728.

[3] B. Baculíková and J. Džurina. *Oscillation theorems for second order neutral differential equations*, Comput. Math. Appl., 61(1), (2011), 94–99.

[4] M. K. Grammatikopoulos, G. Ladas and A. Meimaridou, *Oscillations of second order neutral delay differential equations*, Rad. Mat., 1(2), (1985), 267–274.

[5] Zhenlai Han, Tongxing Li, Shurong Sun and Yibing Sun, *Remarks on the paper [Appl. Math. Comput. 207 (2009) 388–396]*, Appl. Math. Comput., 215(11), (2010), 3998–4007.

[6] G. S. Ladde, V. Lakshmikantham and B. G. Zhang, *Oscillation theory of differential equations with deviating arguments*, Volume 110 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker Inc., New York, 1987.

[7] Tongxing Li, Zhenlai Han, Chenghui Zhang and Li Hua, *Oscillation criteria for second-order superlinear neutral differential equations*, Abstr. Appl. Anal., pages Art. ID 367541, 17, (2011).

[8] Run Xu and Fanwei Meng, *Oscillation criteria for second order quasi-linear neutral delay differential equations*, Appl. Math. Comput., 192(1), (2007), 216–222.

[9] Luhong Ye and Zhiting Xu, *Oscillation criteria for second order quasilinear neutral delay differential equations*, Appl. Math. Comput., 207(2), (2009), 388–396.

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