The Linear Quadratic Tracker on time scales

Martin Bohner

Department of Mathematics and Statistics,
Missouri University of Science and Technology,
400 West 12th Street,
Rolla, MO 65409-0020, USA
E-mail: bohner@mst.edu

Nick Wintz*

Department of Mathematics,
Lindenwood University,
209 S. Kingshighway,
St. Charles, MO 63301, USA
E-mail: nwintz@lindenwood.edu
*Corresponding author

Abstract: In this work, we study a natural extension of the Linear Quadratic Regulator (LQR) on time scales. Here, we unify and extend the Linear Quadratic Tracker (LQT). We seek to find an affine optimal control that minimises a cost functional associated with a completely observable linear system. We then find an affine optimal control for the fixed final state case in terms of the current state. Finally we include an example in disturbance/rejection modelling. A numerical example is also included.

Keywords: time scale; dynamic equation; optimal control; regulator problem; tracking problem; cost functional; riccati equation.

Reference to this paper should be made as follows: Bohner, M. and Wintz, N. (2011) 'The Linear Quadratic Tracker on time scales', *Int. J. Dynamical Systems and Differential Equations*, Vol. 3, No. 4, pp.423–447.

Biographical notes: Martin Bohner is a Professor of Mathematics at Missouri S&T. His research interests center around differential, difference, and dynamic equations as well as their applications to economics, finance, biology, physics, and engineering.

Nick Wintz is an Assistant Professor in Mathematics at Lindenwood University. His research interests include differential and difference equations, dynamic equations on time scales, optimal control and estimation, and game theory.

1 Introduction

In the early 1960s, Kalman among others initiated the Linear Quadratic Regulator (LQR) in the continuous and discrete cases (see Kalman, 1960, 1964; Kalman and Koepcke, 1958). Since then the LQR and its extensions have played a fundamental rôle in control engineering. One such extension is the concept of tracking, first considered as a regulator problem by Kalman (1963). Applications in Linear Quadratic Tracking (LQT) include guidance systems, game theory (Bryson and Ho, 1975), and economics (Pindyck, 1972). For a review of the LQT in the continuous and discrete cases, one can see Tables 1 and 2.

Table 1 The continuous version of the LQT

System: $\dot{x} = Ax + Bu$ Output: $y = Cx$
Cost: $J = \frac{1}{2}(y-z)^T(t_f)P(y-z)(t_f) + \frac{1}{2}\int_{t_0}^{t_f} [(y-z)^T Q(y-z) + u^T Ru](\tau)\mathrm{d}\tau$
Gains: Feedback: $K = R^{-1}B^T S$ Feedforward: $K_v = R^{-1}B^T$
Riccati and Output equations: $-\dot{S} = A^T S + S(A - BK) + C^T QC, \quad S(t_\mathrm{f}) = C^T PC$ $-\dot{v} = (A - BK)^T v + C^T Pz, \quad v(t_\mathrm{f}) = C^T Pz(t_\mathrm{f})$
Affine optimal control: $u = -Kx + K_v v$

Table 2 The discrete version of the LQT

System: $x_{k+1} = Ax_k + Bu_k, \quad k > 1$ Output: $y_k = Cx_k$
Cost: $J_i = \frac{1}{2}(y_N - z_N)^T P(y_N - z_N) + \frac{1}{2}\sum_{k=1}^{N-1} [(y_k - z_k)^T Q(y_k - z_k) + u_k^T Ru_k]$
Gains: Feedback: $K_k = (R + B^T S_{k+1}B)^{-1}B^T S_{k+1}A$ Feedforward: $K_k^v = (R + B^T S_{k+1}B)^{-1}B^T$
Riccati and Output equations: $S_k = A^T S_{k+1}(A - BK_k) + C^T QC, \quad S_N = C^T PC$ $v_k = (A - BK_k)^T v_{k+1} + C^T Pz_k, \quad v_N = C^T Pz_N$
Affine optimal control: $u_k = -K_k x_k + K_k^v v_{k+1}$

In this paper, we seek to extend our results on the LQR (see Bohner and Wintz, 2010) to include applications in tracking and disturbance/rejection. Here, we consider the regressive linear time-invariant system

$$x^{\Delta}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t),$$

associated with the quadratic cost functional

$$J = \frac{1}{2}(Cx - z)^T(t_\mathrm{f})P(Cx - z)(t_\mathrm{f})$$

$$+\frac{1}{2}\int_{t_0}^{t_f}[(Cx-z)^T Q(Cx-z)+u^T Ru](\tau)\Delta\tau,$$

where $P, Q \geq 0$, and $R > 0$ (see Lewis and Syrmos, 1995; Athans and Falb, 1966). The functions x, u, y and z represent the state, control (input), output, and the desired reference signal, respectively. We further assume that our system is completely observable and that the final state is free.

The organisation of this paper is as follows. In Section 2, we provide a brief introduction to dynamic equations on time scales. In Section 3, we offer the variational properties needed such that an optimal control exists. Next, we introduce the Linear Quadratic Tracker (LQT) on time scales in Section 4. In this section, we find an affine optimal control law that drives the plant to track a desired reference signal z. This control can be expressed in two terms. The first term represents the feedback term, which allows the optimal input to be expressed in terms of the current state and a term that anticipates the desired reference signal. The second term represents the feedforward term, which anticipates our desired reference signal. In Section 5, we revisit our results on the LQR for the fixed final state case in Bohner and Wintz (2010). Using our results for the LQT, we now express our minimum control in terms of the current state and a term that anticipates the desired reference signal. Even so, our control law still mirrors the controllability criterion we studied in Bohner and Wintz (2011). Finally, we provide some examples in Section 6. These examples include a scalar version of the LQT as well as a disturbance/rejection model. This work comes from the second author's dissertation (Wintz, 2009).

2 Preliminaries

Here we offer a brief introduction to the theory of dynamic equations on time scales. For a more in-depth study of time scales, see Bohner and Peterson's books (Bohner and Peterson, 2001, 2003) as well as some recent contributions (Atici et al., 2011; Kratz et al., 2011; Jackson et al., 2011; Bohner et al., 2011).

Definition 2.1: A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. We let $\mathbb{T}^\kappa = \mathbb{T} \setminus \{\max \mathbb{T}\}$ if $\max \mathbb{T}$ exists; otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

Example 2.2: The most common examples of time scales are $\mathbb{R}$, $\mathbb{Z}$, $h\mathbb{Z}$ for $h > 0$, and $q^{\mathbb{N}_0}$ for $q > 1$.

Definition 2.3: We define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ and the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \qquad \text{and} \qquad \mu(t) := \sigma(t) - t.$$

For any function $f : \mathbb{T} \to \mathbb{R}$, we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by $f^\sigma = f \circ \sigma$.

Next, we define the delta (or Hilger) derivative as follows.

Definition 2.4: Assume $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^\kappa$. The *delta derivative* $f^\Delta(t)$ is the number (when it exists) such that given any $\varepsilon > 0$, there is a neighbourhood U of t such that

$$\left|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\right| \le \varepsilon|\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

In the next two theorems, we consider some properties of the delta derivative.

Theorem 2.5 (See Bohner and Peterson (2001, Theorem 1.16))**:** *Let $f : \mathbb{T} \to \mathbb{R}$ be a function and $t \in \mathbb{T}^\kappa$. Then we have the following:*

a *If f is differentiable at t, then f is continuous at t.*

b *If f is continuous at t, where t is right-scattered, then f is differentiable at t and*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

c *If f is differentiable at t, where t is right-dense, then*

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

d *If f is differentiable at t, then*

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t). \tag{1}$$

Note that (1) is sometimes called the 'simple useful formula'.

Example 2.6: Note the following examples.

a When $\mathbb{T} = \mathbb{R}$, then (if the limit exists)

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t).$$

b When $\mathbb{T} = \mathbb{Z}$, then

$$f^\Delta(t) = f(t+1) - f(t) =: \Delta f(t).$$

c When $\mathbb{T} = h\mathbb{Z}$ for $h > 0$, then

$$f^\Delta(t) = \frac{f(t+h) - f(t)}{h} =: \Delta_h f(t).$$

d When $\mathbb{T} = q^{\mathbb{Z}}$ for $q > 1$, then

$$f^\Delta(t) = \frac{f(qt) - f(t)}{(q-1)t} =: D_q f(t).$$

Next we consider the linearity property as well as the product rules.

Theorem 2.7 (See Bohner and Peterson (2001, Theorem 1.20))**:** *Let* $f, g : \mathbb{T} \to \mathbb{R}$ *be differentiable at* $t \in \mathbb{T}^\kappa$. *Then we have the following:*

a *For any constants* α *and* β, *the sum* $(\alpha f + \beta g) : \mathbb{T} \to \mathbb{R}$ *is differentiable at* t *with*

$$(\alpha f + \beta g)^\Delta(t) = \alpha f^\Delta(t) + \beta g^\Delta(t).$$

b *The product* $fg : \mathbb{T} \to \mathbb{R}$ *is differentiable at* t *with*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t).$$

Definition 2.8: A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *rd-continuous* on $\mathbb{T}$ when f is continuous in points $t \in \mathbb{T}$ with $\sigma(t) = t$ and it has finite left-sided limits in points $t \in \mathbb{T}$ with $\sup\{s \in \mathbb{T} : \ s < t\} = t$. The class of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathrm{C_{rd}} = \mathrm{C_{rd}}(\mathbb{T}) = \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by $\mathrm{C^1_{rd}}$.

Theorem 2.9 (See Bohner and Peterson (2001, Theorem 1.74))**:** *Any rd-continuous function* $f : \mathbb{T} \to \mathbb{R}$ *has an antiderivative F, i.e.,* $F^\Delta = f$ *on* $\mathbb{T}^\kappa$.

Definition 2.10: Let $f \in \mathrm{C_{rd}}$ and let F be any function such that $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$. Then the Cauchy integral of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a) \quad \text{for all} \quad a, b \in \mathbb{T}.$$

Example 2.11: Let $a, b \in \mathbb{T}$ with $a < b$ and assume that $f \in \mathrm{C_{rd}}$.

a When $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)\mathrm{d}t.$$

b When $\mathbb{T} = \mathbb{Z}$, then

$$\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t).$$

c When $\mathbb{T} = h\mathbb{Z}$ for $h > 0$, then

$$\int_a^b f(t)\Delta t = h \sum_{t=a/h}^{b/h-1} f(th).$$

d When $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)\mathrm{d}_q(t) := (q-1) \sum_{t\in[a,b)\cap\mathbb{T}} tf(t).$$

Definition 2.12: An $m \times n$ matrix-valued function A on $\mathbb{T}$ is rd-continuous if each of its entries are rd-continuous. Furthermore, if $m = n$, A is said to be *regressive* (we write $A \in \mathcal{R}$) if

$$I + \mu(t)A(t) \quad \text{is invertible for all} \quad t \in \mathbb{T}^\kappa.$$

3 Optimisation of linear systems on time scales

Definition 3.1: Let $a, b \in \mathbb{T}$ with $a < b$ and $\alpha, \beta \in \mathbb{R}^n$. A function $\hat{y} \in \mathrm{C}^1_{\mathrm{rd}}$ with $\hat{y}(a) = \alpha$, $\hat{y}(b) = \beta$ is said to be a (weak) *local minimum* to the variational problem

$$\mathcal{J}(y) = \int_a^b L(t, y^\sigma(t), y^\Delta(t))\Delta t \to \min, \tag{2}$$

with $y(a) = \alpha$, $y(b) = \beta$, where $L : \mathbb{T} \times \mathbb{R}^{2n} \to \mathbb{R}$, if there exists $\delta > 0$ such that $\|y - \hat{y}\| < \delta$ and $\mathcal{J}(\hat{y}) \le \mathcal{J}(y)$ for all $y \in \mathrm{C}^1_{\mathrm{rd}}$ satisfying $\hat{y}(a) = \alpha$ and $\hat{y}(b) = \beta$. If $\mathcal{J}(\hat{y}) < \mathcal{J}(y)$ for all $\hat{y} \neq y$, then $\hat{y}$ is said to be *proper*. An $\eta \in \mathrm{C}^1_{\mathrm{rd}}$ is called an *admissible variation* of (2) provided $\eta(a) = \eta(b) = 0$. Let $\eta \in \mathrm{C}^1_{\mathrm{rd}}$ be an admissible variation. We define the function $\Phi : \mathbb{R} \to \mathbb{R}$ by

$$\Phi(\varepsilon) = \Phi(\varepsilon; y, \eta) = \mathcal{J}(y + \varepsilon\eta), \quad \varepsilon \in \mathbb{R}.$$

Then the *first variation* of (2) is defined by $\mathcal{J}_1(y, \eta) = \dot{\Phi}(0; y, \eta)$, while the *second variation* of (2) is defined by $\mathcal{J}_2(y, \eta) = \ddot{\Phi}(0; y, \eta)$.

In the next two theorems, we provide necessary and sufficient conditions for a local minimum.

Theorem 3.2 (See Bohner (2004, Theorem 3.2))**:** *If $\hat{y} \in \mathrm{C}^1_{\mathrm{rd}}$ is a local minimum of* (2)*, then $\mathcal{J}_1(\hat{y}, \eta) = 0$ and $\mathcal{J}_2(\hat{y}, \eta) \ge 0$ for all admissible variations η.*

Thorem 3.3 (See Bohner (2004, Theorem 3.3))**:** *Let $\hat{y} \in \mathrm{C}^1_{\mathrm{rd}}$ with $\hat{y}(a) = \alpha$ and $\hat{y}(b) = \beta$. If $\mathcal{J}_1(\hat{y}, \eta) = 0$ and $\mathcal{J}_2(\hat{y}, \eta) > 0$ for all nontrivial admissible variations η, then $\hat{y} \in \mathrm{C}^1_{\mathrm{rd}}$ is a proper weak local minimum to* (2)*.*

Now we consider the linear time-invariant system

$$x^\Delta(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0, \tag{3}$$

where $x \in \mathbb{R}^n$ represents the state and $u \in \mathbb{R}^m$ represents the input. Associated with (3) is the quadratic cost functional

$$J = \frac{1}{2}(Cx - z)^T(t_f)P(Cx - z)(t_f) + \frac{1}{2}\int_{t_0}^{t_f} [(Cx - z)^T Q(Cx - z) + u^T Ru](\tau)\Delta\tau, \tag{4}$$

where $P, Q \geq 0$ and $R > 0$. To minimise (4), we introduce the augmented cost functional

$$\begin{aligned} J^+ &= \Psi(x(t_f))\alpha + \frac{1}{2}(Cx - z)^T(t_f)P(Cx - z)(t_f) \\ &\quad + \frac{1}{2}\int_{t_0}^{t_f} \left[(Cx - z)^T Q(Cx - z) + u^T Ru\right](\tau)\Delta\tau \\ &\quad + \int_{t_0}^{t_f} \left[(\lambda^\sigma)^T(Ax + Bu - x^\Delta)\right](\tau)\Delta\tau \\ &= \Psi(x(t_f))\alpha + \frac{1}{2}(Cx - z)^T(t_f)P(Cx - z)(t_f) \\ &\quad + \int_{t_0}^{t_f} [H(x, u, \lambda^\sigma) - (\lambda^\sigma)^T x^\Delta](\tau)\Delta\tau, \end{aligned}$$

where the so-called *Hamiltonian* H is given by

$$H(x, u, \lambda) = \frac{1}{2}[(Cx - z)^T Q(Cx - z) + u^T Ru] + \lambda^T(Ax + Bu) \tag{5}$$

while

$$\Psi(x(t_f)) = Cx(t_f) - z(t_f) \tag{6}$$

represents a function of the final state. Here $\alpha \in \mathbb{R}$ and $\lambda \in \mathbb{R}^n$ are multipliers to be determined in later sections. Thus we seek an optimal control that not only minimises (4), but also guarantees that (6) is equal to zero.

Next, we provide necessary conditions for an optimal control. We assume that

$$\frac{d}{d\varepsilon}\int_{t_0}^{t_f} f(\tau, \varepsilon)\Delta\tau = \int_{t_0}^{t_f} \frac{\partial}{\partial\varepsilon} f(\tau, \varepsilon)\Delta\tau \tag{7}$$

for all $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ with $f(\cdot, \varepsilon), \partial f(\cdot, \varepsilon)/\partial\varepsilon \in C_{rd}(\mathbb{T})$.

Lemma 3.4: *Assume* (7) *holds. Then the first variation of* J^+ *is zero provided that* x, λ, *and* u *satisfy*

$$x^\Delta = Ax + Bu, \tag{8a}$$
$$-\lambda^\Delta = A^T\lambda^\sigma + C^T Q(Cx - z), \tag{8b}$$
$$0 = Ru + B^T\lambda^\sigma. \tag{8c}$$

Proof: First note that

$$\begin{aligned}\Phi(\varepsilon) &= J^{+}((x,u,\lambda)+\varepsilon(\eta_1,\eta_2,\eta_3))\\ &= [C(x+\varepsilon\eta_1)-z](t_f)\alpha+\frac{1}{2}[C(x+\varepsilon\eta_1)-z]^T(t_f)P[C(x+\varepsilon\eta_1)-z](t_f)\\ &\quad+\frac{1}{2}\int_{t_0}^{t_f}\left\{[C(x+\varepsilon\eta_1)-z]^TQ[C(x+\varepsilon\eta_1)-z]\right\}(\tau)\Delta\tau\\ &\quad+\frac{1}{2}\int_{t_0}^{t_f}\left\{(u+\varepsilon\eta_2)^TR(u+\varepsilon\eta_2)\right\}(\tau)\Delta\tau\\ &\quad+\int_{t_0}^{t_f}\left\{(\lambda^\sigma+\varepsilon\eta_3^\sigma)^T[A(x+\varepsilon\eta_1)+B(u+\varepsilon\eta_2)-(x+\varepsilon\eta_1)^\Delta]\right\}(\tau)\Delta\tau.\end{aligned}$$

Then

$$\begin{aligned}\dot\Phi(\varepsilon) &= C\eta_1(t_f)\alpha+\eta_1^T(t_f)C^TP[C(x+\varepsilon\eta_1)-z](t_f)\\ &\quad+\int_{t_0}^{t_f}\left\{\eta_1^TC^TQ[C(x+\varepsilon\eta_1)-z]+\eta_2^TR(u+\varepsilon\eta_2)\right\}(\tau)\Delta\tau\\ &\quad+\int_{t_0}^{t_f}\left\{(\eta_3^\sigma)^T[A(x+\varepsilon\eta_1)+B(u+\varepsilon\eta_2)-(x+\varepsilon\eta_1)^\Delta]\right\}(\tau)\Delta\tau\\ &\quad+\int_{t_0}^{t_f}\left\{(\lambda^\sigma+\varepsilon\eta_3^\sigma)^T(A\eta_1+B\eta_2-\eta_1^\Delta)\right\}(\tau)\Delta\tau.\end{aligned}$$

Thus the first variation can be written as

$$\begin{aligned}\dot\Phi(0) &= C\eta_1(t_f)\alpha+[C^TP(Cx-z)(t_f)]^T\eta_1(t_f)\\ &\quad+\int_{t_0}^{t_f}\left\{\eta_1^TC^TQ(Cx-z)+\eta_2^TRu\right\}(\tau)\Delta\tau\\ &\quad+\int_{t_0}^{t_f}\left\{(\eta_3^\sigma)^T(Ax+Bu-x^\Delta)+(\lambda^\sigma)^T(A\eta_1+B\eta_2-\eta_1^\Delta)\right\}(\tau)\Delta\tau\\ &= [C\alpha+(C^TP(Cx-z)-\lambda)^T(t_f)]\eta_1(t_f)+(\lambda^T\eta_1)(t_0)\\ &\quad+\int_{t_0}^{t_f}\left\{[\lambda^\Delta+A^T\lambda^\sigma+C^TQ(Cx-z)]^T\eta_1\right\}(\tau)\Delta\tau\\ &\quad+\int_{t_0}^{t_f}\left\{(Ru+B^T\lambda^\sigma)^T\eta_2+(Ax+Bu-x^\Delta)^T\eta_3^\sigma\right\}(\tau)\Delta\tau.\end{aligned}$$

Now in order for $\dot\Phi(0)=0$, we set each coefficient of independent increments $\eta_1,\eta_2,\eta_3^\sigma$ equal to zero. This yields the necessary conditions for a minimum of (4). Using the Hamiltonian (5), we have state and costate equations

$$x^\Delta = H_\lambda(x,u,\lambda^\sigma) = Ax+Bu$$

and

$$-\lambda^\Delta = H_x(x,u,\lambda^\sigma) = A^T\lambda^\sigma+C^TQ(Cx-z).$$

Similarly, we have the stationary condition

$$0 = H_u(x, u, \lambda^\sigma) = Ru + B^T \lambda^\sigma.$$

This concludes the proof. □

Remark 3.5: We note that x, λ, u solve (8) if and only if they solve

$$x^\Delta = Ax - BR^{-1}B^T\lambda^\sigma, \tag{9a}$$

$$-\lambda^\Delta = A^T\lambda^\sigma + C^TQ(Cx - z), \tag{9b}$$

$$u = -R^{-1}B^T\lambda^\sigma. \tag{9c}$$

Note that in order to find an optimal control, one must determine a value for the costate.

Finally, we give sufficient conditions for a local optimal control.

Lemma 3.6: *Assume* (7) *holds. Then the second variation of* J^+ *is positive provided that* η_1 *and* η_2 *satisfy the constraints* $\eta_1^\Delta = A\eta_1 + B\eta_2$ *and* $\eta_2 \neq 0$.

Proof: Taking the second derivative of Φ, we have

$$\ddot{\Phi}(\varepsilon) = \eta_1^T(t_\mathrm{f})C^TPC\eta_1(t_\mathrm{f}) + \int_{t_0}^{t_\mathrm{f}} \left\{\eta_1^TC^TQC\eta_1 + \eta_2^TR\eta_2\right\}(\tau)\Delta\tau$$
$$+2\int_{t_0}^{t_\mathrm{f}} \left\{[A\eta_1 + B\eta_2 - \eta_1^\Delta]^T\eta_3^\sigma\right\}(\tau)\Delta\tau.$$

If we assume that η_1 and η_2 satisfy the constraint

$$\eta_1^\Delta = A\eta_1 + B\eta_2,$$

then the second variation is given by

$$\ddot{\Phi}(0) = \eta_1^T(t_\mathrm{f})C^TPC\eta_1(t_\mathrm{f}) + \int_{t_0}^{t_\mathrm{f}} \left[\eta_1^TC^TQC\eta_1 + \eta_2^TR\eta_2\right](\tau)\Delta\tau. \tag{10}$$

Note that $P, Q \geq 0$ while $R > 0$. Thus if $\eta_2 \neq 0$, then (10) is guaranteed to be positive. □

4 The Linear Quadratic Tracker

In this section, we seek an affine optimal control that tracks our desired reference signal. Here we consider the state and costate equations (9a)–(9b) subject to $x(t_0) = x_0$ and $\lambda(t_\mathrm{f}) = C^TP(Cx(t_\mathrm{f}) - z(t_\mathrm{f}))$. Here (9a) is associated with the quadratic cost functional (4).

Remark 4.1: To solve the given boundary value problem, we assume that λ can be written as a linear combination of the current state and some term that anticipates the final reference signal. As a result we use the affine sweep condition

$$\lambda(t) = S(t)x(t) - v(t), \tag{11}$$

where v represents an output vector driven by z. Using the terminal condition $S(t_{\mathrm{f}}) = C^T PC \geq 0$, it is natural to assume that $S \geq 0$ as well.

Theorem 4.2: *Assume that* $M = I + \mu BR^{-1}B^T S^\sigma$ *is invertible. Suppose that* S *satisfies*

$$-S^\Delta = C^T QC + A^T S^\sigma + (I + \mu A^T)S^\sigma M^{-1}\left(A - BR^{-1}B^T S^\sigma\right), \tag{12}$$

while v *satisfies*

$$-v^\Delta = [A^T - (I + \mu A^T)S^\sigma M^{-1}BR^{-1}B^T]v^\sigma + C^T Qz. \tag{13}$$

If x *satisfies*

$$x^\Delta = M^{-1}[\left(A - BR^{-1}B^T S^\sigma\right)x + BR^{-1}B^T v^\sigma] \tag{14}$$

and λ *is as given by* (11)*, then*

$$-\lambda^\Delta = A^T\lambda^\sigma + C^T Q(Cx - z). \tag{15}$$

Proof: As λ is given by (11), we use the product rule, (12)–(14), and (1) to get

$$\begin{aligned}
-\lambda^\Delta &= -S^\Delta x - S^\sigma x^\Delta + v^\Delta \\
&= C^T QCx + A^T S^\sigma x + (I + \mu A^T)S^\sigma x^\Delta - S^\sigma x^\Delta - A^T v^\sigma - C^T Qz \\
&= A^T S^\sigma(x + \mu x^\Delta) - A^T v^\sigma + C^T Q(Cx - z) \\
&= A^T(Sx - v)^\sigma + C^T Q(Cx - z) \\
&= A^T\lambda^\sigma + C^T Q(Cx - z).
\end{aligned}$$

This gives (15) as desired. □

We offer another form of the matrix Riccati equation on time scales.

Theorem 4.3: *If both* $R + \mu B^T S^\sigma B$ *and* $I + \mu BR^{-1}B^T S^\sigma$ *are invertible, then* S *solves the Riccati equation* (12) *if and only if it solves*

$$\begin{aligned}
-S^\Delta = C^T QC &+ A^T S^\sigma + (I + \mu A^T)S^\sigma A \\
&-(I + \mu A^T)S^\sigma B(R + \mu B^T S^\sigma B)^{-1}B^T S^\sigma(I + \mu A).
\end{aligned}$$

Proof: The proof follows directly from Bohner and Wintz (2010, Lemma 6.3). □

Now we define our feedback and feedforward gains as follows.

Definition 4.4: Let $R + \mu B^T S^\sigma B$ be invertible. Then the matrix-valued functions

$$K(t) = (R + \mu(t)B^T S^\sigma(t)B)^{-1}B^T S^\sigma(t)(I + \mu(t)A) \tag{16}$$

and

$$K_v(t) = (R + \mu(t)B^T S^\sigma(t)B)^{-1}B^T. \tag{17}$$

are called the *state feedback* (or *Kalman gain*) and the *feedforward gain*, respectively.

Lemma 4.5: *Let $R + \mu B^T S^\sigma B$ be invertible. Then*

$$\mu B^T S^\sigma B R^{-1} B^T = \mu B^T B R^{-1} B^T S^\sigma, \quad (R + \mu B^T S^\sigma B)R^{-1}B^T = B^T M^T,$$

and

$$K^T = (I + \mu A)^T S^\sigma M^{-1} B R^{-1}. \tag{18}$$

Proof: We have

$$\begin{aligned} R^{-1}B^T &= (R + \mu B^T S^\sigma B)^{-1}(R + \mu B^T S^\sigma B)R^{-1}B^T \\ &= (R + \mu B^T S^\sigma B)^{-1}B^T(I + \mu S^\sigma B R^{-1} B^T) \\ &= (R + \mu B^T S^\sigma B)^{-1}B^T M^T, \end{aligned}$$

from which all three formulas follow. □

Next we determine the form of the affine control-tracker law that minimises (4).

Theorem 4.6: *Assume that $R + \mu B^T S^\sigma B$ is invertible and suppose that x, λ, u solve* (9) *such that* (11) *holds. Then u can be written in the form*

$$u(t) = -K(t)x(t) + K_v(t)v^\sigma(t), \tag{19}$$

where K is given by (16) *and K_v is given by* (17).

Proof: Using (9c), (11), (8a), and (1), we have

$$\begin{aligned} u &= -R^{-1}B^T(Sx - v)^\sigma \\ &= -R^{-1}B^T S^\sigma(x + \mu x^\Delta) + R^{-1}B^T v^\sigma \\ &= -R^{-1}B^T S^\sigma[(I + \mu A)x + \mu B u] + R^{-1}B^T v^\sigma. \end{aligned}$$

Combining like terms, we have

$$(I + \mu R^{-1}B^T S^\sigma B)u = -R^{-1}B^T S^\sigma(I + \mu A)x + R^{-1}B^T v^\sigma.$$

Premultiplying by R, we have

$$(R + \mu B^T S^\sigma B)u = -B^T S^\sigma(I + \mu A)x + B^T v^\sigma.$$

Then

$$u = -(R + \mu B^T S^\sigma B)^{-1} B^T S^\sigma (I + \mu A)x + (R + \mu B^T S^\sigma B)^{-1} B^T v^\sigma.$$

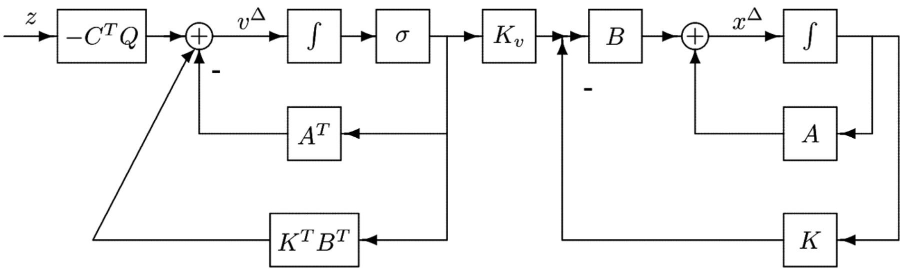
This concludes the proof. □

Now under the control-tracker law (19), the closed-loop plant can be written as

$$x^\Delta = (A - BK)x + BK_v v^\sigma. \tag{20}$$

A block diagram of the affine control scheme is given in Figure 1.

Figure 1 LQT as affine state feedback



Next we rewrite our Riccati and output equations in terms of the closed-loop matrix. We use these equations to determine our optimal cost.

Corollary 4.7: *Let $R + \mu B^T S^\sigma B$ be invertible. Then S solves the Riccati equation* (12) *if and only if it solves*

$$\begin{aligned}-S^\Delta = {} & C^T QC + (A - BK)^T S^\sigma \\ & + (I + \mu(A - BK)^T) S^\sigma (A - BK) + K^T RK.\end{aligned} \tag{21}$$

Similarly v solves the output equation (13) *if and only if it solves*

$$-v^\Delta = (A - BK)^T v^\sigma + C^T Qz. \tag{22}$$

Proof: The proof for S follows from Theorem 4.3 and Bohner and Wintz (2010, Lemma 6.8 and Lemma 6.6). Using (18) in (13), we get (22) directly. □

Note that our Riccati equation (21) is now in Joseph stabilised form (see Lewis and Syrmos, 1995). In the next theorem, we find our optimal cost functional.

Theorem 4.8: *Suppose that S solves* (21) *with*

$$S(t_f) = C^T PC \tag{23}$$

and v solves (22) *with*

$$v(t_f) = C^T Pz(t_f). \tag{24}$$

If x and u satisfy (20) *and* (19)*, then the cost functional* (4) *can be rewritten as*

$$J = \frac{1}{2}x^T(t_0)S(t_0)x(t_0) - x^T(t_0)v(t_0) + w(t_0), \tag{25}$$

where the auxiliary function w satisfies

$$-2w^{\Delta}(t) = z^T(t)Qz(t) - (v^{\sigma})^T(t)BK_v(t)v^{\sigma}(t), \tag{26a}$$

$$w(t_{\mathrm{f}}) = \frac{1}{2}z^T(t_{\mathrm{f}})Pz(t_{\mathrm{f}}). \tag{26b}$$

Proof: We first show

$$\left(x^TSx - 2x^Tv\right)^{\Delta} + (Cx-z)^TQ(Cx-z) + u^TRu = -2w^{\Delta}. \tag{27}$$

To show (27), note that using the product rule, (1), (20), (22), (21), (17), (19), and (26a), we have

$$\begin{aligned}
(x^TSx - 2x^Tv)^{\Delta} &= (x^TS)^{\Delta}x + (x^TS)^{\sigma}x^{\Delta} - 2(x^T)^{\Delta}v^{\sigma} - 2x^Tv^{\Delta} \\
&= ((x^{\Delta})^TS^{\sigma} + x^TS^{\Delta})x + (x+\mu x^{\Delta})^TS^{\sigma}x^{\Delta} \\
&\quad -2(x^T)^{\Delta}v^{\sigma} - 2x^Tv^{\Delta} \\
&= [(A-BK)x + BK_vv^{\sigma}]^TS^{\sigma}x + x^TS^{\Delta}x \\
&\quad +[x+\mu(A-BK)x + \mu BK_vv^{\sigma}]^TS^{\sigma}x^{\Delta} \\
&\quad -2[(A-BK)x + BK_vv^{\sigma}]^Tv^{\sigma} \\
&\quad +2x^T[(A-BK)^Tv^{\sigma} + C^TQz] \\
&= x^T(A-BK)^TS^{\sigma}x + (v^{\sigma})^TK_v^TB^TS^{\sigma}x + x^TS^{\Delta}x \\
&\quad +x^T[I+\mu(A-BK)^T]S^{\sigma}[(A-BK)x + BK_vv^{\sigma}] \\
&\quad +\mu(v^{\sigma})^TK_v^TB^TS^{\sigma}[(A-BK)x + BK_vv^{\sigma}] \\
&\quad -2(v^{\sigma})^TK_v^TB^Tv^{\sigma} + 2x^TC^TQz \\
&= x^T[(A-BK)^TS^{\sigma} + S^{\Delta}]x \\
&\quad +x^T[I+\mu(A-BK)^T]S^{\sigma}(A-BK)x \\
&\quad +x^T[I+\mu(A-BK)^T]S^{\sigma}BK_vv^{\sigma} \\
&\quad +(v^{\sigma})^TK_v^TB^TS^{\sigma}[I+\mu(A-BK)]x + 2x^TC^TQz \\
&\quad +\mu(v^{\sigma})^TK_v^TB^TS^{\sigma}BK_vv^{\sigma} - 2(v^{\sigma})^TK_v^TB^Tv^{\sigma} \\
&= -x^T(C^TQC + K^TRK)x \\
&\quad +2x^T[I+\mu(A-BK)^T]S^{\sigma}BK_vv^{\sigma} - (v^{\sigma})^TK_v^TRK_vv^{\sigma} \\
&\quad -(v^{\sigma})^TBK_vv^{\sigma} + 2x^TC^TQz \\
&= -(Cx-z)^TQ(Cx-z) + z^TQz - x^TK^TRKx \\
&\quad +2x^T[I+\mu(A-BK)^T]S^{\sigma}BK_vv^{\sigma} - (v^{\sigma})^TBK_vv^{\sigma} \\
&\quad -(Kx+u)^TR(Kx+u) \\
&= -(Cx-z)^TQ(Cx-z) + z^TQz - 2x^TK^TRKx \\
&\quad -2x^TK^TRu - u^TRu - (v^{\sigma})^TBK_vv^{\sigma} \\
&\quad +2x^T[I+\mu(A-BK)^T]S^{\sigma}BK_vv^{\sigma} \\
&= -(Cx-z)^TQ(Cx-z) - u^TRu + z^TQz - (v^{\sigma})^TBK_vv^{\sigma} \\
&\quad +2x^T\left\{\left[I+\mu(A-BK)^T\right]S^{\sigma}B - K^TR\right\}K_vv^{\sigma}.
\end{aligned}$$

Now using the feedback gain (16), note that

$$
\begin{aligned}
(I+\mu(A-BK)^T)S^\sigma B - K^T R &= (I+\mu A^T)S^\sigma B - \mu K^T B^T S^\sigma B - K^T R \\
&= (I+\mu A^T)S^\sigma B - K^T(R+\mu B^T S^\sigma B) = 0,
\end{aligned}
$$

and hence (27) holds. Finally, (27), (23), (24), and (26b) imply

$$
\begin{aligned}
J &= \frac{1}{2}(Cx-z)^T(t_\mathrm{f})P(Cx-z)(t_\mathrm{f}) - \frac{1}{2}\int_{t_0}^{t_\mathrm{f}} \left(x^T Sx - 2x^T v\right)^\Delta(\tau)\Delta\tau \\
&\quad + \frac{1}{2}\int_{t_0}^{t_\mathrm{f}} \left[\left(x^T Sx - 2x^T v\right)^\Delta + (Cx-z)^T Q(Cx-z) + u^T Ru\right](\tau)\Delta\tau \\
&= \frac{1}{2}(Cx-z)^T(t_\mathrm{f})P(Cx-z)(t_\mathrm{f}) - \int_{t_0}^{t_\mathrm{f}} \left(\frac{1}{2}x^T Sx - x^T v + w\right)^\Delta(\tau)\Delta\tau \\
&= \frac{1}{2}(Cx-z)^T(t_\mathrm{f})P(Cx-z)(t_\mathrm{f}) - \frac{1}{2}x^T(t_\mathrm{f})S(t_\mathrm{f})x(t_\mathrm{f}) \\
&\quad + x^T(t_\mathrm{f})v(t_\mathrm{f}) - w(t_\mathrm{f}) + \frac{1}{2}x^T(t_0)S(t_0)x(t_0) - x^T(t_0)v(t_0) + w(t_0) \\
&= \frac{1}{2}(Cx-z)^T(t_\mathrm{f})P(Cx-z)(t_\mathrm{f}) - \frac{1}{2}x^T(t_\mathrm{f})C^T PCx(t_\mathrm{f}) \\
&\quad + x^T(t_\mathrm{f})C^T Pz(t_\mathrm{f}) - \frac{1}{2}z^T(t_\mathrm{f})Pz(t_\mathrm{f}) \\
&\quad + \frac{1}{2}x^T(t_0)S(t_0)x(t_0) - x^T(t_0)v(t_0) + w(t_0).
\end{aligned}
$$

This shows (25). □

Remark 4.9: Note that when z is removed, the LQT reduces down to the output quadratic regulator.

5 Linear Quadratic Regulator with final state fixed

In this section, we revisit our results (Bohner and Wintz, 2010) for the LQR. In the fixed final state case, we sought an open-loop control in terms of a final state difference. This in turn required the existence of the inverse of a weighted controllability Gramian. Now using our results on the LQT, we will rewrite this input in terms of the current state. As a result, our optimal control resembles the control-tracker law (19). Here we consider the linear system (9) with $C = I$ and $z = 0$, i.e.,

$$
x^\Delta = Ax - BR^{-1}B^T\lambda^\sigma, \quad -\lambda^\Delta = Qx + A^T\lambda^\sigma, \quad u = -R^{-1}B^T\lambda^\sigma. \tag{28}
$$

Note that (28) is associated with the cost functional

$$
J = \frac{1}{2}x^T(t_\mathrm{f})S(t_\mathrm{f})x(t_\mathrm{f}) + \frac{1}{2}\int_{t_0}^{t_\mathrm{f}} \left(x^T Qx + u^T Ru\right)(\tau)\Delta\tau, \tag{29}
$$

where $R > 0$ and $S(t_f), Q \geq 0$. We let $z(t_f) \in \mathbb{R}^p$ and a $p \times n$-matrix C be given. Moreover, we consider (28) subject to $x(t_0) = x_0$ and $\lambda(t_f) = S(t_f)x(t_f) + C^T\alpha$. Here, we seek an optimal control that not only minimises (29), but also guarantees

$$\Psi(x(t_f), t_f) = Cx(t_f) - z(t_f) = 0. \tag{30}$$

Remark 5.1: In order to solve this two-point boundary value problem, we introduce, as in (11), the affine sweep condition

$$\lambda(t) = S(t)x(t) + V(t)\alpha, \tag{31}$$

where V is not necessarily a square matrix. Again V represents an output matrix.

Theorem 5.2: *Assume that $M = I + \mu BR^{-1}B^T S^\sigma$ is invertible. Suppose that S satisfies*

$$-S^\Delta = Q + A^T S^\sigma + (I + \mu A^T)S^\sigma M^{-1}\left(A - BR^{-1}B^T S^\sigma\right) \tag{32}$$

and V satisfies the output equation

$$-V^\Delta = \left[A^T - (I + \mu A^T)S^\sigma M^{-1}BR^{-1}B^T\right]V^\sigma. \tag{33}$$

If x satisfies

$$x^\Delta = M^{-1}\left[(A - BR^{-1}B^T S^\sigma)x - BR^{-1}B^T V^\sigma\alpha\right] \tag{34}$$

and λ is as in (31)*, then*

$$-\lambda^\Delta = Qx + A^T\lambda^\sigma.$$

Proof: This follows from Theorem 4.2 by using $C = I$, $z = 0$, and $v = -V\alpha$ in Theorem 4.2. □

Next, we find an optimal control that minimises our cost functional.

Theorem 5.3: *Let $R + \mu B^T S^\sigma B$ be invertible and suppose that x, u, λ satisfy* (28) *such that* (31) *holds. Then*

$$u(t) = -K(t)x(t) - K_v(t)V^\sigma(t)\alpha, \tag{35}$$

where K is given by (16) *and K_v is given by* (17)*.*

Proof: This follows from Theorem 4.6 by using $C = I$, $z = 0$, and $v = -V\alpha$ in Theorem 4.6. □

Now under this control law, the closed plant can be written as

$$x^\Delta = (A - BK)x - BK_v V^\sigma\alpha. \tag{36}$$

Next we want to rewrite our Riccati and output equations in terms of the Kalman gain.

Corollary 5.4: *Let* $R + \mu B^T S^\sigma B$ *be invertible. Then* S *solves the Riccati equation* (32) *if and only if it solves*

$$-S^\Delta = Q + (A - BK)^T S^\sigma + (I + \mu(A - BK)^T) S^\sigma (A - BK) + K^T RK.$$

Similarly, V *solves the output equation* (33) *if and only if it solves*

$$-V^\Delta = (A - BK)^T V^\sigma.$$

Proof: This follows from Corollary 4.7 by using $C = I$, $z = 0$, and $v = -V\alpha$ in Corollary 4.7. □

Now looking back at (35), note that the feedforward term represents the term that anticipates a final reference signal. As a result, we want to rewrite the Lagrange multiplier α in terms of this final reference signal. This gives us the following form of our optimal control.

Theorem 5.5: *Suppose that* x *and* u *satisfy* (36) *and* (35). *Furthermore, assume that* (30) *holds and that* V *satisfies* (33) *with* $V(t_f) = C^T$. *If the weighted controllability Gramian*

$$G(t) := -\int_t^{t_f} \left\{ (V^\sigma)^T B (R + \mu B^T S^\sigma B)^{-1} B^T V^\sigma \right\} (\tau) \Delta\tau \tag{37}$$

is invertible, then u *can be written in the form*

$$u(t) = -[K(t) - K_v(t) V^\sigma(t) G^{-1}(t) V^T(t)] x(t) - K_v(t) V^\sigma(t) G^{-1}(t) z(t_f). \tag{38}$$

Proof: We let $\tilde{z} = V^T x + G\alpha$ and use the product rule, (33), (18), (36), (37), and (17) to find

$$\begin{aligned} \tilde{z}^\Delta &= (V^T)^\Delta x + (V^\sigma)^T x^\Delta + G^\Delta \alpha \\ &= -(V^\sigma)^T (A - BK) x + (V^\sigma)^T B K_v V^\sigma \alpha \\ &\quad + (V^\sigma)^T [(A - BK) x - B K_v V^\sigma \alpha] \\ &= 0 \end{aligned}$$

and thus

$$\tilde{z}(t) = \tilde{z}(t_f) = V^T(t_f) x(t_f) + G(t_f)\alpha = Cx(t_f) = z(t_f)$$

by (30). Then

$$z(t_f) = V^T(t) x(t) + G(t)\alpha,$$

which implies that

$$\alpha = G^{-1}(t)[z(t_f) - V^T(t) x(t)].$$

Finally, plugging α into (35) yields (38) as desired. □

Remark 5.6: Note that just as in Bohner and Wintz (2010), the optimal control depends on the inverse of a weighted controllability Gramian. If $\det G(t) = 0$ for all $t \in [t_0, t_\mathrm{f}]$, then the problem is said to be *abnormal* and there is no solution. If we pick $C = 0$, then the problem reduces to the free final state case. On the other hand, if we pick $C = I$, the problem reduces to the fixed final state. However, in Bohner and Wintz (2010), we found an optimal control where $S(t_\mathrm{f}) = Q = 0$. Consequently, our result here is more general.

6 Examples

Example 6.1 (The Continuous LQT): Let $\mathbb{T} = \mathbb{R}$ and consider

$$x'(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t),$$

associated with the cost functional

$$J = \frac{1}{2}(Cx - z)^T(t_\mathrm{f})P(Cx - z)(t_\mathrm{f}) + \frac{1}{2}\int_{t_0}^{t_\mathrm{f}} [(Cx - z)^T Q(Cx - z) + u^T Ru](\tau)\mathrm{d}\tau$$

(observe part (a) of Examples 2.6 and 2.11). Then the state, costate, and stationary equations (9) are given by

$$x' = Ax - BR^{-1}B^T\lambda, \quad -\lambda' = A^T\lambda + C^TQ(Cx - z), \quad u = -R^{-1}B^T\lambda.$$

In this case, our feedback and feedforward gains (16) and (17) are given as

$$K(t) = R^{-1}B^TS(t) \qquad \text{and} \qquad K_v(t) = R^{-1}B^T.$$

Now the control-tracker law (19) and the closed-loop plant (20) can be written as

$$u(t) = -K(t)x(t) + K_v(t)v(t)$$

and

$$x' = (A - BK)x + BK_v v,$$

respectively, and the closed-loop Riccati and output equations (21) and (22) can be written as

$$-S' = C^TQC + K^TRK + S(A - BK) + (A - BK)^TS$$

and

$$-v' = (A - BK)^Tv + C^TQz,$$

respectively. The optimal cost is given by (25), where the function w satisfies

$$-2w' = z^TQz - v^TBK_v v.$$

A summary of these well-known results can be found in Table 1.

Example 6.2 (The Discrete LQT): Let $\mathbb{T} = \mathbb{Z}$ and consider

$$\Delta x(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t).$$

By observing Example 2.6(b) and introducing

$$\tilde{A} = I + A,$$

we can rewrite the system as

$$x(t+1) = \tilde{A}x(t) + Bu(t), \quad y(t) = Cx(t),$$

and the associated cost functional takes the form (observe Example 2.11(b))

$$J = \frac{1}{2}(Cx - z)^T(t_\mathrm{f})P(Cx - z)(t_\mathrm{f}) + \frac{1}{2}\sum_{\tau=t_0}^{t_\mathrm{f}-1}[(Cx - z)^T Q(Cx - z) + u^T Ru](\tau).$$

Then the state, costate, and stationary equations (9) are given by

$$\begin{aligned} x(t+1) &= \tilde{A}x(t) - BR^{-1}B^T\lambda(t+1), \\ \lambda(t) &= \tilde{A}^T\lambda(t+1) + C^TQ(Cx(t) - z(t)), \\ u(t) &= -R^{-1}B^T\lambda(t+1). \end{aligned}$$

In this case, our feedback and feedforward gains (16) and (17) are given as

$$K(t) = (R + B^TS(t+1)B)^{-1}B^TS(t+1)\tilde{A}$$

and

$$K_v(t) = (R + B^TS(t+1)B)^{-1}B^T.$$

Now the control-tracker law (19) and the closed-loop plant (20) can be written as

$$u(t) = -K(t)x(t) + K_v(t)v(t+1)$$

and

$$x(t+1) = (\tilde{A} - BK(t))x(t) + BK_v(t)v(t+1),$$

respectively, and the closed-loop Riccati and output equations (21) and (22) can be written as

$$S(t) = C^TQC + K^T(t)RK(t) + (\tilde{A} - BK(t))^TS(t+1)(\tilde{A} - BK(t))$$

and

$$v(t) = (\tilde{A} - BK(t))^Tv(t+1) + C^TQz(t),$$

respectively. The optimal cost is given by (25), where the function w satisfies

$$w(t) = w(t+1) + \frac{1}{2}z^T(t)Qz(t) - \frac{1}{2}v^T(t+1)BK_v(t)v(t+1).$$

A summary of these well-known results can be found in Table 2.

Example 6.3 (The h-Quantum LQT)**:** Let $\mathbb{T} = h\mathbb{Z}$ with $h > 0$ and consider

$$\Delta_h x(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t),$$

By observing Example 2.6(c) and introducing

$$\tilde{A} = I + hA, \quad \tilde{B} = hB, \quad \tilde{Q} = hQ, \quad \tilde{R} = hR,$$

we can rewrite the system as

$$x(t+h) = \tilde{A}x(t) + \tilde{B}u(t), \quad y(t) = Cx(t),$$

and the associated cost functional takes the form (observe Example 2.11(c))

$$J = \frac{1}{2}(Cx - z)^T(t_f)P(Cx - z)(t_f) + \frac{1}{2}\sum_{\tau=t_0/h}^{t_f/h-1}[(Cx - z)^T\tilde{Q}(Cx - z) + u^T\tilde{R}u](\tau h).$$

Then the state, costate, and stationary equations (9) are given by

$$x(t+h) = \tilde{A}x(t) - \tilde{B}\tilde{R}^{-1}\tilde{B}^T\lambda(t+h),$$
$$\lambda(t) = \tilde{A}^T\lambda(t+h) + C^T\tilde{Q}(Cx(t) - z(t)),$$
$$u(t) = -\tilde{R}^{-1}\tilde{B}^T\lambda(t+h).$$

In this case, our feedback and feedforward gains (16) and (17) are given as

$$K(t) = (\tilde{R} + \tilde{B}^TS(t+h)\tilde{B})^{-1}\tilde{B}^TS(t+h)\tilde{A}$$

and

$$K_v(t) = (\tilde{R} + \tilde{B}^TS(t+h)\tilde{B})^{-1}\tilde{B}^T.$$

Now the control-tracker law (19) and the closed-loop plant (20) can be written as

$$u(t) = -K(t)x(t) + K_v(t)v(t+h)$$

and

$$x(t+h) = (\tilde{A} - \tilde{B}K(t))x(t) + \tilde{B}K_v(t)v(t+h),$$

respectively, and the closed-loop Riccati and output equations (21) and (22) can be written as

$$S(t) = C^T\tilde{Q}C + K^T(t)\tilde{R}K(t) + (\tilde{A} - \tilde{B}K(t))^TS(t+h)(\tilde{A} - \tilde{B}K(t))$$

and

$$v(t) = (\tilde{A} - \tilde{B}K(t))^Tv(t+h) + C^T\tilde{Q}z(t),$$

respectively. The optimal cost is given by (25), where the function w satisfies

$$w(t) = w(t+h) + \frac{1}{2}z^T(t)\tilde{Q}z(t) - \frac{1}{2}v^T(t+h)\tilde{B}K_v(t)v(t+h).$$

Example 6.4 (The q-Quantum LQT)**:** Let $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$ and consider

$$D_q x(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t).$$

By observing Example 2.6(d) and introducing

$$\tilde{A}(t) = I + (q-1)tA, \quad \tilde{B}(t) = (q-1)tB,$$
$$\tilde{Q}(t) = (q-1)tQ, \quad \tilde{R}(t) = (q-1)tR,$$

we can rewrite the system as

$$x(qt) = \tilde{A}(t)x(t) + \tilde{B}(t)u(t), \quad y(t) = Cx(t),$$

and the associated cost functional takes the form (observe Example 2.11(d))

$$J = \frac{1}{2}(Cx - z)^T(t_\mathrm{f})P(Cx - z)(t_\mathrm{f}) + \frac{1}{2}\sum_{\tau\in[t_0,t_\mathrm{f})\cap\mathbb{T}}[(Cx - z)^T\tilde{Q}(Cx - z) + u^T\tilde{R}u](\tau).$$

Then the state, costate, and stationary equations (9) are given by

$$x(qt) = \tilde{A}(t)x(t) - \tilde{B}(t)\tilde{R}^{-1}(t)\tilde{B}^T(t)\lambda(qt),$$
$$\lambda(t) = \tilde{A}^T(t)\lambda(qt) + C^T\tilde{Q}(t)(Cx(t) - z(t)),$$
$$u(t) = -\tilde{R}^{-1}(t)\tilde{B}^T(t)\lambda(qt).$$

In this case, our feedback and feedforward gains (16) and (17) are given as

$$K(t) = (\tilde{R}(t) + \tilde{B}^T(t)S(qt)\tilde{B}(t))^{-1}\tilde{B}^T(t)S(qt)\tilde{A}(t)$$

and

$$K_v(t) = (\tilde{R}(t) + \tilde{B}^T(t)S(qt)\tilde{B}(t))^{-1}\tilde{B}^T(t).$$

Now the control-tracker law (19) and the closed-loop plant (20) can be written as

$$u(t) = -K(t)x(t) + K_v(t)v(qt)$$

and

$$x(qt) = (\tilde{A}(t) - \tilde{B}(t)K(t))x(t) + \tilde{B}(t)K_v(t)v(qt),$$

respectively, and the closed-loop Riccati and output equations (21) and (22) can be written as

$$S(t) = C^T\tilde{Q}(t)C + K^T(t)\tilde{R}(t)K(t) + (\tilde{A}(t) - \tilde{B}(t)K(t))^T S(qt)(\tilde{A}(t) - \tilde{B}(t)K(t))$$

and

$$v(t) = (\tilde{A}(t) - \tilde{B}(t)K(t))^T v(qt) + C^T \tilde{Q}(t)z(t),$$

respectively. The optimal cost is given by (25), where the function w satisfies

$$w(t) = w(qt) + \frac{1}{2}z^T(t)\tilde{Q}(t)z(t) - \frac{1}{2}v^T(qt)\tilde{B}(t)K_v(t)v(qt).$$

Example 6.5 (The Scalar LQT)**:** Consider the scalar control system

$$x^{\Delta}(t) = ax(t) + bu(t), \quad y(t) = cx(t),$$

associated with the cost functional

$$J = \frac{1}{2}p(cx - z))^2(t_{\mathrm{f}}) + \frac{1}{2}\int_{t_0}^{t_{\mathrm{f}}} [q(cx - z)^2 + ru^2](\tau)\Delta\tau.$$

Then the state, costate, and stationary equations (9) are given by

$$x^{\Delta} = ax - \frac{b^2}{r}\lambda^{\sigma}, \quad -\lambda^{\Delta} = a\lambda^{\sigma} + c^2qx - cqz, \quad u = -\frac{b}{r}\lambda^{\sigma}.$$

In this case, our feedback and feedforward gains (16) and (17) are given as

$$k(t) = \frac{b(1 + a\mu(t))s(\sigma(t))}{r + \mu(t)s(\sigma(t))b^2} \quad \text{and} \quad k_v(t) = \frac{b}{r + \mu(t)s(\sigma(t))b^2}.$$

Now the control-tracker law (19) and the closed-loop plant (20) can be written as

$$u(t) = -k(t)x(t) + k_v(t)v(\sigma(t))$$

and

$$x^{\Delta} = (a - bk)x + bk_v v^{\sigma},$$

respectively, and the closed-loop Riccati and output equations (21) and (22) can be written as

$$-s^{\Delta} = qc^2 + rk^2 + (2 + \mu(a - bk))(a - bk)s^{\sigma} = qc^2 + rk^2 + (2 \odot (a - bk))s^{\sigma},$$

(where $2 \odot \alpha := \alpha \oplus \alpha := 2\alpha + \mu\alpha^2$ for $\alpha \in \mathbb{R}$) and

$$-v^{\Delta} = (a - bk)v^{\sigma} + cqz,$$

respectively. The optimal cost (25) is given by

$$J = \frac{1}{2}s(t_0)x^2(t_0) - x(t_0)v(t_0) + w(t_0),$$

where the auxiliary function w satisfies

$$-w^{\Delta}(t) = \frac{1}{2}qz^2(t) - \frac{1}{2}bk_v(t)v^2(\sigma(t)).$$

Next, we extend our results to consider a disturbance/rejection model. In this case, we have a known disturbance (see Lewis and Syrmos, 1995; Dorato et al., 1994) in our state equation.

Example 6.6: In Section 4, we found an affine optimal control in terms of the current state. However, there are circumstances where it is more convenient to express the input in terms of an error term. In this example, we consider the state equation

$$\ell^{\Delta} = A\ell + Bu,$$

where ℓ represents the given state that is possibly corrupt, outdated, or incomplete. Suppose that we want a more desirable state z that contains more information on the process being modelled. Assuming that z is known, when we plug the substitution $x = \ell - z$ into the state equation, we have

$$x^{\Delta} = \ell^{\Delta} - z^{\Delta} = A\ell + Bu - z^{\Delta} = A(x+z) + Bu - z^{\Delta} = Ax + Bu + d,$$

where $d = Az - z^{\Delta}$ is a known disturbance. Then picking $C = I$ and $z = 0$ in (4) and (23), we use the cost functional

$$J = \frac{1}{2}x^T(t_{\mathrm{f}})Px(t_{\mathrm{f}}) + \frac{1}{2}\int_{t_0}^{t_{\mathrm{f}}}(x^TQx + u^TRu)(\tau)\Delta\tau.$$

Similarly, we use here the Hamiltonian

$$H(x,u,\lambda) = \frac{1}{2}(x^TQx + u^TRu) + \lambda^T(Ax + Bu + d)$$

and the state, costate, and stationary equations

$$x^{\Delta} = Ax + Bu + d, \quad -\lambda^{\Delta} = A^T\lambda^{\sigma} + Qx, \quad u = -R^{-1}B^T\lambda^{\sigma},$$

subject to $x(t_0) = x_0$ and $\lambda(t_{\mathrm{f}}) = S(t_{\mathrm{f}})x(t_{\mathrm{f}})$. Again we use the affine sweep condition (11). Suppose that S satisfies (12) (with $C = I$) subject to $S(t_{\mathrm{f}}) = P$ while v satisfies

$$\begin{aligned}-v^{\Delta} &= [A^T - (I + \mu A^T)S^{\sigma}M^{-1}BR^{-1}B^T]v^{\sigma} - (I + \mu A^T)S^{\sigma}M^{-1}d \\ &= (A - BK)^Tv^{\sigma} - [I + \mu(A - BK)^T]S^{\sigma}d\end{aligned}$$

subject to $v(t_{\mathrm{f}}) = 0$. If x satisfies

$$x^{\Delta} = M^{-1}[\left(A - BR^{-1}B^TS^{\sigma}\right)x + d + BR^{-1}B^Tv^{\sigma}]$$

and λ is as given by (11), then

$$\begin{aligned}-\lambda^{\Delta} &= -S^{\Delta}x - S^{\sigma}x^{\Delta} + v^{\Delta} \\ &= Qx + A^TS^{\sigma}x + (I + \mu A^T)S^{\sigma}x^{\Delta} - S^{\sigma}x^{\Delta} - A^Tv^{\sigma} \\ &= A^TS^{\sigma}(x + \mu x^{\Delta}) - A^Tv^{\sigma} + Qx \\ &= A^T(Sx - v)^{\sigma} + Qx \\ &= A^T\lambda^{\sigma} + Qx.\end{aligned}$$

Note that the disturbance d is known and is already accounted for by the output equation. It is customary to pick $d = 0$ when solving for u. This leads to

$$\begin{aligned} u &= -R^{-1}B^T(Sx - v)^\sigma = -R^{-1}B^T S^\sigma(x + \mu x^\Delta) + R^{-1}B^T v^\sigma \\ &= -Kx + K_v v^\sigma, \end{aligned}$$

where the last equation follows as in the proof of Theorem 4.6.

Example 6.7: In this last example, we include a numerical example of the LQT. We consider a tracking model that can be represented by the SISO (single-input, single-output) dynamic system

$$\begin{aligned} x^\Delta &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} u, \quad x_0 = \begin{bmatrix} 7.1 \\ 0 \\ 0 \\ 4.5 \end{bmatrix} \\ y &= \begin{bmatrix} 5 & 0 & 0 & 0 \end{bmatrix} x. \end{aligned}$$

We pick our state variables x_1, x_2, x_3, x_4 to represent the position, velocity, reference angle θ, and θ^Δ, respectively. Given the dynamics of our system, only x_1 is observed. We set the weights in (4) to be $P = Q = R = 1$. Here we use the methods given in Section 4 to find a scalar, affine control that forces the above system to track the deterministic trajectory

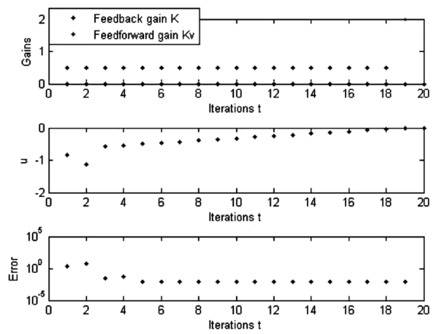
$$z(t) = 0.09(t - 20)^2 + 1.$$

For convenience, we consider only isolated time scales, where it is assumed that the time scale is known *a priori*. We implemented our tracking scheme for 20 iterations. Note since the Riccati and output equations as well as the feedback and feedforward gains do not depend on the current state, these equations can be pre-computed and stored offline. In the first two cases, we use the same time scale throughout the entire iteration. In the third case, we let $\mathbb{T} = \overline{2^{\mathbb{Z}}}$ for $t < 10$ and $\mathbb{T} = 3\mathbb{Z}$ when $t \geq 10$. As a result, the Riccati and output equations are altered midway through the implementation of the tracking scheme. It follows that the gains are also changed as the time scale changes. This is an example of a useful engineering technique called gain scheduling. In Figure 2, we plot the gains, control, and error for each case.

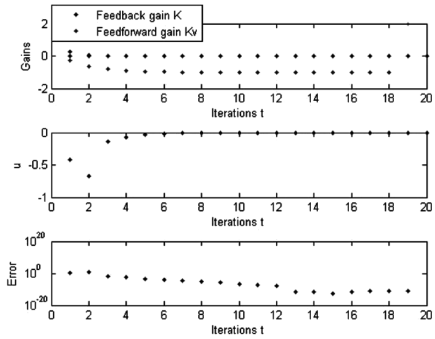
7 Concluding remarks and future work

Example 6.7 offers a potential application for implementing time scales in radar analysis. From a numerical standpoint, our results represent a generalised sampling technique to study flight dynamics of an aircraft, where there are continuous, discrete, or possibly uneven measurements. When considering the flight plan of an aircraft, we can sample the aircraft as it takes off, is in flight, and as it lands as three

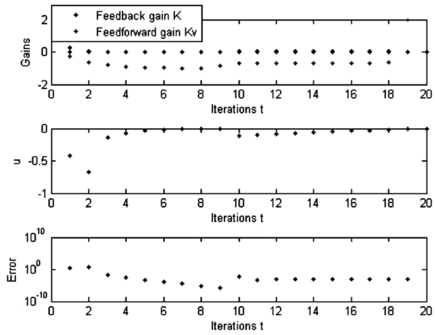
Figure 2 LQT control schemes to track $z(t) = 0.09(t-20)^2 + 1$: (a) Case 1: $\mathbb{T} = \mathbb{Z}$; (b) Case 2: $\mathbb{T} = 2^{\mathbb{Z}}$ and (c) Case 3: $\mathbb{T} = 2^{\mathbb{Z}}$, $t < 10$ and $\mathbb{T} = 3\mathbb{Z}$, $t \geq 10$



(a)



(b)



(c)

distinct time scales. Assuming that the dynamics are stationary and the flight plan is known in advance, we can schedule the gains accordingly. For future research, we seek to track a trajectory, where the time scale is not known in advance but created instead by the dynamics of the aircraft. As a result, the aircraft can be tracked 'on the fly'.

Throughout this paper, we assumed that each component of the state and reference vectors are on the same time scale. However, this is not always realistic. In future work, we seek to study regulator problems, where the components of the states and inputs have different measurements. Note that Example 6.7 represents an ideal scenario when we are tracking a deterministic trajectory. We can also track a stochastic trajectory when the state is also corrupted by noise. This leads us to the development of the Kalman filter on time scales in a forthcoming paper.

References

Athans, M. and Falb, P.L. (1966) *Optimal Control. An Introduction to the Theory and its Applications*, McGraw-Hill Book Co., New York.

Atici, F.M., Biles, D.C. and Lebedinsky, A. (2011) 'A utility maximisation problem on multiple time scales', *Int. J. Dyn. Syst. Differ. Equ.*, Vol. 3, Nos. 1–2, pp.38–47.

Bohner, M. (2004) 'Calculus of variations on time scales', *Dynam. Systems Appl.*, Vol. 13, Nos. 3–4, pp.339–349.

Bohner, M., Matthews, T. and Tuna, A. (2011) 'Diamond-alpha Grüss type inequalities on time scales', *Int. J. Dyn. Syst. Differ. Equ.*, Vol. 3, Nos. 1–2, pp.234–247.

Bohner, M. and Peterson, A. (2001) *Dynamic Equations on Time Scales*, Birkhäuser Boston Inc., Boston, MA.

Bohner, M. and Peterson, A. (Eds.) (2003) *Advances in Dynamic Equations on Time Scales*, Birkhäuser Boston Inc., Boston, MA.

Bohner, M. and Wintz, N. (2010) 'The linear quadratic regulator on time scales', *Int. J. Difference Equ.*, Vol. 5, No. 2, pp.149–174.

Bohner, M. and Wintz, N. (2011) 'Controllability and observability of time-invariant linear dynamic systems', *Math. Bohem.*, To appear.

Bryson Jr., A.E. and Ho, Y.C. (1975) *Applied Optimal Control*, Hemisphere Publishing Corp., Washington DC, Optimization, Estimation, and Control, Revised Printing.

Dorato, P., Cerone, V. and Abdallah, C. (1994). *Linear-Quadratic Control: An Introduction*, Simon & Schuster.

Jackson, B.J., Davis, J.M., Gravagne, I.A. and Marks II, R.J. (2011) 'Linear state feedback stabilisation on time scales', *Int. J. Dyn. Syst. Differ. Equ.*, Vol. 3, Nos. 1–2, pp.163–177.

Kalman, R.E. (1960) 'Contributions to the theory of optimal control', *Bol. Soc. Mat. Mexicana*, Vol. 2, No. 5, pp.102–119.

Kalman, R.E. (1963) 'The theory of optimal control and the calculus of variations', *Mathematical Optimization Techniques*, Univ. California Press, Berkeley, California, pp.309–331.

Kalman, R.E. (1964) 'When is a linear control system optimal?', *Trans. ASME Ser. D. J. Basic Engineering*, Vol. 86, pp.81–90.

Kalman, R.E. and Koepcke, R.W. (1958) 'Optimal synthesis of linear sampling control systems using generalized performance indexes', *Trans. ASME Ser. D. J. Basic Engineering*, Vol. 80, pp.1820–1826.

Kratz, W., Hilscher, R.S. and Zeidan, V. (2011) 'Eigenvalue and oscillation theorems for time scale symplectic systems', *Int. J. Dyn. Syst. Differ. Equ.*, Vol. 3, Nos. 1–2, pp.84–131.

Lewis, F.L. and Syrmos, V.L. (1995) *Optimal Control*, 2nd ed., John Wiley & Sons Inc., New York.

Pindyck, R.S. (1972) 'An application of the linear quadratic tracking problem to economic stabilization policy', *IEEE Trans. Automatic Control*, Vol. AC-17, No. 3, pp.287–300.

Wintz, N. (2009) *The Kalman Filter on Time Scales*, PhD Thesis, Missouri University of Science and Technology, Rolla, Missouri, USA.