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The logarithm on time scales

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We briefly present the well-studied exponential function on a time scale and pose the problem of finding an appropriate logarithm function on a time scale.

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1. Introduction

An introduction to dynamic equations on time scales can be found in [1,2]. Consider the initial value problem

$$y^{\Delta} = p(t)y, \quad y(t_0) = 1. \tag{1}$$

(Note: If the time scale $\mathbb{T}$ is equal to $\mathbb{R}$, then $y^{\Delta} = y'$, while if $\mathbb{T} = \mathbb{Z}$, then $y^{\Delta} = \Delta y$, but $\mathbb{T}$ is allowed to be any nonempty closed subset of $\mathbb{R}$.) It is well known that (1) has a unique solution (denoted by $e_p(t, t_0)$ and sometimes abbreviated by e_p) provided $p{:}\mathbb{T} \to \mathbb{R}$ is rd-continuous and regressive. Regressive means that $1 + \mu(t)p(t) \neq 0$ holds for all $t \in \mathbb{T}$. (Note: If $\mathbb{T} = \mathbb{R}$, then $\mu(t) \equiv 0$, while if $\mathbb{T} = \mathbb{Z}$, then $\mu(t) \equiv 1$.) The set of all regressive functions is an Abelian group under the circle plus addition:

$$p \oplus q = p + q + \mu pq, \quad \ominus p = -\frac{p}{1 + \mu p}, \quad p \ominus q = \frac{p - q}{1 + \mu q}.$$

Next, defining scalar multiplication by $\alpha \odot p = \alpha p \int_0^1 (1 + \mu(t)p(t)h)^{\alpha - 1}\,\mathrm{d}h$ for $\alpha \in \mathbb{R}$, the set of all positively regressive functions (i.e., $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}$) is made into a vector space. We have:

$$e_p e_q = e_{p \oplus q}, \quad \frac{e_p}{e_q} = e_{p \ominus q}, \quad e_p^{\alpha} = e_{\alpha \odot p}.$$

2. Open Problem

Define a "nice" logarithm function on time scales and present its properties.

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3. First Approach

The Euler–Cauchy differential equation $t^2y'' - 3ty' + 4y = 0$ has $y_1(t) = t^2$ and $y_2(t) = t^2 \ln t$ as two solutions. It can be checked that two solutions of the Euler–Cauchy dynamic equation $t\sigma(t)y^{\Delta\Delta} - 3ty^{\Delta} + 4y = 0$ are

$$y_1(t) = e_{2/t}(t, t_0) \quad \text{and} \quad y_2(t) = e_{2/t}(t, t_0)\int_{t_0}^{t} \frac{\Delta\tau}{\tau + 2\mu(\tau)}, \tag{2}$$

where $t_0 \in \mathbb{T}$. Note that $e_{2/t}$ is the time scales analogue of t^2 so that we somehow could view the integral in (2) as the time scales analogue of $\ln t$.

4. Second Approach

It could also be natural to define a logarithm by

$$L_p(t, t_0) = \int_{t_0}^{t} \frac{p^{\Delta}(\tau)}{p(\tau)}\Delta\tau. \tag{3}$$

Pertinent to the definition (3), the following three formulas hold:

$$\frac{(pq)^{\Delta}}{pq} = \frac{p^{\Delta}}{p} \oplus \frac{q^{\Delta}}{q}, \quad \frac{(p/q)^{\Delta}}{p/q} = \frac{p^{\Delta}}{p} \ominus \frac{q^{\Delta}}{q}, \quad \alpha \odot \frac{p^{\Delta}}{p} = \frac{(p^{\alpha})^{\Delta}}{p^{\alpha}}.$$

But now the resulting formulas are not “nice”, e.g.:

$$L_{pq}(t, t_0) = L_p(t, t_0) + L_q(t, t_0) + \int_{t_0}^{t} \frac{\mu(\tau)p^{\Delta}(\tau)q^{\Delta}(\tau)}{p(\tau)q(\tau)}\Delta\tau.$$

References

[1] Bohner, M. and Peterson, A., 2001, *Dynamic Equations on Time Scales: An Introduction with Applications* (Boston: Birkhäuser).

[2] Bohner, M. and Peterson, A., 2003, *Advances in Dynamic Equations on Time Scales* (Boston: Birkhäuser).