# Symplectic Systems and Related Discrete Quadratic Functionals

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September 4, 1996

**Abstract**

We establish a characterization of positive definiteness of a certain discrete quadratic functional associated to a symplectic system via disconjugacy of this system and positive definiteness of some related matrix. This generalized discrete version of Jacobi's condition also applies to linear Hamiltonian difference systems and to Sturm-Liouville difference equations of higher order for those objects are special cases of symplectic systems.

**Keywords.** Symplectic System, Disconjugacy, Controllability.

**AMS Subject Classifications.** 39A10, 39A12, 93C55.

# 1. Introduction

In this work we wish to examine so-called symplectic systems of the form

$$\text{(S)} \qquad z_{k+1} = S_k z_k, \qquad M \le k \le N,$$

as well as related discrete quadratic functionals

$$\mathcal{F}(z) = \sum_{k=M}^{N} z_k^T \left\{ S_k^T \mathcal{M} S_k - \mathcal{M} \right\} z_k + z_*^T T z_*.$$

Here, $S_k$ are given symplectic $2n \times 2n$-matrices for $M \le k \le N$ ($M, N \in \mathbb{Z}$, $n \in \mathbb{N}$) while $z_M, z_* := \mathcal{M}^T \mathcal{J} z_M + \mathcal{M} z_{N+1} \in \mathbb{R}^{2n}$, $z = (z_k)_{M \le k \le N+1}$,

$$\mathcal{M} = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

with $n \times n$-zero-matrices and with $n \times n$-identity-matrices, and $T$ is some given $2n \times 2n$-matrix. A $2n \times 2n$-matrix $S$ is called symplectic provided $S^T \mathcal{J} S = \mathcal{J}$ holds.

Systems (S) and functionals $\mathcal{F}$ have been introduced by the author and Ondřej Došlý in [8], and systems of the form (S) are also studied in the recently appeared book by Calvin Ahlbrandt and Allan Peterson [3]. These systems have the advantage that they include so-called linear Hamiltonian difference systems (see [1, 6, 9, 10, 11]) and hence also discrete Sturm-Liouville difference equations (see [2, 7]) as special cases. Thus our main result below applies to all those important objects also.

Since functionals of the form $\mathcal{F}$ arise as so-called second variations when trying to solve certain discrete variational problems (see e.g. [4]), we are mainly concerned with the notion of positive definiteness for functionals $\mathcal{F}$. Here, $\mathcal{F}$ is called positive definite - and we write $\mathcal{F} > 0$ - provided

$$\mathcal{F}(z) > 0 \text{ for all } z \text{ with } \mathcal{M} z \neq 0,\ \mathcal{M} S_k z_k = \mathcal{M} z_{k+1} (M \le k \le N),\ z_* \in \operatorname{Im} R^T$$

holds with some given $2n \times 2n$-matrix $R$.

We will proceed in the following manner. The next section will contain our main result that characterizes positive definiteness as described above; however, its proof will be postponed to Section 6. While Sections 3 and 4 are concerned with preliminaries for symplectic systems and discrete quadratic functionals, respectively, Section 5 is devoted to the study of some special, to (S) related, symplectic system which is needed for proving our main result in the last section.

## 2. The Main Result on Positive Definiteness

First, we provide the notation that is necessary to state our main result. We let $\mathcal{Z}_k$, $M \le k \le N+1$, be the solution of

$$\mathcal{Z}_{k+1} = S_k \mathcal{Z}_k, \qquad M \le k \le N, \qquad \mathcal{Z}_M = \mathcal{J}^T.$$

Furthermore, we put $X_k = (I\ 0)\mathcal{Z}_k\binom{I}{0}$, $B_k = (I\ 0)S_k\binom{0}{I}$, and

$$Q^*_{N+1} = (\mathcal{M} + \mathcal{Z}^T_{N+1}\mathcal{M}^T)^\dagger(\mathcal{Z}^T_{N+1}\mathcal{M}\mathcal{Z}_{N+1} + \mathcal{M}^T)(\mathcal{M}^T + \mathcal{M}\mathcal{Z}_{N+1})^\dagger,$$

where $M^\dagger$ denotes the Moore-Penrose inverse of the matrix $M$, i.e., the unique matrix that satisfies $MM^\dagger M = M$ and $M^\dagger M M^\dagger = M^\dagger$ such that both $MM^\dagger$ and $M^\dagger M$ are symmetric. Let $\mathrm{Ker}M$ and $\mathrm{Im}M$ denote the kernel and the image of the matrix $M$, respectively. We say that the system (S) is disconjugate (on $[M, N+1] \cap \mathbb{Z}$) provided

$$\mathrm{Ker}X_{k+1} \subset \mathrm{Ker}X_k, \quad X_k X^\dagger_{k+1} B_k \ge 0 \quad \text{for all} \quad M \le k \le N$$

holds, where $M \ge 0$ means positive definiteness of the matrix $M$. Finally, condition (B) means

$$R(T + Q^*_{N+1})R^T > 0 \text{ on } \mathrm{Im}R$$

while condition (C) stands for

$$\mathrm{Im}R^T \subset \mathrm{Im}(\mathcal{M}^T + \mathcal{M}\mathcal{Z}_{N+1}).$$

With this notation our main result now reads as follows.

**Theorem 1.**

(i) If (S) is disconjugate and if (B) holds, then $\mathcal{F} > 0$.

(ii) If $\mathcal{F} > 0$ and if (C) holds, then (B) holds and (S) is disconjugate.

If we impose the condition of controllability (on $[M, N+1] \cap \mathbb{Z}$) on the system (S) which means that

$$\mathcal{M}z = 0 \text{ implies } z = 0 \text{ for all solutions } z \text{ of (S)},$$

then we may rewrite Theorem 1 above as follows.

**Theorem 2.** Suppose that (S) is controllable.
Then (S) is disconjugate and (B) holds if and only if $\mathcal{F}$ is positive definite.

The proofs of Theorem 1 and Theorem 2 are performed in the last section of this paper. For them we need some preliminaries on both symplectic systems and discrete quadratic functionals.

## 3. Symplectic Systems

We start this section with the following basic lemma on symplectic matrices.

**Lemma 1.** If $S$ is symplectic, then so are $S^{-1}$ and $S^T$.
**Proof.** Let $S$ be symplectic, i.e., $S^T\mathcal{J}S = \mathcal{J}$ hold. Then $S$ is invertible since $\mathcal{J}$ is, and we have

$$\mathcal{J} = (S^T)^{-1}\mathcal{J}S^{-1} = (S^{-1})^T\mathcal{J}S^{-1}$$

so that $S^{-1}$ is symplectic. Moreover, $\mathcal{J}^{-1} = \mathcal{J}^T = -\mathcal{J}$ implies $S = \mathcal{J}^{-1}(S^T)^{-1}\mathcal{J}$ and hence

$$(S^T)^T\mathcal{J}S^T = \mathcal{J}^{-1}(S^T)^{-1}\mathcal{J}\mathcal{J}S^T = -\mathcal{J}^{-1}(S^T)^{-1}S^T = -\mathcal{J}^{-1} = \mathcal{J}$$

so that $S^T$ is symplectic also. ■

For a $2n \times n$-matrix $Z_M$ the sequence $Z = (Z_k)_{M \le k \le N+1}$ with $Z_{k+1} = S_k Z_k$, $M \le k \le N$, is called a *conjoined basis* of (S) provided

$$Z_M^T\mathcal{J}Z_M = 0 \quad \text{and} \quad \text{rank} Z_M = n$$

hold. Let $Z$ and $\tilde{Z}$ be two conjoined bases of (S). Then the sequence

$$\mathcal{Z} = (Z_k)_{M \le k \le N+1} \quad \text{with} \quad \mathcal{Z}_k = (Z_k \; \tilde{Z}_k), \; M \le k \le N+1,$$

is called a *normalized conjoined basis* of (S) if $\mathcal{Z}_M$ is symplectic.

**Lemma 2** *(Wronskian Identity).* Let $F_k, G_k$, $M \le k \le N+1$, be any matrices of size $2n \times m_f$ and $2n \times m_g$ with $m_f, m_g \in \mathbb{N}$, respectively, such that

$$F_{k+1} = S_k F_k, \quad G_{k+1} = S_k G_k, \qquad M \le k \le N.$$

Then $F_k^T\mathcal{J}G_k = F_M^T\mathcal{J}G_M$ for all $M \le k \le N+1$.
**Proof.** For $M \le k \le N$ we have

$$F_{k+1}^T\mathcal{J}G_{k+1} = F_k^T S_k^T\mathcal{J}S_k G_k = F_k^T\mathcal{J}G_k$$

so that our assertion follows immediately. ■

Of course, the Wronskian identity implies that $\mathcal{Z}_k$ is symplectic for all $M \le k \le N+1$ whenever $\mathcal{Z}$ is a normalized conjoined basis of (S). Also, if $Z$ is a conjoined basis of (S) we have that $Z_k^T\mathcal{J}Z_k = 0$ holds for all $M \le k \le N+1$. Finally, $\text{rank} Z_k = n$ for all $M \le k \le N+1$ in this case is trivial since the matrices $S_k$ are all symplectic and hence of full rank. We conclude this section with the following easy observation.

**Lemma 3.** If $Z$ is a conjoined basis of (S), then there exists another conjoined basis $\tilde{Z}$ of (S) such that $\mathcal{Z} = (Z\ \tilde{Z})$ is a normalized conjoined basis of (S).
**Proof.** Let $Z$ be a conjoined basis of (S) and put

$$\tilde{Z}_M = \mathcal{J}^{-1} Z_M (Z_M^T Z_M)^{-1}.$$

Observe that the occuring inverse exists due to the fact of $\operatorname{rank} Z_M = n$. Hence $\operatorname{rank}\tilde{Z}_M = \operatorname{rank} Z_M = n$ and

$$\begin{aligned} \tilde{Z}_M^T \mathcal{J} \tilde{Z}_M &= (Z_M^T Z_M)^{-1} Z_M^T \mathcal{J} \mathcal{J} \mathcal{J}^{-1} Z_M (Z_M^T Z_M)^{-1} \\ &= (Z_M^T Z_M)^{-1} Z_M^T \mathcal{J} Z_M (Z_M^T Z_M)^{-1} = 0 \end{aligned}$$

so that $\tilde{Z}$ defined by $\tilde{Z}_{k+1} = S_k \tilde{Z}_k$, $M \le k \le N$, is indeed a conjoined basis of (S). Moreover, we have

$$Z_M^T \mathcal{J} \tilde{Z}_M = Z_M^T \mathcal{J} \mathcal{J}^{-1} Z_M (Z_M^T Z_M)^{-1} = I$$

and hence $\mathcal{Z}_M = (Z_M\ \tilde{Z}_M)$ yields

$$\mathcal{Z}_M^T \mathcal{J} \mathcal{Z}_M = \begin{pmatrix} Z_M^T \\ \tilde{Z}_M^T \end{pmatrix} \mathcal{J} (Z_M\ \tilde{Z}_M) = \begin{pmatrix} Z_M^T \mathcal{J} Z_M & Z_M^T \mathcal{J} \tilde{Z}_M \\ \tilde{Z}_M^T \mathcal{J} Z_M & \tilde{Z}_M^T \mathcal{J} \tilde{Z}_M \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = \mathcal{J}.$$

Thus $\mathcal{Z}$ with $\mathcal{Z}_k = (Z_k\ \tilde{Z}_k)$, $M \le k \le N+1$, is a normalized conjoined basis of (S). ■

# 4. Discrete Quadratic Functionals

**Proposition 1.** If $z$ solves (S), then we have

$$\mathcal{F}(z) = z_{N+1}^T \mathcal{M} z_{N+1} - z_M^T \mathcal{M} z_M + z_*^T T z_*.$$

**Proof.** Let $z$ solve (S). This implies

$$\begin{aligned} \sum_{k=M}^{N} z_k^T \left\{ S_k^T \mathcal{M} S_k - \mathcal{M} \right\} z_k &= \sum_{k=M}^{N} \left\{ z_{k+1}^T \mathcal{M} z_{k+1} - z_k^T \mathcal{M} z_k \right\} \\ &= \sum_{k=M}^{N} \Delta \left\{ z_k^T \mathcal{M} z_k \right\} = z_{N+1}^T \mathcal{M} z_{N+1} - z_M^T \mathcal{M} z_M, \end{aligned}$$

and hence our simple result follows. ■

Our next result may be viewed as a symplectic version of Picone's identity. Under certain assumptions it provides a formula for $\mathcal{F}(z)$ also for these cases when $z$ is not a solution of (S).

**Proposition 2** *(Picone's Identity).* Let $M \leq k \leq N$. Suppose there exist $c_k, c_{k+1} \in \mathbb{R}^n$ such that

$$\mathcal{M} z_k = \mathcal{M} Z_k c_k \quad \text{and} \quad \mathcal{M} S_k z_k = \mathcal{M} Z_{k+1} c_{k+1}$$

hold, where $Z$ is any conjoined basis of (S). Then we have

$$z_k^T \left\{ S_k^T \mathcal{M} S_k - \mathcal{M} \right\} z_k = \Delta \left\{ c_k^T Z_k^T \mathcal{M}^T Z_k c_k \right\} + z_k^T \mathcal{J}^T Z_k \left\{ \Delta c_k \right\}.$$

**Proof.** We assume that such $c_k, c_{k+1}$ exist. This implies

$$\begin{aligned}
&\Delta \left\{ c_k^T Z_k^T \mathcal{M}^T Z_k c_k \right\} + z_k^T \mathcal{J}^T Z_k \left\{ \Delta c_k \right\} \\
&\quad = c_{k+1}^T Z_{k+1}^T \mathcal{M}^T Z_{k+1} c_{k+1} - c_k^T Z_k^T \mathcal{M}^T Z_k c_k + z_k^T \mathcal{J}^T Z_k c_{k+1} - z_k^T \mathcal{J}^T Z_k c_k \\
&\quad = z_k^T S_k^T \mathcal{M}^T Z_{k+1} c_{k+1} - z_k^T \mathcal{M}^T Z_k c_k - z_k^T \mathcal{J} Z_k c_{k+1} + z_k^T \mathcal{J} Z_k c_k \\
&\quad = z_k^T S_k^T \mathcal{M}^T S_k Z_k c_{k+1} - z_k^T \mathcal{J} Z_k c_{k+1} - \left\{ z_k^T \mathcal{M}^T Z_k c_k - z_k^T \mathcal{J} Z_k c_k \right\} \\
&\quad = z_k^T \left\{ S_k^T \mathcal{M}^T S_k - \mathcal{J} \right\} Z_k c_{k+1} - z_k^T \left\{ \mathcal{M}^T - \mathcal{J} \right\} Z_k c_k \\
&\quad = z_k^T \left\{ S_K^T (\mathcal{M} + \mathcal{J}) S_k - \mathcal{J} \right\} Z_k c_{k+1} - z_k^T \mathcal{M} Z_k c_k \\
&\quad = z_k^T S_k^T \mathcal{M} S_k Z_k c_{k+1} - z_k^T \mathcal{M} Z_k c_k \\
&\quad = z_k^T S_k^T \mathcal{M} S_k z_k - z_k^T \mathcal{M} z_k = z_k^T \left\{ S_k^T \mathcal{M} S_k - \mathcal{M} \right\} z_k
\end{aligned}$$

which is our desired result. ■

**Proposition 3.** Let $\operatorname{Ker}(\mathcal{M} S_k Z_k) \subset \operatorname{Ker}(\mathcal{M} Z_k)$ hold for some $M \leq k \leq N$, where $Z$ is any conjoined basis of (S). Then we have with $\mathcal{L} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$

$$\operatorname{Im}(\mathcal{M} S_k \mathcal{L}) \subset \operatorname{Im}(S_k Z_k).$$

Next, if $\mathcal{M} z_k = \mathcal{M} Z_k c_k$ holds for some $z_k, c_k \in \mathbb{R}^n$, then there exists $c_{k+1} \in \mathbb{R}^n$ with $\mathcal{M} S_k z_k = \mathcal{M} Z_{k+1} c_{k+1}$. Finally, in this case we have for some $s_k \in \mathbb{R}^n$

$$z_k^T \mathcal{J}^T Z_k \left\{ \Delta c_k \right\} = s_k^T D_k s_k,$$

where $D_k = X_k X_{k+1}^{\dagger} B_k$ is symmetric.

**Proof.** Let $Z$ be a conjoined basis of (S) with $\operatorname{Ker}(\mathcal{M} S_k Z_k) \subset \operatorname{Ker}(\mathcal{M} Z_k)$. We then pick according to Lemma 3 a conjoined basis $\tilde{Z}$ of (S) such that $\mathcal{Z} = (Z \; \tilde{Z})$

is a normalized conjoined basis of (S). Then (see Lemma 2) $\mathcal{Z}_k$ is symplectic and hence (see Lemma 1) so is $\mathcal{Z}_k^T$. Thus

$$\begin{aligned}
\mathcal{J} &= \mathcal{Z}_k \mathcal{J} \mathcal{Z}_k^T = (Z_k\ \tilde{Z}_k) \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} Z_k^T \\ \tilde{Z}_k^T \end{pmatrix} \\
&= (-\tilde{Z}_k\ Z_k) \begin{pmatrix} Z_k^T \\ \tilde{Z}_k^T \end{pmatrix} = Z_k \tilde{Z}_k^T - \tilde{Z}_k Z_k^T.
\end{aligned}$$

Let $c \in \operatorname{Ker}(Z_k^T S_k^T \mathcal{M}^T)$. Then

$$\begin{aligned}
0 &= \mathcal{M} S_k \tilde{Z}_k Z_k^T S_k^T \mathcal{M}^T c = \mathcal{M} S_k (Z_k \tilde{Z}_k^T - \mathcal{J}) S_k^T \mathcal{M}^T c \\
&= \mathcal{M} S_k Z_k \tilde{Z}_k^T S_k^T \mathcal{M}^T c - \mathcal{M} \mathcal{J} \mathcal{M}^T c \\
&= \mathcal{M} S_k Z_k \tilde{Z}_k^T S_k^T \mathcal{M}^T c.
\end{aligned}$$

Thus $\tilde{Z}_k^T S_k^T \mathcal{M}^T c \in \operatorname{Ker}(\mathcal{M} S_k Z_k) \subset \operatorname{Ker}(\mathcal{M} Z_k)$ so that

$$\begin{aligned}
0 &= \mathcal{M} Z_k \tilde{Z}_k^T S_k^T \mathcal{M}^T c = \mathcal{M} (\tilde{Z}_k Z_k^T + \mathcal{J}) S_k^T \mathcal{M}^T c \\
&= \mathcal{M} \mathcal{J} S_k^T \mathcal{M}^T c = \mathcal{L}^T S_k^T \mathcal{M}^T c
\end{aligned}$$

follows. This proves $\operatorname{Ker}(\mathcal{M} S_k Z_k)^T \subset \operatorname{Ker}(\mathcal{M} S_k \mathcal{L})^T$, and hence our first claim is established. Now, $\mathcal{M} z_k = \mathcal{M} Z_k c_k$ for some $c_k \in \mathbb{R}^n$ implies

$$\begin{aligned}
\mathcal{M} S_k z_k &= \mathcal{M} S_k \mathcal{J} (\mathcal{M} - \mathcal{M}^T) z_k = \mathcal{M} S_k \mathcal{J} \mathcal{M} Z_k c_k - \mathcal{M} S_k \mathcal{J} \mathcal{M}^T z_k \\
&= \mathcal{M} S_k (I - \mathcal{L}) Z_k c_k + \mathcal{M} S_k \mathcal{L} z_k \\
&= \mathcal{M} S_k Z_k c_k + \mathcal{M} S_k \mathcal{L} (z_k - Z_k c_k) \in \operatorname{Im}(\mathcal{M} S_k Z_k),
\end{aligned}$$

and hence there exists $c_{k+1} \in \mathbb{R}^n$ with $\mathcal{M} S_k z_k = \mathcal{M} S_k Z_k c_{k+1}$ which is our second assertion. To prove the remainder of the proposition we make use of the well-known result (see e.g. [6, Remark 2 (iii)])

$$\operatorname{Ker} V \subset \operatorname{Ker} W \quad \text{iff} \quad W = W V^\dagger V.$$

Hence, with $Z_k = \binom{X_k}{U_k}$, $z_k = \binom{x_k}{u_k}$, $S_k = \begin{pmatrix} A_k & B_k \\ C_k & E_k \end{pmatrix}$, and $s_k = u_k - U_k X_k^\dagger x_k$ we have

$$s_k^T D_k s_k = s_k^T X_k X_{k+1}^\dagger X_{k+1} \Delta c_k = s_k^T X_k \Delta c_k = z_k^T \mathcal{J}^T Z_k \Delta c_k,$$

and $D_k = B_k^T E_k - B_k^T (X_{k+1}^\dagger)^T X_{k+1}^T U_{k+1} X_{k+1}^\dagger B_k$ is symmetric. ■

## 5. The “Big” Symplectic System

In order to prove our main result, Theorem 1 from Section 2 above, we will apply Picone’s identity together with the preceding Proposition 3. However, we will not apply it to our system (S) but to another symplectic system (S$^*$) related to (S). The purpose of this section is to introduce system (S$^*$) and to study its properties. To do so we define $4n \times 4n$-matrices $S_k^*$, $M \le k \le N$, by

$$S_k^* = \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} S_k \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix}^T + \begin{pmatrix} \mathcal{J}\mathcal{M} & 0 \\ 0 & \mathcal{J}\mathcal{M} \end{pmatrix}.$$

First of all we have the following.

**Proposition 4.** $S_k^*$ are symplectic for all $M \le k \le N$.
**Proof.** Let $M \le k \le N$. The computation

$$\begin{aligned} S_k^{*^T} \mathcal{J}^* S_k^* &= \left\{ \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} S_k^T (\mathcal{M}^T \; \mathcal{L}) + \begin{pmatrix} \mathcal{J}\mathcal{M} & 0 \\ 0 & \mathcal{J}\mathcal{M} \end{pmatrix} \right\} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} S_k^* \\ &= \left\{ \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} S_k^T (-\mathcal{L} \; \mathcal{M}^T) + \begin{pmatrix} 0 & \mathcal{J}\mathcal{M} \\ -\mathcal{J}\mathcal{M} & 0 \end{pmatrix} \right\} \left\{ \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} S_k \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix}^T + \begin{pmatrix} \mathcal{J}\mathcal{M} & 0 \\ 0 & \mathcal{J}\mathcal{M} \end{pmatrix} \right\} \\ &= \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} S_k^T (-\mathcal{M} + \mathcal{M}^T) S_k \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix}^T + \begin{pmatrix} 0 & \mathcal{J}\mathcal{M} \\ -\mathcal{J}\mathcal{M} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} S_k^T \mathcal{J} S_k \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix}^T + \begin{pmatrix} 0 & \mathcal{J}\mathcal{M} \\ -\mathcal{J}\mathcal{M} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} \mathcal{J} \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix}^T + \begin{pmatrix} 0 & \mathcal{J}\mathcal{M} \\ -\mathcal{J}\mathcal{M} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{M}\mathcal{J}\mathcal{M}^T & \mathcal{M}\mathcal{J}\mathcal{L} + \mathcal{J}\mathcal{M} \\ \mathcal{L}\mathcal{J}\mathcal{M}^T - \mathcal{J}\mathcal{M} & \mathcal{L}\mathcal{J}\mathcal{L} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = \mathcal{J}^*, \end{aligned}$$

where we used $\mathcal{J}^*$ for the “big” $4n \times 4n$-matrix $\mathcal{J}$, shows that $S_k^*$ is symplectic.■

Our next two results illustrate the relationship between the symplectic systems (S) and

$$\text{(S}^*\text{)} \qquad z_{k+1}^* = S_k^* z_k^*, \qquad M \le k \le N.$$

**Proposition 5.** If $\mathcal{Z}$ is a normalized conjoined basis, then $Z^*$ defined by

$$Z_k^* = \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} \mathcal{Z}_k + \begin{pmatrix} \mathcal{M}^T \\ \mathcal{J}\mathcal{M} \end{pmatrix}, \qquad M \le k \le N+1,$$

is a conjoined basis of (S$^*$).

**Proof.** Let $\mathcal{Z}$ and $Z^*$ be as above. First we have for $M \le k \le N$

$$\begin{aligned}
S_k^* Z_k^* &= \left\{ \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} S_k (\mathcal{M}^T \; \mathcal{L}) + \begin{pmatrix} \mathcal{J}\mathcal{M} & 0 \\ 0 & \mathcal{J}\mathcal{M} \end{pmatrix} \right\} \left\{ \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} \mathcal{Z}_k + \begin{pmatrix} \mathcal{M}^T \\ \mathcal{J}\mathcal{M} \end{pmatrix} \right\} \\
&= \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} S_k (\mathcal{M}^T \mathcal{M} + \mathcal{L}) \mathcal{Z}_k + \begin{pmatrix} \mathcal{J}\mathcal{M}\mathcal{M}^T \\ \mathcal{J}\mathcal{M} \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} S_k \mathcal{Z}_k + \begin{pmatrix} \mathcal{M}^T \\ \mathcal{J}\mathcal{M} \end{pmatrix} = \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} \mathcal{Z}_{k+1} + \begin{pmatrix} \mathcal{M}^T \\ \mathcal{J}\mathcal{M} \end{pmatrix} = Z_{k+1}^*.
\end{aligned}$$

Next,

$$\begin{aligned}
Z_M^{*^T} Z_M^* &= \left\{ \mathcal{Z}_M^T (\mathcal{M}^T \; \mathcal{L}) + (\mathcal{M} \; \mathcal{J}\mathcal{M}) \right\} \left\{ \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} \mathcal{Z}_M + \begin{pmatrix} \mathcal{M}^T \\ \mathcal{J}\mathcal{M} \end{pmatrix} \right\} \\
&= \mathcal{Z}_M^T \mathcal{Z}_M + \mathcal{M}\mathcal{M}^T + \mathcal{J}\mathcal{M} = \mathcal{Z}_M^T \mathcal{Z}_M + I
\end{aligned}$$

is invertible, i.e., $\operatorname{rank} Z_M^* = 2n$ holds, and finally

$$\begin{aligned}
Z_M^{*^T} \mathcal{J}^* Z_M^* &= \left\{ \mathcal{Z}_M^T (\mathcal{M}^T \; \mathcal{L}) + (\mathcal{M} \; \mathcal{J}\mathcal{M}) \right\} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} Z_M^* \\
&= \left\{ \mathcal{Z}_M^T (-\mathcal{L} \; \mathcal{M}^T) + (-\mathcal{J}\mathcal{M} \; \mathcal{M}) \right\} \left\{ \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} \mathcal{Z}_M + \begin{pmatrix} \mathcal{M}^T \\ \mathcal{J}\mathcal{M} \end{pmatrix} \right\} \\
&= \mathcal{Z}_M^T (-\mathcal{M} + \mathcal{M}^T) \mathcal{Z}_M - \mathcal{J}\mathcal{M}\mathcal{M}^T + \mathcal{M}\mathcal{J}\mathcal{M} \\
&= \mathcal{Z}_M^T \mathcal{J} \mathcal{Z}_M - \mathcal{M}^T + \mathcal{M} = \mathcal{J} - \mathcal{M}^T + \mathcal{M} = 0.
\end{aligned}$$

Hence $Z^*$ is indeed a conjoined basis of (S$^*$). ■

**Proposition 6.** Let $\alpha \in \mathbb{R}^{2n}$. If $z = (z_k)_{M \le k \le N+1}$ satisfies $\mathcal{M} z_{k+1} = \mathcal{M} S_k z_k$, $M \le k \le N$, then $z^* = (z_k^*)_{M \le k \le N+1}$ defined by

$$z_k^* = \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} z_k + \begin{pmatrix} \mathcal{M}^T \mathcal{J} \alpha \\ 0 \end{pmatrix}, \qquad M \le k \le N+1,$$

satisfies $\mathcal{M}^* z_{k+1}^* = \mathcal{M}^* S_k^* z_k^*$, $M \le k \le N$. Moreover, in this case we have

$$z_k^{*^T} \left\{ S_k^{*^T} \mathcal{M}^* S_k^* - \mathcal{M}^* \right\} z_k^* = z_k^T \left\{ S_k^T \mathcal{M} S_k - \mathcal{M} \right\} z_k.$$

**Proof.** Let $z$ and $z^*$ be as above. Then we have for $M \le k \le N$

$$\mathcal{M}^* S_k^* z_k^* = \mathcal{M}^* \left\{ \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} S_k (\mathcal{M}^T \; \mathcal{L}) + \begin{pmatrix} \mathcal{J}\mathcal{M} & 0 \\ 0 & \mathcal{J}\mathcal{M} \end{pmatrix} \right\} \left\{ \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} z_k + \begin{pmatrix} \mathcal{M}^T \mathcal{J} \alpha \\ 0 \end{pmatrix} \right\}$$

$$
\begin{aligned}
&= \mathcal{M}^* \left\{ \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} S_k (\mathcal{M}^T \mathcal{M} + \mathcal{L}) z_k + \begin{pmatrix} \mathcal{J}\mathcal{M}\mathcal{M}^T \mathcal{J}\alpha \\ 0 \end{pmatrix} \right\} \\
&= \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \left\{ \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} S_k z_k + \begin{pmatrix} \mathcal{M}^T \mathcal{J}\alpha \\ 0 \end{pmatrix} \right\} \\
&= \begin{pmatrix} 0 \\ \mathcal{M} S_k z_k + \mathcal{M}^T \mathcal{J}\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ \mathcal{M} z_{k+1} + \mathcal{M}^T \mathcal{J}\alpha \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \left\{ \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} z_{k+1} + \begin{pmatrix} \mathcal{M}^T \mathcal{J}\alpha \\ 0 \end{pmatrix} \right\} = \mathcal{M}^* z^*_{k+1} \qquad .
\end{aligned}
$$

Now,

$$
\begin{aligned}
&S_k^{*^T} \mathcal{M}^* S_k^* = S_k^{*^T} \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \left\{ \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} S_k \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix}^T + \begin{pmatrix} \mathcal{J}\mathcal{M} & 0 \\ 0 & \mathcal{J}\mathcal{M} \end{pmatrix} \right\} \\
&= \left\{ \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} S_k^T (\mathcal{M}^T \ \mathcal{L}) + \begin{pmatrix} \mathcal{J}\mathcal{M} & 0 \\ 0 & \mathcal{J}\mathcal{M} \end{pmatrix} \right\} \left\{ \begin{pmatrix} 0 \\ \mathcal{M} \end{pmatrix} S_k \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix}^T + \begin{pmatrix} 0 & 0 \\ \mathcal{J}\mathcal{M} & 0 \end{pmatrix} \right\} \\
&= \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} S_k^T \mathcal{L}\mathcal{M} S_k \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix}^T + \begin{pmatrix} 0 & 0 \\ \mathcal{J}\mathcal{M} & 0 \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} S_k^T \mathcal{M} S_k \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix}^T + \begin{pmatrix} 0 & 0 \\ \mathcal{J}\mathcal{M} & 0 \end{pmatrix}
\end{aligned}
$$

implies because of $\binom{\mathcal{M}}{\mathcal{L}}^T \binom{\mathcal{M}}{\mathcal{L}} = I$

$$
\begin{aligned}
&z_k^{*^T} \left\{ S_k^{*^T} \mathcal{M}^* S_k^* - \mathcal{M}^* \right\} z_k^* = z_k^{*^T} \left\{ \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} S_k^T \mathcal{M} S_k \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix}^T - \begin{pmatrix} 0 & 0 \\ \mathcal{L} & 0 \end{pmatrix} \right\} z_k^* \\
&= z_k^T S_k^T \mathcal{M} S_k z_k - z_k^T (\mathcal{M}^T \ \mathcal{L}) \begin{pmatrix} 0 & 0 \\ \mathcal{L} & 0 \end{pmatrix} \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} z_k \\
&= z_k^T \left\{ S_k^T \mathcal{M} S_k - \mathcal{M} \right\} z_k
\end{aligned}
$$

so that all our assertions are shown. ■

To conclude this section we wish to remark the following. We have $Z_k^* = \binom{X_k^*}{U_k^*}$ with

$$
X_k^* = \begin{pmatrix} 0 & I \\ X_k & \tilde{X}_k \end{pmatrix} \quad \text{and} \quad U_k^* = \begin{pmatrix} I & 0 \\ U_k & \tilde{U}_k \end{pmatrix}
$$

provided we denote $\mathcal{Z}_k = \begin{pmatrix} X_k & \tilde{X}_k \\ U_k & \tilde{U}_k \end{pmatrix}$. In [5, Lemma A6], the Moore-Penrose inverse of $X^* = \begin{pmatrix} 0 & I \\ X & \tilde{X} \end{pmatrix}$ is computed to be

$$X^{*^\dagger} = \begin{pmatrix} -(S^{-\frac{1}{2}}X)^\dagger S^{-\frac{1}{2}}X & (S^{-\frac{1}{2}}X)^\dagger S^{-\frac{1}{2}} \\ I - \tilde{X}^T S^{-1}\left\{I - X(S^{-\frac{1}{2}}X)^\dagger S^{-\frac{1}{2}}\right\}\tilde{X} & \tilde{X}^T S^{-1}\left\{I - X(S^{-\frac{1}{2}}X)^\dagger S^{-\frac{1}{2}}\right\} \end{pmatrix}$$

with $S = I + \tilde{X}\tilde{X}^T$, and this formula may be also checked directly using the definition of the Moore-Penrose inverse. We hence have

$$D_k^* = B_k^{*^T} E_k^* - B_k^{*^T} U_{k+1}^* X_{k+1}^{*^\dagger} B_k^* = \begin{pmatrix} 0 & 0 \\ 0 & B_k^T E_k - F_k \end{pmatrix}$$

provided $\mathrm{Ker} X_{k+1} \subset \mathrm{Ker} X_k$ holds, where

$$\begin{aligned} F_k &= B_k^T \Big\{U_{k+1}(S^{-\frac{1}{2}}X_{k+1})^\dagger S^{-\frac{1}{2}} + \tilde{U}_{k+1}\tilde{X}_{k+1}^T S^{-1} \\ &\quad -\tilde{U}_{k+1}\tilde{X}_{k+1}^T S^{-1} X_{k+1}(S^{-\frac{1}{2}}X_{k+1})^\dagger S^{-\frac{1}{2}}\Big\} B_k^T \\ &= B_k^T \Big\{(X_{k+1}^\dagger)^T U_{k+1}^T X_{k+1}(S^{-\frac{1}{2}}X_{k+1})^\dagger S^{-\frac{1}{2}} X_{k+1} X_{k+1}^\dagger + \tilde{U}_{k+1}\tilde{X}_{k+1}^T S^{-1} \\ &\quad -\tilde{U}_{k+1}\tilde{X}_{k+1} S^{-1} X_{k+1}(S^{-\frac{1}{2}}X_{k+1})^\dagger S^{-\frac{1}{2}} X_{k+1} X_{k+1}^\dagger\Big\} B_k^T \\ &= B_k^T U_{k+1} X_{k+1}^\dagger B_k \end{aligned}$$

so that $D_k^* = \begin{pmatrix} 0 & 0 \\ 0 & D_k \end{pmatrix}$ holds for all $M \le k \le N$.

# 6. The Proof of the Main Result

We use the notation from Section 2 and start with proving part (i) of Theorem 1. Let (S) be disconjugate and assume that (B) holds. We pick $z$ with $\mathcal{M}z_{k+1} = \mathcal{M}S_k z_k$, $M \le k \le N$, and $z_* = R^T c \in \mathrm{Im} R^T$. Then $z^*$ defined as in Proposition 6 with $\alpha = z_M$ satisfies $\mathcal{M}^* z_{k+1}^* = \mathcal{M}^* S_k^* z_k^*$, $M \le k \le N$, where $S_k^*$ is defined as in the previous section. Then we have

$$z_{N+1}^* = \begin{pmatrix} \mathcal{M} \\ \mathcal{L} \end{pmatrix} z_{N+1} + \begin{pmatrix} \mathcal{M}^T \mathcal{J}\alpha \\ 0 \end{pmatrix} = \begin{pmatrix} z_* \\ \mathcal{L} z_{N+1} \end{pmatrix}$$

and

$$\begin{aligned} \mathcal{M}^* z_M^* &= \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \begin{pmatrix} \mathcal{M} + \mathcal{M}^T \mathcal{J} \\ \mathcal{L} \end{pmatrix} z_M = \begin{pmatrix} 0 \\ \mathcal{M} + \mathcal{M}^T \mathcal{J} \end{pmatrix} z_M \\ &\in \mathrm{Im}\begin{pmatrix} 0 \\ \mathcal{M}^T + \mathcal{M}\mathcal{J}^T \end{pmatrix} = \mathrm{Im}(\mathcal{M}^* \mathcal{Z}_M^*). \end{aligned}$$

Proposition 3, applied to the "big" system (S$^*$), yields that

$$\mathcal{M}^* z_k^* \in \operatorname{Im}(\mathcal{M}^* \mathcal{Z}_k) \quad \text{holds for all} \quad M \le k \le N+1.$$

We then have, again by applying Proposition 2 and Proposition 3 to the "big" system (use also Propositions 5 and 6)

$$\begin{aligned}
\mathcal{F}(z) &= \sum_{k=M}^{N} z_k^T \left\{ S_k^T \mathcal{M} S_k - \mathcal{M} \right\} z_k + z_*^T T z_* \\
&= \sum_{k=M}^{N} z_k^{*^T} \left\{ S_k^{*^T} \mathcal{M}^* S_k^* - \mathcal{M}^* \right\} z_k^* + z_*^T T z_* \\
&= z_*^T Q_{N+1}^* z_* + z_*^T T z_* + \sum_{k=M}^{N} s_k^{*^T} D_k^* s_k^* \\
&= c^T R (T + Q_{N+1}^*) R^T c + \sum_{k=M}^{N} s_k^{*^T} \begin{pmatrix} 0 & 0 \\ 0 & D_k \end{pmatrix} s_k^* \ge 0.
\end{aligned}$$

If $\mathcal{F}(z) = 0$, then $z_* = R^T c = 0$ and $D_k^* s_k^* = 0$ for all $M \le k \le N$ imply $x_{N+1}^* = 0$ and hence $x_k^* = 0$ for all $M \le k \le N$ also (by putting $z_k^* = \binom{x_k^*}{u_k^*}$) and therefore $\mathcal{M} z = 0$. Thus we have $\mathcal{F} > 0$ and the proof of part (i) is complete.

To show (ii) also, we now assume that $\mathcal{F} > 0$ and (C) hold. Suppose $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$ for all $M \le k < m \le N$ but $\operatorname{Ker} X_{m+1} \not\subset \operatorname{Ker} X_m$. With $c \in \operatorname{Ker} X_{m+1} \setminus \operatorname{Ker} X_m$ we put

$$z_k := \begin{cases} Z_k c & M \le k \le m \\ 0 & m < k \le N+1. \end{cases}$$

Then $z_* = \mathcal{M} z_{N+1} + \mathcal{M}^T \mathcal{J} z_M = 0$, $\mathcal{M} z_m \ne 0$, and

$$\mathcal{M} S_m z_m = \mathcal{M} S_m Z_m c = \mathcal{M} Z_{m+1} c = 0 = \mathcal{M} z_{m+1}.$$

However, Proposition 1 yields

$$\begin{aligned}
\mathcal{F}(z) &= z_m^T \mathcal{M} z_m - z_M^T \mathcal{M} z_M + z_m^T \left\{ S_m^T \mathcal{M} S_m - \mathcal{M} \right\} z_m \\
&= z_m^T S_m^T \mathcal{M} S_m z_m = 0
\end{aligned}$$

so that $\mathcal{F} \not> 0$. We hence have $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$ for all $M \le k \le N$. Thus $D_k$ are symmetric for all $M \le k \le N$ according to Proposition 3. We now assume the existence of some $M < m \le N$ and some $d \in \mathbb{R}^n$ with $d^T D_m d < 0$. We put $c := -X_{m+1}^\dagger B_m d$ and

$$z_k := \begin{cases} Z_k c & M \le k < m \\ Z_k c + \binom{0}{d} & k = m \\ 0 & m < k \le N+1. \end{cases}$$

Again we have $z_* = 0$, $\mathcal{M}z \neq 0$, and

$$\mathcal{M}S_{m-1}z_{m-1} = \mathcal{M}S_{m-1}Z_{m-1}c = \mathcal{M}Z_m c = \mathcal{M}z_m,$$

$$\mathcal{M}S_m z_m = \mathcal{M}S_m Z_m c + \begin{pmatrix} 0 \\ B_m d \end{pmatrix} = \mathcal{M}Z_{m+1}c + \begin{pmatrix} 0 \\ -X_{m+1}c \end{pmatrix} = 0 = \mathcal{M}z_{m+1}.$$

However, Proposition 2 yields

$$\begin{aligned} \mathcal{F}(z) &= c_{m+1}^T Z_{m+1}^T \mathcal{M}^T Z_{m+1} c_{m+1} - c_M^T Z_M^T \mathcal{M}^T Z_M c_M + \sum_{k=M}^{m} z_k^T \mathcal{J}^T Z_k \{\Delta c_k\} \\ &= -z_m^T \mathcal{J}^T Z_m c = -\left\{ Z_m c + \begin{pmatrix} 0 \\ d \end{pmatrix} \right\}^T \mathcal{J}^T Z_m c \\ &= c^T Z_m^T \mathcal{J} Z_m c - d^T X_m c = -d^T X_m c \\ &= d^T D_m d < 0 \end{aligned}$$

so that again $\mathcal{F} \not> 0$. Hence (S) is indeed disconjugate. Finally, to prove that (B) holds also, we assume that it doesn't, i.e., that there exists $c \in \mathrm{Im} R \setminus \{0\}$ with

$$c^T R(T + Q_{N+1}^*) R^T c < 0.$$

Then we have $d := R^T c \neq 0$, and (C) yields

$$d = R^T c \in \mathrm{Im} R^T \subset \mathrm{Im}(\mathcal{M}^T + \mathcal{M}\mathcal{Z}_{N+1}).$$

Let $\beta \in \mathbb{R}^{2n}$ with $d = (\mathcal{M}^T + \mathcal{M}\mathcal{Z}_{N+1})\beta$ and put $\gamma := (\mathcal{M}^T + \mathcal{M}\mathcal{Z}_{N+1})^\dagger d$ and

$$z_k := \mathcal{Z}_k \gamma, \qquad M \leq k \leq N+1.$$

Then we have

$$\begin{aligned} z_* &= \mathcal{M}^T \mathcal{J} z_0 + \mathcal{M} z_{N+1} = (\mathcal{M}^T \mathcal{J} \mathcal{Z}_0 + \mathcal{M}\mathcal{Z}_{N+1})\gamma \\ &= (\mathcal{M}^T + \mathcal{M}\mathcal{Z}_{N+1})\gamma = (\mathcal{M}^T + \mathcal{M}\mathcal{Z}_{N+1})(\mathcal{M}^T + \mathcal{M}\mathcal{Z}_{N+1})^\dagger(\mathcal{M}^T + \mathcal{M}\mathcal{Z}_{N+1})\beta \\ &= (\mathcal{M}^T + \mathcal{M}\mathcal{Z}_{N+1})\beta = d = R^T c \in \mathrm{Im} R^T. \end{aligned}$$

Since $z$ is a solution of (S) we may apply Proposition 1 to obtain

$$\begin{aligned} \mathcal{F}(z) &= z_{N+1}^T \mathcal{M} z_{N+1} - z_M^T \mathcal{M} z_M + z_*^T T z_* \\ &= \gamma^T \left\{ \mathcal{Z}_{N+1}^T \mathcal{M} \mathcal{Z}_{N+1} - \mathcal{Z}_M^T \mathcal{M} \mathcal{Z}_M \right\} \gamma + d^T T d \\ &= d^T Q_{N+1}^* d + d^T T d = c^T R(T + Q_{N+1}^*) R^T c < 0 \end{aligned}$$

so that again $\mathcal{F} \not> 0$. Hence (B) holds and the proof of Theorem 1 is complete.

For the proof of Theorem 2 we just remark that disconjugacy of (S) on $[M, N+1] \cap \mathbb{Z}$ together with controllability of (S) on $[M, N+1] \cap \mathbb{Z}$ imply invertibility of $\mathcal{M}^T + \mathcal{M}\mathcal{Z}_{N+1}$ (so that condition (C) is automatically satisfied in this case), and this can easily be seen as follows: Supppose (S) is controllable and disconjugate on $[M, N+1] \cap \mathbb{Z}$ and let $c \in \operatorname{Ker} X_{N+1}$. Then $c \in \operatorname{Ker} X_k$ for all $M \le k \le N+1$ due to disconjugacy. We define a solution $z$ of (S) by

$$z_k := Z_k c, \qquad M \le k \le N+1.$$

Then $\mathcal{M}z = 0$ and hence controllability implies $z = 0$ so that $c = 0$. This consideration yields invertibility of $X_{N+1}$ and hence of $\mathcal{M}^T + \mathcal{M}\mathcal{Z}_{N+1} = \begin{pmatrix} 0 & I \\ X_{N+1} & \tilde{X}_{N+1} \end{pmatrix}$.

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