

The Beverton–Holt q -difference equation

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(Received 13 December 2012; final version received 1 May 2013)

The Beverton–Holt model is a classical population model which has been considered in the literature for the discrete-time case. Its continuous-time analogue is the well-known logistic model. In this paper, we consider a quantum calculus analogue of the Beverton–Holt equation. We use a recently introduced concept of periodic functions in quantum calculus in order to study the existence of periodic solutions of the Beverton–Holt q -difference equation. Moreover, we present proofs of quantum calculus versions of two so-called Cushing–Henson conjectures.

Keywords: Beverton–Holt equation; Cushing–Henson conjecture; time scale; dynamic equation; logistic equation; Jensen inequality

AMS Subject Classifications: 39A10; 39A11; 39A12; 39A20; 34C25

1. Introduction

The Beverton–Holt difference equation has wide applications in population growth [1] and is given by

$$x(t+1) = \frac{\nu K(t)x(t)}{K(t) + (\nu - 1)x(t)}, \quad t \in \mathbb{N}_0, \quad (1)$$

where $\nu > 1$, $K(t) > 0$ for all $t \in \mathbb{N}_0$, and $x(0) > 0$. We call the sequence K the *carrying capacity* and ν the *inherent growth rate* [8]. The periodically forced Beverton–Holt equation, which is obtained by letting the carrying capacity be a periodic positive sequence $K(t)$ with period $\omega \in \mathbb{N}$, i.e., $K(t + \omega) = K(t)$ for all $t \in \mathbb{N}_0$, has been treated with the methods found in [8–10]. For the Beverton–Holt dynamic equation on time scales, one article has been presented by Bohner and Warth [7]. In [7], a general Beverton–Holt equation is given, which reduces to Equation (1) in the discrete case and to the well-known logistic equation in the continuous case. The approach given in [7] opened a new path to the study of the discrete Beverton–Holt equation, which was pursued

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by Bohner *et al.* in [6]. The crucial idea in [6,7] was to rewrite Equation (1) as

$$\Delta x(t) = \alpha x(t+1) \left(1 - \frac{x(t)}{K(t)}\right), \quad t \in \mathbb{N}_0, \quad \text{where } \alpha := \frac{\nu-1}{\nu}, \quad (2)$$

and thus identify the continuous version of the discrete Beverton–Holt equation (2) as

$$x'(t) = \alpha x(t) \left(1 - \frac{x(t)}{K(t)}\right), \quad t \in \mathbb{R}, \quad (3)$$

which turned out to be the usual logistic equation.¹ This approach was generalized to any so-called dynamic equation of the form

$$x^\Delta(t) = \alpha x^\sigma(t) \left(1 - \frac{x(t)}{K(t)}\right), \quad t \in \mathbb{T}, \quad (4)$$

hence accommodating both the continuous and discrete equations (2) and (3). However, the restriction on the time scale $\mathbb{T}$ was that it should be periodic. Hence, $\mathbb{N}_0$ and $\mathbb{R}$ (and also $h\mathbb{N}_0$ with $h > 0$) were allowed, but $q^{\mathbb{N}_0}$ for $q > 1$ was not.

In this paper, we are filling this gap by studying a quantum calculus version of the Beverton–Holt equation, namely, a Beverton–Holt q -difference equation. This became possible by using a new definition of periodic functions in quantum calculus which was introduced by the authors in [3, Definition 3.1] (see also [4]). Using this concept, we are interested in seeking ω -periodic solutions of the Beverton–Holt q -difference equation given by

$$x^\Delta(t) = a(t)x^\sigma(t) \left(1 - \frac{x(t)}{K(t)}\right), \quad t \in q^{\mathbb{N}_0}, \quad (5)$$

where a is 1-periodic and K is ω -periodic, and

$$x^\Delta(t) = \frac{x(qt) - x(t)}{(q-1)t}, \quad x^\sigma(t) = x(\sigma(t)), \quad \sigma(t) = qt, \quad t \in q^{\mathbb{N}_0}.$$

Using this notation and also our Assumptions (7) below, we can easily rewrite Equation (5) as

$$x(qt) = \frac{\nu x(t)K(t)}{K(t) + (\nu-1)x(t)}, \quad t \in q^{\mathbb{N}_0}, \quad \text{where } \nu := \frac{1}{1 - (q-1)\alpha}. \quad (6)$$

One can now observe the similarity of the discrete (additive) recursion (1) and the quantum (multiplicative) recursion (6).

The set-up of this paper is as follows. Section 2 contains some preliminaries on quantum calculus. We approach the periodic solutions of the Beverton–Holt q -difference equation (5) by some strategies presented in Section 3. In Sections 4 and 5, we formulate and prove the first and the second Cushing–Henson conjectures for the q -difference equations case, respectively.

2. Some auxiliary results

DEFINITION 2.1 *We say that a function $p : q^{\mathbb{N}_0} \rightarrow \mathbb{R}$ is regressive provided*

$$1 + \mu(t)p(t) \neq 0 \text{ for all } t \in q^{\mathbb{N}_0}, \quad \text{where } \mu(t) = (q-1)t.$$

The set of all regressive functions will be denoted by $\mathcal{R}$. Moreover, $p \in \mathcal{R}$ is called positively regressive and we write $p \in \mathcal{R}^+$ provided

$$1 + \mu(t)p(t) > 0 \text{ for all } t \in q^{\mathbb{N}_0}.$$

DEFINITION 2.2 Exponential function Let $p \in \mathcal{R}$ and $t_0 \in q^{\mathbb{N}_0}$. The exponential function $e_p(\cdot, t_0)$ on $q^{\mathbb{N}_0}$ is defined by

$$e_p(t, t_0) = \prod_{s \in [t_0, t)} [1 + (q-1)sp(s)] \text{ for } t \geq t_0, \quad t \in q^{\mathbb{N}_0}.$$

Remark 2.3 See [5, Theorem 2.44] If $p \in \mathcal{R}^+$, then $e_p(t, t_0) > 0$ for all $t \geq t_0, t \in q^{\mathbb{N}_0}$.

DEFINITION 2.4 See [3, Definition 3.1] A function $f : q^{\mathbb{N}_0} \rightarrow \mathbb{R}$ is called ω -periodic if

$$f(t) = q^\omega f(q^\omega t) \text{ for all } t \in q^{\mathbb{N}_0}.$$

THEOREM 2.5 See [5, Theorem 2.36] If $p \in \mathcal{R}$, then

- (i) $e_0(t, s) = 1$ and $e_p(t, t) = 1$;
- (ii) $e_p(t, s) = 1/e_p(s, t)$;
- (iii) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
- (iv) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
- (v) $(1/e_p(\cdot, s))^\Delta(t) = -p(t)/e_p(\sigma(t), s)$.

The integral on $q^{\mathbb{N}_0}$ is defined as follows.

DEFINITION 2.6 Let $m, n \in \mathbb{N}_0$ with $m < n$. For $f : q^{\mathbb{N}_0} \rightarrow \mathbb{R}$, we define

$$\int_{q^m}^{q^n} f(t) \Delta t := (q-1) \sum_{k=m}^{n-1} q^k f(q^k).$$

THEOREM 2.7 Integration by parts, see [5, Theorem 1.77] For $a, b \in q^{\mathbb{N}_0}$ and $f, g : q^{\mathbb{N}_0} \rightarrow \mathbb{R}$, we have

$$\int_a^b f^\sigma(t) g^\Delta(t) \Delta t = f(b)g(b) - f(a)g(a) + \int_a^b f^\Delta(t) g(t) \Delta t$$

and

$$\int_a^b f(t) g^\Delta(t) \Delta t = f(b)g(b) - f(a)g(a) + \int_a^b f^\Delta(t) g^\sigma(t) \Delta t.$$

THEOREM 2.8 Jensen's inequality, see [12, Theorem 2.2] Let $a, b \in q^{\mathbb{N}_0}$ and $c, d \in \mathbb{R}$. Suppose $g, h : ([a, b] \cap q^{\mathbb{N}_0}) \rightarrow (c, d)$ and $\int_a^b |h(s)| \Delta s > 0$. If $F \in C((c, d), \mathbb{R})$ is convex, then

$$F \left(\frac{\int_a^b |h(s)| g(s) \Delta s}{\int_a^b |h(s)| \Delta s} \right) \leq \frac{\int_a^b |h(s)| F(g(s)) \Delta s}{\int_a^b |h(s)| \Delta s}.$$

If F is strictly convex, then ' $\leq$ ' can be replaced by ' $<$ '.

3. The Beverton–Holt equation

Throughout this paper, we use the following assumptions and notation:

$$\begin{cases} q > 1, & t_0 \in q^{\mathbb{N}_0}, & K \text{ is positive and } \omega\text{-periodic, } \omega \in \mathbb{N}, \\ a(t) = \frac{\alpha}{t}, & t \in q^{\mathbb{N}_0}, & 0 < \alpha < \frac{1}{q-1}, \\ \lambda := 1 - (q-1)\alpha, & \beta := \frac{1}{q^\omega \lambda^{-\omega} - 1}. \end{cases} \quad (7)$$

Note that Assumption (7) implies that $0 < \lambda < 1$, $-a \in \mathcal{R}^+$, and

$$e_{-a}(t, s) = \lambda^{\log_q(t/s)} \text{ for all } t, s \in q^{\mathbb{N}_0}. \quad (8)$$

Note also that β is well defined as $q/\lambda \notin \{-1, 1\}$ since $\lambda \notin \{-q, q\}$ due to $0 < \lambda < 1$.

In the q -difference equation (5), we assume $x(t) > 0$ for all $t \in q^{\mathbb{N}_0}$ and substitute

$$u := \frac{1}{x}.$$

Then, using the quotient rule [5, Theorem 1.20 (v)], Equation (5) becomes

$$u^\Delta(t) = -a(t)u(t) + \frac{a(t)}{K(t)}. \quad (9)$$

The general solution of Equation (9) is given by applying variation of parameters [5, Theorem 2.77] twice as

$$u(t) = e_{-a}(t, t_0)u(t_0) + \int_{t_0}^t e_{-a}(t, \sigma(s)) \frac{a(s)}{K(s)} \Delta s \quad (10)$$

$$= e_{-a}(t, q^\omega t_0)u(q^\omega t_0) + \int_{q^\omega t_0}^t e_{-a}(t, \sigma(s)) \frac{a(s)}{K(s)} \Delta s, \quad (11)$$

where $t \in q^{\mathbb{N}_0}$. Now, we require an ω -periodic solution $\bar{x}$ of Equation (5). This means that $\bar{x}$ satisfies $\bar{x}(t) = q^\omega \bar{x}(q^\omega t)$ for all $t \in q^{\mathbb{N}_0}$. This in turn means that a solution $\bar{u} = 1/\bar{x}$ of Equation (9) satisfies

$$q^\omega \bar{u}(t) = \bar{u}(q^\omega t) \text{ for all } t \in q^{\mathbb{N}_0}. \quad (12)$$

LEMMA 3.1 Assume Assumption (7). If Equation (9) has a solution $\bar{u}$ satisfying Equation (12), then

$$\bar{u}(t_0) = \beta \int_{t_0}^{q^\omega t_0} e_{-a}(t_0, \sigma(s)) \frac{a(s)}{K(s)} \Delta s.$$

Proof Assume Equation (9) has a solution $\bar{u}$ satisfying Equation (12). Then,

$$\begin{aligned} \bar{u}(t_0) &= q^{-\omega} \bar{u}(q^\omega t_0) \\ &= q^{-\omega} e_{-a}(q^\omega t_0, t_0) \bar{u}(t_0) + q^{-\omega} \int_{t_0}^{q^\omega t_0} e_{-a}(q^\omega t_0, \sigma(s)) \frac{a(s)}{K(s)} \Delta s \end{aligned}$$

$$\begin{aligned}
&= \frac{q^{-\omega}}{1 - q^{-\omega} e_{-a}(q^{\omega} t_0, t_0)} \int_{t_0}^{q^{\omega} t_0} e_{-a}(q^{\omega} t_0, t_0) e_{-a}(t_0, \sigma(s)) \frac{a(s)}{K(s)} \Delta s \\
&= \frac{q^{-\omega} \lambda^{\omega}}{1 - q^{-\omega} \lambda^{\omega}} \int_{t_0}^{q^{\omega} t_0} e_{-a}(t_0, \sigma(s)) \frac{a(s)}{K(s)} \Delta s \\
&= \frac{1}{q^{\omega} \lambda^{-\omega} - 1} \int_{t_0}^{q^{\omega} t_0} e_{-a}(t_0, \sigma(s)) \frac{a(s)}{K(s)} \Delta s.
\end{aligned}$$

Thus, $\bar{u}$ satisfies the required initial condition. ■

4. The first Cushing–Henson conjecture

Now we state and prove the first Cushing–Henson conjecture for the Beverton–Holt q -difference equation (5).

Conjecture 4.1 First Cushing–Henson conjecture The Beverton–Holt q -difference model (5) with an ω -periodic carrying capacity K has a unique ω -periodic solution $\bar{x}$ that globally attracts all solutions.

Using Equation (10) and Lemma 3.1, the solution $\bar{u}$ of Equation (9) can be written as

$$\begin{aligned}
\bar{u}(t) &= e_{-a}(t, t_0) \bar{u}(t_0) + \int_{t_0}^t e_{-a}(t, \sigma(s)) \frac{a(s)}{K(s)} \Delta s \\
&= \beta \int_{t_0}^{q^{\omega} t_0} e_{-a}(t, \sigma(s)) \frac{a(s)}{K(s)} \Delta s + \int_{t_0}^t e_{-a}(t, \sigma(s)) \frac{a(s)}{K(s)} \Delta s.
\end{aligned} \tag{13}$$

THEOREM 4.2 Assume Assumption (7) and let $\bar{u}$ be given by Equation (13). Then, $\bar{x} := 1/\bar{u}$ is an ω -periodic solution of the Beverton–Holt q -difference equation (5).

Proof By Equation (11), we have

$$\bar{u}(t) = e_{-a}(t, q^{\omega} t_0) \bar{u}(q^{\omega} t_0) + \int_{q^{\omega} t_0}^t e_{-a}(t, \sigma(s)) \frac{a(s)}{K(s)} \Delta s$$

so that

$$\begin{aligned}
q^{-\omega} \bar{u}(q^{\omega} t) &= e_{-a}(q^{\omega} t, q^{\omega} t_0) q^{-\omega} \bar{u}(q^{\omega} t_0) + \int_{q^{\omega} t_0}^{q^{\omega} t} e_{-a}(q^{\omega} t, \sigma(s)) \frac{a(s)}{q^{\omega} K(s)} \Delta s \\
&= e_{-a}(t, t_0) \bar{u}(t_0) + \int_{t_0}^t e_{-a}(t, \sigma(s)) \frac{a(s)}{K(s)} \Delta s \\
&= \bar{u}(t)
\end{aligned}$$

since by putting $t_0 = q^m$ and $t = q^n$, we have

$$\begin{aligned}
\int_{q^{\omega} t_0}^{q^{\omega} t} e_{-a}(q^{\omega} t, \sigma(s)) \frac{a(s)}{q^{\omega} K(s)} \Delta s &= (q-1) \sum_{k=\omega+m}^{\omega+n-1} q^k e_{-a}(q^{\omega+n}, q^{k+1}) \frac{a(q^k)}{q^{\omega} K(q^k)} \\
&= (q-1) \sum_{k=m}^{n-1} q^{k+\omega} e_{-a}(q^{\omega+n}, q^{k+\omega+1}) \frac{a(q^{k+\omega})}{q^{\omega} K(q^{k+\omega})}
\end{aligned}$$

$$\begin{aligned}
&= (q-1) \sum_{k=m}^{n-1} q^k e_{-a}(q^n, q^{k+1}) \frac{a(q^k)}{K(q^k)} \\
&= \int_{t_0}^t e_{-a}(t, \sigma(s)) \frac{a(s)}{K(s)} \Delta s.
\end{aligned}$$

Hence, $\bar{u}$ satisfies Equation (12) and thus $\bar{x}$ is indeed ω -periodic. ■

Now we are ready to prove the validity of the first Cushing–Henson conjecture.

THEOREM 4.3 *Assume Assumption (7). The solution $\bar{x}$ of Equation (5) given in Theorem 4.2 is globally attractive.*

Proof First note that K is bounded. Indeed, define

$$\tilde{K} := \max_{0 \leq k \leq \omega-1} K(q^k).$$

For any $m \in \mathbb{N}_0$, there exist $\ell \in \mathbb{N}_0$ and $0 \leq k \leq \omega-1$ such that $m = \ell\omega + k$, and thus

$$K(q^m) = K(q^{\ell\omega+k}) = q^{-\ell\omega} K(q^k) \leq K(q^k) \leq \tilde{K}.$$

Now let x be any solution of Equation (5) with $x(t) > 0$ for all $t \in q^{\mathbb{N}_0}$. We have

$$\begin{aligned}
|x(t) - \bar{x}(t)| &= \left| \frac{1}{e_{-a}(t, t_0)u(t_0) + \int_{t_0}^t e_{-a}(t, \sigma(s))a(s)/K(s) \Delta s} \right. \\
&\quad \left. - \frac{1}{e_{-a}(t, t_0)\bar{u}(t_0) + \int_{t_0}^t e_{-a}(t, \sigma(s))a(s)/K(s) \Delta s} \right| \\
&= \left| \frac{1}{e_{-a}(t, t_0)/x(t_0) + \int_{t_0}^t e_{-a}(t, \sigma(s))a(s)/K(s) \Delta s} \right. \\
&\quad \left. - \frac{1}{e_{-a}(t, t_0)/\bar{x}(t_0) + \int_{t_0}^t e_{-a}(t, \sigma(s))a(s)/K(s) \Delta s} \right| \\
&= \frac{\left| \frac{1}{\bar{x}(t_0)} - \frac{1}{x(t_0)} \right| |e_{-a}(t, t_0)|}{\left| \frac{e_{-a}(t, t_0)}{x(t_0)} + \int_{t_0}^t e_{-a}(t, \sigma(s)) \frac{a(s)}{K(s)} \Delta s \right| \left| \frac{e_{-a}(t, t_0)}{\bar{x}(t_0)} + \int_{t_0}^t e_{-a}(t, \sigma(s)) \frac{a(s)}{K(s)} \Delta s \right|} \\
&\leq \frac{|1/\bar{x}(t_0) - 1/x(t_0)| e_{-a}(t, t_0)}{\left(\int_{t_0}^t e_{-a}(t, \sigma(s))a(s)/K(s) \Delta s \right)^2} \\
&\leq \frac{\tilde{K}^2 |1/\bar{x}(t_0) - 1/x(t_0)| e_{-a}(t, t_0)}{\left(\int_{t_0}^t e_{-a}(t, \sigma(s))a(s) \Delta s \right)^2} \\
&\leq \tilde{K}^2 \frac{|1/\bar{x}(t_0) - 1/x(t_0)| e_{-a}(t, t_0)}{(1 - e_{-a}(t, t_0))^2},
\end{aligned}$$

which due to [2, Theorem 2] tends to zero as $t \rightarrow \infty$. ■

5. The second Cushing–Henson conjecture

Now we state and prove the second Cushing–Henson conjecture for the Beverton–Holt q -difference equation (5).

Conjecture 5.1 Second Cushing–Henson conjecture The average of the ω -periodic solution $\bar{x}$ of Equation (5) is strictly less than the average of the ω -periodic carrying capacity K times the constant $(q - \lambda)/(1 - \lambda)$.

In order to prove the second Cushing–Henson conjecture, we use the following series of auxiliary results.

LEMMA 5.2 Assume Assumption (7). Then, for any $t, u, v \in q^{\mathbb{N}_0}$, we have

$$\int_u^v e_{-a}(t, \sigma(s)) \Delta s = \frac{ve_{-a}(t, v) - ue_{-a}(t, u)}{1 + \alpha}. \quad (14)$$

Proof Using Theorems 2.5 and 2.7, we get

$$\begin{aligned} \int_u^v e_{-a}(t, \sigma(s)) \Delta s &= \int_u^v \frac{e_{-a}^{\Delta_s}(t, s)}{a(s)} \Delta s \\ &= \frac{1}{\alpha q} \int_u^v \sigma(s) e_{-a}^{\Delta_s}(t, s) \Delta s \\ &= \frac{1}{\alpha q} \left\{ ve_{-a}(t, v) - ue_{-a}(t, u) - \int_u^v e_{-a}(t, s) \Delta s \right\} \\ &= \frac{1}{\alpha q} \left\{ ve_{-a}(t, v) - ue_{-a}(t, u) - \int_u^v \lambda e_{-a}(t, \sigma(s)) \Delta s \right\} \\ &= \frac{ve_{-a}(t, v) - ue_{-a}(t, u)}{\alpha q + \lambda} \\ &= \frac{ve_{-a}(t, v) - ue_{-a}(t, u)}{1 + \alpha}, \end{aligned}$$

which shows Equation (14). ■

LEMMA 5.3 Assume Assumption (7). Then, for any $s, u, v \in q^{\mathbb{N}_0}$, we have

$$\int_u^v \frac{e_{-a}(t, s)}{t^2} \Delta t = \frac{q}{1 + \alpha} \left\{ \frac{e_{-a}(u, s)}{u} - \frac{e_{-a}(v, s)}{v} \right\}. \quad (15)$$

Proof Using Theorems 2.5 and 2.7, we get

$$\begin{aligned} \int_u^v \frac{e_{-a}(t, s)}{t^2} \Delta t &= \frac{-1}{\alpha} \int_u^v \frac{-a(t) e_{-a}(t, s)}{t} \Delta t \\ &= \frac{-1}{\alpha} \int_u^v \frac{e_{-a}^{\Delta_t}(t, s)}{t} \Delta t \\ &= \frac{-1}{\alpha} \left\{ \frac{e_{-a}(v, s)}{v} - \frac{e_{-a}(u, s)}{u} - \int_u^v e_{-a}(\sigma(t), s) \left(\frac{-1}{t\sigma(t)} \right) \Delta t \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\alpha} \left\{ \frac{e_{-a}(u, s)}{u} - \frac{e_{-a}(v, s)}{v} - \frac{\lambda}{q} \int_u^v \frac{e_{-a}(t, s)}{t^2} \Delta t \right\} \\
&= \frac{q}{\alpha + 1} \left\{ \frac{e_{-a}(u, s)}{u} - \frac{e_{-a}(v, s)}{v} \right\},
\end{aligned}$$

which shows Equation (15). ■

Now note that Equation (13) implies that for $t_0 \leq t < q^\omega t_0$, we have

$$\bar{u}(t) = \int_{t_0}^{q^\omega t_0} \frac{h(t, s)}{sK(s)} \Delta s, \quad (16)$$

where

$$h(t, s) := \alpha e_{-a}(t, \sigma(s))(\beta + \chi(t, s)) \quad \text{with } \chi(t, s) := \begin{cases} 1 & \text{if } s < t \\ 0 & \text{if } s \geq t. \end{cases} \quad (17)$$

LEMMA 5.4 Assume Assumptions (7) and (17). Then, for $t_0 \leq s < q^\omega t_0$, we have

$$\int_{t_0}^{q^\omega t_0} \frac{h(t, s)}{t^2} \Delta t = \frac{\alpha}{(\alpha + 1)s}. \quad (18)$$

Proof Using Lemma 5.3 and $\beta q^\omega \lambda^{-\omega} - \beta - 1 = 0$, we obtain

$$\begin{aligned}
\int_{t_0}^{q^\omega t_0} \frac{h(t, s)}{t^2} \Delta t &= \alpha \left\{ \beta \int_{t_0}^{q^\omega t_0} \frac{e_{-a}(t, \sigma(s))}{t^2} \Delta t + \int_{\sigma(s)}^{q^\omega t_0} \frac{e_{-a}(t, \sigma(s))}{t^2} \Delta t \right\} \\
&\stackrel{(15)}{=} \frac{\alpha q}{\alpha + 1} \left\{ \beta \left(\frac{e_{-a}(t_0, \sigma(s))}{t_0} - \frac{e_{-a}(q^\omega t_0, \sigma(s))}{q^\omega t_0} \right) + \frac{1}{qs} - \frac{e_{-a}(q^\omega t_0, \sigma(s))}{q^\omega t_0} \right\} \\
&\stackrel{(8)}{=} \frac{\alpha q}{\alpha + 1} \left\{ \frac{1}{qs} + \frac{e_{-a}(q^\omega t_0, \sigma(s))}{q^\omega t_0} (\beta q^\omega \lambda^{-\omega} - \beta - 1) \right\} \\
&= \frac{\alpha}{(\alpha + 1)s},
\end{aligned}$$

which shows Equation (18). ■

LEMMA 5.5 Assume Assumptions (7) and (17). Then, for $t_0 \leq t < q^\omega t_0$, we have

$$\int_{t_0}^{q^\omega t_0} h(t, s) \Delta s = \frac{\alpha t}{1 + \alpha}. \quad (19)$$

Proof Using Lemma 5.2 and $\beta q^\omega \lambda^{-\omega} - \beta - 1 = 0$, we obtain

$$\begin{aligned}
\int_{t_0}^{q^\omega t_0} h(t, s) \Delta s &= \alpha \left\{ \beta \int_{t_0}^{q^\omega t_0} e_{-a}(t, \sigma(s)) \Delta s + \int_{t_0}^t e_{-a}(t, \sigma(s)) \Delta s \right\} \\
&\stackrel{(14)}{=} \alpha \left\{ \beta \left(\frac{q^\omega t_0 e_{-a}(t, q^\omega t_0) - t_0 e_{-a}(t, t_0)}{1 + \alpha} \right) + \frac{t - t_0 e_{-a}(t, t_0)}{1 + \alpha} \right\} \\
&\stackrel{(8)}{=} \frac{\alpha}{1 + \alpha} \{ t + t_0 e_{-a}(t, t_0) (\beta q^\omega \lambda^{-\omega} - \beta - 1) \} \\
&= \frac{\alpha t}{1 + \alpha},
\end{aligned}$$

which shows Equation (19). ■

Now we are ready to prove the validity of the second Cushing–Henson conjecture.

THEOREM 5.6 *Let $\bar{x}$ be the unique ω -periodic solution of Equation (5). If $\omega \neq 1$, then*

$$\frac{1}{\omega} \int_{t_0}^{q^\omega t_0} \bar{x}(t) \Delta t < \frac{q - \lambda}{1 - \lambda} \left\{ \frac{1}{\omega} \int_{t_0}^{q^\omega t_0} K(t) \Delta t \right\}. \quad (20)$$

Proof Since K is ω -periodic with $\omega \neq 1$, $tK(t)$ cannot be a constant. In addition, $F(x) = 1/x$ is strictly convex. Thus, we may use Jensen's inequality (Theorem 2.8) for the single inequality in the forthcoming calculation to obtain

$$\begin{aligned} \int_{t_0}^{q^\omega t_0} \bar{x}(t) \Delta t &= \int_{t_0}^{q^\omega t_0} \frac{1}{\bar{u}(t)} \Delta t \\ &= \int_{t_0}^{q^\omega t_0} \frac{1}{\int_{t_0}^{q^\omega t_0} h(t, s)/sK(s) \Delta s} \Delta t \\ &= \int_{t_0}^{q^\omega t_0} F \left(\frac{\int_{t_0}^{q^\omega t_0} h(t, s)/sK(s) \Delta s}{\int_{t_0}^{q^\omega t_0} h(t, s) \Delta s} \right) \frac{1}{\int_{t_0}^{q^\omega t_0} h(t, s) \Delta s} \Delta t \\ &< \int_{t_0}^{q^\omega t_0} \frac{\int_{t_0}^{q^\omega t_0} h(t, s) F(1/sK(s)) \Delta s}{\left(\int_{t_0}^{q^\omega t_0} h(t, s) \Delta s \right)^2} \Delta t \\ &= \int_{t_0}^{q^\omega t_0} \frac{\int_{t_0}^{q^\omega t_0} h(t, s) sK(s) \Delta s}{\left(\int_{t_0}^{q^\omega t_0} h(t, s) \Delta s \right)^2} \Delta t \\ &\stackrel{(19)}{=} \int_{t_0}^{q^\omega t_0} \frac{\int_{t_0}^{q^\omega t_0} h(t, s) sK(s) \Delta s}{(\alpha t / (1 + \alpha))^2} \Delta t \\ &= \left(\frac{1 + \alpha}{\alpha} \right)^2 \int_{t_0}^{q^\omega t_0} \int_{t_0}^{q^\omega t_0} \frac{h(t, s) sK(s)}{t^2} \Delta s \Delta t \\ &= \left(\frac{1 + \alpha}{\alpha} \right)^2 \int_{t_0}^{q^\omega t_0} sK(s) \int_{t_0}^{q^\omega t_0} \frac{h(t, s)}{t^2} \Delta t \Delta s \\ &\stackrel{(18)}{=} \left(\frac{1 + \alpha}{\alpha} \right)^2 \int_{t_0}^{q^\omega t_0} sK(s) \frac{\alpha}{(\alpha + 1)s} \Delta s \\ &= \frac{1 + \alpha}{\alpha} \int_{t_0}^{q^\omega t_0} K(s) \Delta s, \end{aligned}$$

which shows Inequality (20). The proof is complete. ■

THEOREM 5.7 *If K is 1-periodic, then Inequality (20) becomes an equality, i.e.,*

$$\frac{1}{\omega} \int_{t_0}^{q^\omega t_0} \bar{x}(t) \Delta t = \frac{q - \lambda}{1 - \lambda} \left\{ \frac{1}{\omega} \int_{t_0}^{q^\omega t_0} K(t) \Delta t \right\}. \quad (21)$$

Proof Since K is 1-periodic, we have

$$K(t) = \frac{C}{t} \text{ for some } C > 0.$$

Now it is easy to check that $\bar{x}$ given by

$$\bar{x}(t) := \frac{1 + \alpha}{\alpha} K(t) = \frac{(1 + \alpha)C}{\alpha t}$$

is 1-periodic and satisfies

$$\bar{x}^\Delta(t) = a(t)\bar{x}^\sigma(t) \left(1 - \frac{\bar{x}(t)}{K(t)} \right) \text{ for all } t \in q^{\mathbb{N}_0}.$$

Hence, $\bar{x}$ is the unique 1-periodic solution of Equation (5). Thus, (21) holds. $\blacksquare$

Remark 5.8 Note that the factor

$$\frac{q - \lambda}{1 - \lambda} \quad (22)$$

is not present in the statements of the Cushing–Henson conjectures for the continuous and the discrete cases. However, in q -calculus, the presence of such quantities is common, and when replacing q by 1 (i.e., letting $q \rightarrow 1^+$) in these terms, the continuous case is usually recovered. Note that replacing q by 1 in the factor (22) yields the corresponding ‘continuous’ (and also ‘discrete’) factor 1.

Notes

1. Note that, as Jim Cushing points out, the terminology ‘discrete logistic equation’ differs, due to Robert May [11], in a slight but essential way, from the discrete model (2).

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