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THE GRÜSS INEQUALITY ON TIME SCALES

MARTIN BOHNER AND THOMAS MATTHEWS[*]
Department of Mathematics and Statistics, University of Missouri-Rolla,
Rolla, MO 65409-0020, USA

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Abstract

We prove the Grüss inequality on time scales and thus unify corresponding continuous and discrete versions from the literature. We also apply our results to the quantum calculus case.

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1 Introduction

Gerhard Grüss derived a formula to estimate the deviation of the integral mean of the product of two functions from the product of the integral means. As shown in [4], the so-called Grüss inequality

$$\left|\frac{1}{b-a}\int_a^b f^\sigma(s)g^\sigma(s)\Delta s-\frac{1}{(b-a)^2}\int_a^b f^\sigma(s)\Delta s\int_a^b g^\sigma(s)\Delta s\right| \le \frac{1}{4}(M_1-m_1)(M_2-m_2). \quad (1.1)$$

holds. The setup of this paper is as follows. In Section 2 we first give some preliminary results on time scales that are needed in the remainder of this paper. Next, in Section 3 we prove the time scales version of the Grüss inequality (1.1) and apply our result to the special cases of continuous, discrete, and quantum calculus.

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[*] E-mail address Martin Bohner: bohner@umr.edu, E-mail address Thomas Matthews: tmnqb@umr.edu

2 Time Scales Essentials

Definition 2.1. A *time scale* is an arbitrary nonempty closed subset of the real numbers.

The most important examples of time scales are $\mathbb{R}$, $\mathbb{Z}$ and $q^{\mathbb{N}_0} := \{q^k \mid k \in \mathbb{N}_0\}$.

Definition 2.2. If $\mathbb{T}$ is a time scale, then we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} \mid s > t\}$ for all $t \in \mathbb{T}$, the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T} \mid s < t\}$ for all $t \in \mathbb{T}$, and the *graininess function* $\mu : \mathbb{T} \to [0,\infty)$ by $\mu(t) := \sigma(t) - t$ for all $t \in \mathbb{T}$. Furthermore for a function $f : \mathbb{T} \to \mathbb{R}$, we define $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$ and $f^\rho(t) = f(\rho(t))$ for all $t \in \mathbb{T}$. In this definition we use $\inf\emptyset = \sup\mathbb{T}$ (i.e., $\sigma(t) = t$ if t is the maximum of $\mathbb{T}$) and $\sup\emptyset = \inf\mathbb{T}$ (i.e., $\rho(t) = t$ if t is the minimum of $\mathbb{T}$).

These definitions allow us to characterize every point in a time scale as displayed in Table 1.

Table 1. Classification of Points

t right-scattered	$t < \sigma(t)$
t right-dense	$t = \sigma(t)$
t left-scattered	$\rho(t) < t$
t left-dense	$\rho(t) = t$
t isolated	$\rho(t) < t < \sigma(t)$
t dense	$\rho(t) = t = \sigma(t)$

Definition 2.3. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* (denoted by C_{rd}) if it is continuous at right-dense points of $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points of $\mathbb{T}$.

Theorem 2.4 (Existence of Antiderivatives)**.** *Let f be rd-continuous and $t_0 \in \mathbb{T}$. Then f has an antiderivative F satisfying $F^\Delta = f$.*

Proof: See [2, Theorem 1.74]. □

Definition 2.5. If f is rd-continuous and $t_0 \in \mathbb{T}$, then we define the *integral*

$$F(t) = \int_{t_0}^{t} f(\tau)\Delta\tau \quad \text{for} \quad t \in \mathbb{T}. \tag{2.1}$$

Therefore for $f \in C_{\mathrm{rd}}$ we have $\int_a^b f(\tau)\Delta\tau = F(b) - F(a)$, where $F^\Delta = f$.

Theorem 2.6. *Let f, g be rd-continuous, $a, b, c \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$. Then*

i. $\int_a^b [\alpha f(t) + \beta g(t)]\Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t$, *ii.* $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$,

iii. $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$,

iv. $\int_a^a f(t)\Delta t = 0$.

Proof: See [2, Theorem 1.77]. □

Theorem 2.7 (Jensen's Inequality)**.** *Let $a,b \in \mathbb{T}$ and $c,d \in \mathbb{R}$. If $g : [a,b] \to (c,d)$ is rd-continuous and $F : (c,d) \to \mathbb{R}$ is continuous and convex, then*

$$F\left(\frac{\int_a^b g(t)\Delta t}{b-a}\right) \le \frac{\int_a^b F(g(t))\Delta t}{b-a}. \tag{2.2}$$

Proof: See [2, Theorem 6.17]. □

3 The Grüss Inequality on Time Scales

Similarly as in [4], the Grüss inequality can be shown for general time scales.

Theorem 3.1. *Let $a,b,s \in \mathbb{T}$, $f, g \in C_{\mathrm{rd}}$ and $f, g : [a,b] \to \mathbb{R}$. Then for*

$$m_1 \le f(s) \le M_1, \quad m_2 \le g(s) \le M_2, \tag{3.1}$$

we have

$$\left| \frac{1}{b-a} \int_a^b f^\sigma(s) g^\sigma(s)\Delta s - \frac{1}{(b-a)^2} \int_a^b f^\sigma(s)\Delta s \int_a^b g^\sigma(s)\Delta s \right| \le \frac{1}{4}(M_1 - m_1)(M_2 - m_2). \tag{3.2}$$

Proof: To prove this theorem we first consider $f(s) = g(s)$ and

$$\frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s = 0.$$

If we define $\nu(s) = \frac{f(s)-m_1}{M_1-m_1}$, then we get

$$f(s) = m_1 + (M_1 - m_1)\nu(s)$$

with $\nu(s) \in [0,1]$. Since

$$\begin{aligned}
\int_a^b (\nu^\sigma(s))^2 \Delta s &\le \int_a^b \nu^\sigma(s)\Delta s \\
&= \int_a^b \frac{f^\sigma(s) - m_1}{M_1 - m_1}\Delta s \\
&= -\frac{m_1}{M_1 - m_1}(b-a),
\end{aligned}$$

we have

$$\begin{aligned} D(f,f) &:= \frac{1}{b-a}\int_a^b (f^\sigma(s))^2\Delta s - \left(\frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s\right)^2 \\ &= \frac{1}{b-a}\int_a^b [m_1+(M_1-m_1)\nu^\sigma(s)]^2\Delta s \\ &\leq m_1^2+2m_1(M_1-m_1)\left(-\frac{m_1}{M_1-m_1}\right)+(M_1-m_1)^2\left(-\frac{m_1}{M_1-m_1}\right) \\ &= -m_1M_1 \\ &= \frac{1}{4}\left((M_1-m_1)^2-(M_1+m_1)^2\right) \\ &\leq \frac{1}{4}(M_1-m_1)^2. \end{aligned}$$

Now consider the case

$$\frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s = F(b-a) \neq 0,$$

where $F \in \mathbb{R}$ and let $f_1(s) = f(s) - F(b-a)$. Therefore

$$\begin{aligned} \frac{1}{b-a}\int_a^b f_1^\sigma(s)\Delta s &= \frac{1}{b-a}\int_a^b (f^\sigma(s)-F(b-a))\Delta s \\ &= \frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s - F(b-a) \\ &= 0 \end{aligned}$$

with

$$f_1(s) \in [m_1-F(b-a), M_1-F(b-a)].$$

Hence f_1 satisfies the assumption from the earlier part of this proof so that

$$D(f_1,f_1) \leq \frac{1}{4}[M_1-F(b-a)-(m_1-F(b-a))]^2 = \frac{1}{4}(M_1-m_1)^2.$$

Moreover we have

$$\begin{aligned} D(f_1,f_1) &= \frac{1}{b-a}\int_a^b (f^\sigma(s)-F(b-a))^2\Delta s \\ &= \frac{1}{b-a}\left(\int_a^b (f^\sigma(s))^2\Delta s - 2F^2(b-a)^3 + F^2(b-a)^3\right) \\ &= \frac{1}{b-a}\int_a^b (f^\sigma(s))^2\Delta s - F^2(b-a)^2 \\ &= D(f,f) \end{aligned}$$

and thus

$$D(f,f) = D(f_1,f_1) \le \frac{1}{4}(M_1 - m_1)^2.$$

Now we consider the case of general functions f and g and assume (3.1). Then

$$\begin{aligned} D(f,g) &:= \frac{1}{b-a}\int_a^b f^\sigma(s)g^\sigma(s)\Delta s - \frac{1}{(b-a)^2}\int_a^b f^\sigma(s)\Delta s \int_a^b g^\sigma(s)\Delta s \\ &= \frac{1}{4}\left[\frac{1}{b-a}\int_a^b ((f^\sigma(s)+g^\sigma(s))^2 - (f^\sigma(s)-g^\sigma(s))^2)\Delta s \right. \\ &\quad \left. - \frac{4}{(b-a)^2}\int_a^b f^\sigma(s)\Delta s \int_a^b g^\sigma(s)\Delta s\right]. \end{aligned}$$

Note that

$$D(f+g, f+g) \le \frac{1}{4}(M_1 + M_2 - m_1 - m_2)^2$$

by what we have shown earlier. With the help of the notation

$$\frac{1}{b-a}\int_a^b f^\sigma(s)\Delta s = \overline{f}, \quad \frac{1}{b-a}\int_a^b g^\sigma(s)\Delta s = \overline{g}$$

which results in

$$\left(\frac{1}{b-a}\int_a^b (f^\sigma(s) - g^\sigma(s))\Delta s\right)^2 = (\overline{f} - \overline{g})^2$$

and Jensen's inequality (see Theorem 2.7) for the convex quadratic function

$$\frac{1}{b-a}\int_a^b (f^\sigma(s) - g^\sigma(s))^2\Delta s \ge \left(\frac{\int_a^b (f^\sigma(s) - g^\sigma(s))\,\Delta s}{b-a}\right)^2 = (\overline{f} - \overline{g})^2,$$

we get

$$\begin{aligned}
D(f,g) &\leq \frac{1}{4}\left[\frac{1}{4}(M_1+M_2-m_1-m_2)^2+\frac{1}{(b-a)^2}\left(\int_a^b (f^\sigma(s)+g^\sigma(s))\Delta s\right)^2\right.\\
&\qquad \left.-\frac{1}{b-a}\int_a^b (f^\sigma(s)-g^\sigma(s))^2\Delta s-4\overline{f}\overline{g}\right]\\
&= \frac{1}{16}(M_1+M_2-m_1-m_2)^2\\
&\qquad +\frac{1}{4}\left[(\overline{f}-\overline{g})^2-\frac{1}{b-a}\int_a^b (f^\sigma(s)-g^\sigma(s))^2\Delta s\right]\\
&\leq \frac{1}{16}(M_1+M_2-m_1-m_2)^2\\
&= \frac{1}{4}\left[(M_1-m_1)(M_2-m_2)+\frac{1}{4}(M_1-m_1-M_2+m_2)^2\right].
\end{aligned}$$

If $M_1-m_1=M_2-m_2$, then clearly

$$D(f,g)\leq \frac{1}{4}(M_1-m_1)(M_2-m_2),$$

but if $M_1-m_1\neq M_2-m_2$, then we define

$$p=\sqrt{\frac{M_2-m_2}{M_1-m_1}},\quad q=\sqrt{\frac{M_1-m_1}{M_2-m_2}}$$

and let $f_1(s)=pf(s)$, $g_1(s)=qg(s)$ and obtain

$$\overline{m}_1=pm_1\leq f_1(s)\leq pM_1=\overline{M}_1,\quad \overline{m}_2=qm_2\leq g_1(s)\leq qM_2=\overline{M}_2.$$

Now we get

$$\begin{aligned}
\overline{M}_1-\overline{m}_1 &= \sqrt{\frac{M_2-m_2}{M_1-m_1}}(M_1-m_1)=\sqrt{(M_1-m_1)(M_2-m_2)}\\
&= \sqrt{\frac{M_1-m_1}{M_2-m_2}}(M_2-m_2)=\overline{M}_2-\overline{m}_2
\end{aligned}$$

and

$$\begin{aligned}
D(f,g) &= pqD(f,g)=D(f_1,g_1)\leq \frac{1}{4}(\overline{M}_1-\overline{m}_1)(\overline{M}_2-\overline{m}_2)\\
&= \frac{1}{4}pq(M_1-m_1)(M_2-m_2)=\frac{1}{4}(M_1-m_1)(M_2-m_2).
\end{aligned}$$

If we consider the case of $-f$, then we can conclude

$$D(-f,g)=-D(f,g)\leq \frac{1}{4}(-m_1+M_1)(M_2-m_2),$$

and finally we get

$$|D(f,g)| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2).$$

Thus (3.2) holds. $\square$

If we apply the Grüss inequality to different time scales, we will get some well-known and some new results.

Corollary 3.2 (continuous case)**.** *Let* $\mathbb{T} = \mathbb{R}$. *We have*

$$\left| \frac{1}{b-a} \int_a^b f(s)g(s)ds - \frac{1}{(b-a)^2} \int_a^b f(s)ds \int_a^b g(s)ds \right| \tag{3.3}$$
$$\leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2),$$

where

$$m_1 \leq f(s) \leq M_1, \quad m_2 \leq g(s) \leq M_2,$$

which is exactly the Grüss inequality shown in [4]. The constant $\frac{1}{4}$ *in the right-hand side of* (3.3) *is the best possible.*

Corollary 3.3 (discrete case)**.** *Let* $\mathbb{T} = \mathbb{Z}$ *and* $a = 0$, $b = n$, $s = j$ *and* $f(k) = x_k$. *Then*

$$\left| \frac{1}{n} \sum_{j=1}^{n} x_j y_j - \frac{1}{n^2} \sum_{j=1}^{n} x_j \sum_{j=1}^{n} y_j \right| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2),$$

where

$$m_1 \leq x_j \leq M_1, \quad m_2 \leq y_j \leq M_2.$$

This corresponds to the result obtained in [3, (1.3)].

In the discrete case the sharpness of this inequality holds only for even n. In [1] it was shown, that

$$\left| \frac{1}{n} \sum_{j=1}^{n} x_j y_j - \frac{1}{n^2} \sum_{j=1}^{n} x_j \sum_{j=1}^{n} y_j \right| \leq \frac{1}{n} \left[\frac{n}{2}\right] \left(1 - \frac{1}{n}\left[\frac{n}{2}\right]\right) (M_1 - m_1)(M_2 - m_2)$$

is sharp in our sense. This inequality now holds for even and odd n.

Corollary 3.4 (quantum calculus case)**.** *Let* $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, $a = q^m$ *and* $b = q^n$. *Then*

$$\left| \frac{\sum\limits_{k=m}^{n-1} q^k f(q^{k+1}) g(q^{k+1})}{\sum\limits_{k=m}^{n-1} q^k} - \frac{1}{\left(\sum\limits_{k=m}^{n-1} q^k\right)^2} \left(\sum_{k=m}^{n-1} q^k f(q^{k+1})\right) \left(\sum_{k=m}^{n-1} q^k g(q^{k+1})\right) \right|$$
$$\leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2),$$

where

$$m_1 \leq f(q^k) \leq M_1, \quad m_2 \leq g(q^k) \leq M_2.$$

As in the discrete case, the $\frac{1}{4}$ *in the right-hand side may not be sharp.*

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References

[1] M. Biernacki, H. Pidek and C. Ryll-Nardzewski: Sur une inégalité entre des intégrales définies, *Ann. Univ. Mariae Curie-Skłodowska. Sect. A.*, vol. 4 (1950), pp. 1–4.

[2] M. Bohner and A. Peterson: Dynamic equations on time scales, Boston, MA: *Birkhäuser Boston Inc.* (2001). An introduction with applications.

[3] S. S. Dragomir: A Grüss type discrete inequality in inner product spaces and applications, *J. Math. Anal. Appl.*, vol. 250, no. 2 (2000), pp. 494–511.

[4] G. Grüss: Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int\limits_a^b f(x)g(x)\,dx - \frac{1}{(b-a)^2}\int\limits_a^b f(x)\,dx \int\limits_a^b g(x)\,dx$, *Math. Z.*, vol. 39, no. 1 (1935), pp. 215–226.