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THE LAPLACE TRANSFORM ON ISOLATED TIME SCALES

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ABSTRACT. Starting with a general definition of the Laplace transform on arbitrary time scales, we specify the Laplace transform on isolated time scales, prove several properties of the Laplace transform in this case, and establish a formula for the inverse Laplace transform. The concept of convolution is considered in more detail by proving the convolution theorem and a discrete analogue of the classical theorem of Titchmarsh for the usual continuous convolution.

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1. INTRODUCTION

A time scale is an arbitrary nonempty closed subset of the real numbers. Time scales analysis unifies and extends continuous and discrete analysis, see [3, 5]. The Laplace transform on time scales was introduced by Hilger in [6], but in a form that tries to unify the (continuous) Laplace transform and the (discrete) Z-transform. For arbitrary time scales, the Laplace transform was introduced and investigated by Bohner and Peterson in [4] (see also [3, Section 3.10]). It was further developed by the authors in [1, 2].

Let $\mathbb{T}$ be a time scale with the forward jump operator σ and the delta differentiation operator Δ. Let $\mu(t) = \sigma(t) - t$ for $t \in \mathbb{T}$ (the so-called graininess of the time scale). A function $p : \mathbb{T} \to \mathbb{C}$ is called *regressive* if

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}.$$

The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{C}$ will be denoted by $\mathcal{R}$. Suppose $p \in \mathcal{R}$ and fix $s \in \mathbb{T}$. Then the initial value problem

$$y^{\Delta}(t) = p(t)y(t), \quad y(s) = 1 \tag{1.1}$$

has a unique solution on $\mathbb{T}$. This solution is called the *exponential function* and is denoted by $e_p(t, s)$.

Assume that $\sup \mathbb{T} = \infty$ and fix $t_0 \in \mathbb{T}$. Below we assume that z is a complex constant which is regressive, i.e., $1 + \mu(t)z \neq 0$ for all $t \in \mathbb{T}$. Therefore $e_z(\cdot, t_0)$ is well defined on $\mathbb{T}$. Suppose $x : [t_0, \infty)_{\mathbb{T}} \to \mathbb{C}$ is a locally Δ-integrable function, i.e., it is Δ-integrable over each compact subinterval of $[t_0, \infty)_{\mathbb{T}}$. Then the *Laplace transform* of x is defined by

$$\mathcal{L}\{x\}(z) = \int_{t_0}^{\infty} \frac{x(t)}{e_z(\sigma(t), t_0)} \Delta t \quad \text{for} \quad z \in \mathcal{D}\{x\}, \tag{1.2}$$

where $\mathcal{D}\{x\}$ consists of all complex numbers $z \in \mathcal{R}$ for which the improper integral exists.

The following two concepts are introduced and investigated by the authors in [1]. For a function $f : [t_0, \infty)_{\mathbb{T}} \to \mathbb{C}$, its *shift* (or *delay*) $\widehat{f}(t, s)$ is defined as the solution of the problem

$$\begin{gathered} \widehat{f}^{\Delta_t}(t, \sigma(s)) = -\widehat{f}^{\Delta_s}(t, s), \quad t, s \in \mathbb{T}, \quad t \geq s \geq t_0, \\ \widehat{f}(t, t_0) = f(t), \quad t \in \mathbb{T}, \quad t \geq t_0. \end{gathered} \tag{1.3}$$

For given functions $f, g : [t_0, \infty)_{\mathbb{T}} \to \mathbb{C}$, their *convolution* $f * g$ is defined by

$$(f * g)(t) = \int_{t_0}^{t} \widehat{f}(t, \sigma(s)) g(s) \Delta s, \quad t \in \mathbb{T}, \quad t \geq t_0. \tag{1.4}$$

This paper is organized as follows. In Section 2, we specify and investigate the above concept of Laplace transform for time scales which have graininess that is bounded below by a strictly positive number. These are special cases of so-called isolated time scales. The concept of time scales is actually not needed there and in the remainder of this paper, since all statements and proofs are given directly without referring to this theory. Only this present Section 1 contains time scales concepts and hence shows the origin of this development. However, for a reader to follow the rest of this paper, it is not necessary to be familiar with time scales theory. In Section 3, we present the convolution theorem and a discrete analogue of a classical theorem of Titchmarsh, while Section 4 features a formula for the calculation of the inverse Laplace transform. Finally, in Section 5, we discuss several examples of time scales for which our theory applies, e.g., (see [7, 8]) $h\mathbb{Z}$ with $h > 0$, $q^{\mathbb{N}_0}$ with $q > 1$, and $\mathbb{N}_0^p$ with $p \geq 1$.

2. THE LAPLACE TRANSFORM

Throughout we let t_n be real numbers for all $n \in \mathbb{N}_0$ such that

$$\lim_{n \to \infty} t_n = \infty \quad \text{and} \quad \omega_n := t_{n+1} - t_n > 0 \text{ for all } n \in \mathbb{N}_0, \tag{2.1}$$

while we assume in the main results of this paper that

$$\lim_{n \to \infty} t_n = \infty \quad \text{and} \quad \omega := \inf_{n \in \mathbb{N}_0} \omega_n > 0, \quad \text{where} \quad \omega_n := t_{n+1} - t_n \text{ for } n \in \mathbb{N}_0 \tag{2.2}$$

holds. Note that, for example, the numbers

$$t_n = hn,\ n \in \mathbb{N}_0 \quad \text{and} \quad t_n = q^n,\ n \in \mathbb{N}_0,$$

where $h > 0$ and $q > 1$, respectively, satisfy our assumption (2.2), while the numbers

$$t_n = \sqrt{n},\ n \in \mathbb{N}_0 \quad \text{and} \quad t_n = \ln n,\ n \in \mathbb{N}$$

do not satisfy our assumption (2.2).

Let z be a complex number such that

$$z \neq -\frac{1}{\omega_n} \quad \text{for all} \quad n \in \mathbb{N}_0. \tag{2.3}$$

Then the solution $e_z(t_n, t_m)$ of the (see (1.1)) problem

$$y(t_{n+1}) = (1 + \omega_n z) y(t_n), \quad y(t_m) = 1, \quad m, n \in \mathbb{N}_0$$

satisfies

$$e_z(t_n, t_m) = \prod_{k=m}^{n-1} (1 + \omega_k z) \quad \text{if} \quad n \geq m \tag{2.4}$$

and

$$e_z(t_n, t_m) = \frac{1}{\prod_{k=n}^{m-1} (1 + \omega_k z)} \quad \text{if} \quad n \leq m,$$

where the products for $m = n$ are understood, as usual, to be 1. Thus, in conformance with (1.2), we make the following definition.

Definition 2.1. Assume (2.1). If $x : \{t_n : n \in \mathbb{N}_0\} \to \mathbb{C}$ is a function, then its *Laplace transform* is defined by

$$\widetilde{x}(z) = \mathcal{L}\{x\}(z) = \sum_{n=0}^{\infty} \frac{\omega_n x(t_n)}{\prod_{k=0}^{n} (1 + \omega_k z)} \tag{2.5}$$

for those values $z \in \mathbb{C}$ satisfying (2.3) for which this series converges.

Let us recall our assumptions (2.2) and (2.3). Define

$$P_n(z) := \prod_{k=0}^{n} (1 + \omega_k z), \quad n \in \mathbb{N}_0, \tag{2.6}$$

which is a polynomial in z of degree $n + 1$. It is easily verified that

$$P_n(z) - P_{n-1}(z) = z \omega_n P_{n-1}(z), \quad n \in \mathbb{N}_0 \tag{2.7}$$

and

$$\frac{1}{P_{n-1}(z)} - \frac{1}{P_n(z)} = z \frac{\omega_n}{P_n(z)}, \quad n \in \mathbb{N}_0 \tag{2.8}$$

hold, where $P_{-1}(z) \equiv 1$.

The numbers $\alpha_n = -\omega_n^{-1}$, $n \in \mathbb{N}_0$, belong to the real axis interval $[-\omega^{-1}, 0)$. For any $\delta > 0$ and $n \in \mathbb{N}_0$, we set

$$D_\delta := \mathbb{C} \setminus \bigcup_{n=0}^{\infty} D_\delta^n, \quad \text{where} \quad D_\delta^n := \{z \in \mathbb{C} : \ |z - \alpha_n| < \delta\}, \ n \in \mathbb{N}_0 \tag{2.9}$$

so that D_δ is a closed domain of the complex plane $\mathbb{C}$, and the points of D_δ are in distance not less than δ from the set $\{\alpha_n : n \in \mathbb{N}_0\}$.

Lemma 2.2. *Assume* (2.2), (2.3), (2.6), *and* (2.9). *For any* $z \in D_\delta$, *we have*

$$|P_n(z)| \geq (\delta\omega)^{n+1} \quad \text{and} \quad |P_n(z)| \geq \delta(\delta\omega)^n \omega_n \quad \text{for all} \quad n \in \mathbb{N}_0. \tag{2.10}$$

Moreover,

$$\lim_{n\to\infty} P_n(z) = \infty \quad \text{for all} \quad z \in D_\delta \quad \text{provided} \quad \delta > \omega^{-1}. \tag{2.11}$$

Proof. For any $z \in D_\delta$ and $n \in \mathbb{N}_0$, we have

$$\begin{aligned} |P_n(z)| &= \left| \prod_{k=0}^{n} (1 + \omega_k z) \right| = \left| \prod_{k=0}^{n} (\omega_k (z - \alpha_k)) \right| \\ &= \omega_n \left(\prod_{k=0}^{n-1} \omega_k \right) \left(\prod_{k=0}^{n} |z - \alpha_k| \right) \\ &\geq \omega_n \omega^n \delta^{n+1} = \delta(\delta\omega)^n \omega_n. \end{aligned}$$

Thus the proof of the second statement in (2.10) is complete. The first statement in (2.10) follows from the second statement in (2.10), and (2.11) follows from (2.10). $\square$

Example 2.3. Let us show that

$$\mathcal{L}\{1\}(z) = \frac{1}{z} \quad \text{and} \quad \mathcal{L}\{e_\alpha(\cdot, t_0)\}(z) = \frac{1}{z - \alpha}.$$

We have for $z \in D_\delta$, with $\delta > \omega^{-1}$, using (2.5), (2.6), (2.8), and (2.11),

$$\begin{aligned} \mathcal{L}\{1\}(z) &= \sum_{n=0}^{\infty} \frac{\omega_n}{P_n(z)} = \frac{1}{z} \sum_{n=0}^{\infty} \left[\frac{1}{P_{n-1}(z)} - \frac{1}{P_n(z)} \right] \\ &= \frac{1}{z} \lim_{m\to\infty} \left[1 - \frac{1}{P_m(z)} \right] = \frac{1}{z}. \end{aligned}$$

Now we find the Laplace transform of the function $e_\alpha(t) = e_\alpha(t, t_0)$, for which we have by (2.4) and (2.6),

$$e_\alpha(t_n) = \prod_{k=0}^{n-1} (1 + \omega_k \alpha) = P_{n-1}(\alpha) \quad \text{for} \quad n \in \mathbb{N}_0.$$

It follows that

$$\widetilde{e_\alpha}(z) = \mathcal{L}\{e_\alpha\}(z) = \sum_{n=0}^{\infty} \frac{\omega_n e_\alpha(t_n)}{P_n(z)} = \sum_{n=0}^{\infty} \frac{\omega_n P_{n-1}(\alpha)}{P_n(z)}$$

$$= \sum_{n=0}^{\infty} \frac{\omega_n}{1+\omega_n z} \prod_{k=0}^{n-1} \frac{1+\omega_k \alpha}{1+\omega_k z} = \sum_{n=0}^{\infty} \frac{\omega_n}{1+\omega_n z} \prod_{k=0}^{n-1} \frac{\alpha-\alpha_k}{z-\alpha_k}. \tag{2.12}$$

Since the numbers α_k, $k \in \mathbb{N}_0$, are contained in the finite interval $[-\omega^{-1}, 0)$, there is a sufficiently large number $R_0 > 0$ such that

$$\left| \frac{\alpha-\alpha_k}{z-\alpha_k} \right| \le \frac{1}{2} \quad \text{for all} \quad |z| \ge R_0 \quad \text{and} \quad k \in \mathbb{N}_0. \tag{2.13}$$

Therefore the series (2.12) converges for $|z| \ge R_0$, because

$$\left| \frac{\omega_n}{1+\omega_n z} \right| = \frac{1}{|z-\alpha_n|} \le \frac{1}{\delta}$$

is bounded. Next, we can write, using (2.8),

$$\begin{aligned}
\widetilde{e_\alpha}(z) &= \sum_{n=0}^{\infty} \frac{\omega_n P_{n-1}(\alpha)}{P_n(z)} = \frac{\omega_0}{P_0(z)} + \sum_{n=1}^{\infty} \frac{\omega_n P_{n-1}(\alpha)}{P_n(z)} \\
&= \frac{\omega_0}{P_0(z)} + \frac{1}{z} \sum_{n=1}^{\infty} \left[\frac{P_{n-1}(\alpha)}{P_{n-1}(z)} - \frac{P_{n-1}(\alpha)}{P_n(z)} \right] \\
&= \frac{\omega_0}{P_0(z)} + \frac{1}{z} \sum_{n=1}^{\infty} \left[\frac{(1+\omega_{n-1}\alpha) P_{n-2}(\alpha)}{P_{n-1}(z)} - \frac{P_{n-1}(\alpha)}{P_n(z)} \right] \\
&= \frac{\omega_0}{P_0(z)} + \frac{1}{z} \sum_{n=1}^{\infty} \left[\frac{P_{n-2}(\alpha)}{P_{n-1}(z)} - \frac{P_{n-1}(\alpha)}{P_n(z)} \right] + \frac{\alpha}{z} \sum_{n=1}^{\infty} \frac{\omega_{n-1} P_{n-2}(\alpha)}{P_{n-1}(z)} \\
&= \frac{\omega_0}{P_0(z)} + \frac{1}{z P_0(z)} - \frac{1}{z} \lim_{m\to\infty} \frac{P_{m-1}(\alpha)}{P_m(z)} + \frac{\alpha}{z} \widetilde{e_\alpha}(z) \\
&= \frac{1}{z} + \frac{\alpha}{z} \widetilde{e_\alpha}(z),
\end{aligned}$$

where we have used the fact that

$$\lim_{m\to\infty} \frac{P_{m-1}(\alpha)}{P_m(z)} = 0$$

because of

$$\frac{P_{m-1}(\alpha)}{P_m(z)} = \frac{1}{1+\omega_m z} \prod_{k=0}^{m-1} \frac{\alpha-\alpha_k}{z-\alpha_k}$$

and (2.13). Thus we have obtained the equality

$$\widetilde{e_\alpha}(z) = \frac{1}{z} + \frac{\alpha}{z} \widetilde{e_\alpha}(z).$$

Hence

$$\widetilde{e_\alpha}(z) = \frac{1}{z-\alpha}.$$

Theorem 2.4. *Assume* (2.2). *If the function* $x : \{t_n : n \in \mathbb{N}_0\} \to \mathbb{C}$ *satisfies the condition*

$$|x(t_n)| \le C R^n \quad \textit{for all} \quad n \in \mathbb{N}_0, \tag{2.14}$$

where C and R are some positive constants, then the series in (2.5) *converges uniformly with respect to z in the region D_δ with $\delta > R\omega^{-1}$ and therefore its sum $\widetilde{x}(z)$ is an analytic (holomorphic) function in D_δ.*

Proof. By Lemma 2.2 and (2.14), for the general term of the series in (2.5), we have the estimate

$$\left|\frac{\omega_n x(t_n)}{P_n(z)}\right| \le \frac{\omega_n C R^n}{\delta(\delta\omega)^n\omega_n} = \frac{C}{\delta}\left(\frac{R}{\delta\omega}\right)^n \quad \text{for} \quad n \in \mathbb{N}_0 \quad \text{and} \quad z \in D_\delta.$$

The series

$$\sum_{n=0}^{\infty}\left(\frac{R}{\delta\omega}\right)^n$$

converges if $\delta > R\omega^{-1}$. This completes the proof. □

A large class of functions for which the Laplace transform exists is the class $\mathcal{F}_\delta$ of functions $x : \{t_n : n \in \mathbb{N}_0\} \to \mathbb{C}$ satisfying the condition

$$\sum_{n=0}^{\infty}(\delta\omega)^{-n}|x(t_n)| < \infty. \tag{2.15}$$

Theorem 2.5. *Assume* (2.2). *For any $x \in \mathcal{F}_\delta$, the series in* (2.5) *converges uniformly with respect to z in the region D_δ, and therefore its sum $\widetilde{x}(z)$ is an analytic function in D_δ.*

Proof. The proof follows from the second inequality in (2.10) and from (2.15). □

Theorem 2.6. *Assume* (2.2). *Let $x : \{t_n : n \in \mathbb{N}_0\} \to \mathbb{C}$ be a function and define a new function $x^\Delta : \{t_n : n \in \mathbb{N}_0\} \to \mathbb{C}$ by*

$$x^\Delta(t_n) = \frac{x(t_n + \omega_n) - x(t_n)}{\omega_n}.$$

Suppose that $x \in \mathcal{F}_\delta$. Then $x^\Delta \in \mathcal{F}_\delta$, too, and

$$\mathcal{L}\{x^\Delta\}(z) = z\widetilde{x}(z) - x(t_0). \tag{2.16}$$

Moreover, defining $x^{\Delta\Delta} = (x^\Delta)^\Delta$, we have that $x^{\Delta\Delta} \in \mathcal{F}_\delta$ and that

$$\mathcal{L}\{x^{\Delta\Delta}\}(z) = z^2\widetilde{x}(z) - zx(t_0) - x^\Delta(t_0). \tag{2.17}$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty}(\delta\omega)^{-n}|x^\Delta(t_n)| &= \sum_{n=0}^{\infty}(\delta\omega)^{-n}\frac{|x(t_{n+1}) - x(t_n)|}{\omega_n} \\
&\le \omega^{-1}\sum_{n=0}^{\infty}(\delta\omega)^{-n}\left[|x(t_{n+1})| + |x(t_n)|\right] \\
&= \delta\sum_{n=0}^{\infty}(\delta\omega)^{-n-1}|x(t_{n+1})| + \omega^{-1}\sum_{n=0}^{\infty}(\delta\omega)^{-n}|x(t_n)| \;<\; \infty
\end{aligned}$$

and therefore $x^\Delta \in \mathcal{F}_\delta$. Next, using the definition (2.5) of the Laplace transform, we find

$$\begin{aligned}
\mathcal{L}\{x^\Delta\}(z) &= \sum_{n=0}^{\infty} \frac{\omega_n x^\Delta(t_n)}{P_n(z)} = \sum_{n=0}^{\infty} \frac{x(t_{n+1}) - x(t_n)}{P_n(z)} \\
&= \sum_{n=0}^{\infty} \frac{x(t_{n+1})}{P_n(z)} - \sum_{n=0}^{\infty} \frac{x(t_n)}{P_n(z)} = \sum_{n=0}^{\infty} \frac{x(t_{n+1})}{P_{n+1}(z)}(1+\omega_{n+1}z) - \sum_{n=0}^{\infty} \frac{x(t_n)}{P_n(z)} \\
&= \sum_{n=0}^{\infty} \frac{x(t_{n+1})}{P_{n+1}(z)} - \sum_{n=0}^{\infty} \frac{x(t_n)}{P_n(z)} + z\sum_{n=0}^{\infty} \frac{\omega_{n+1}x(t_{n+1})}{P_{n+1}(z)} \\
&= -\frac{x(t_0)}{P_0(z)} + z\left[\widetilde{x}(z) - \frac{\omega_0 x(t_0)}{P_0(z)}\right] \\
&= -\frac{(1+\omega_0 z)x(t_0)}{P_0(z)} + z\widetilde{x}(z) = -x(t_0) + z\widetilde{x}(z)
\end{aligned}$$

so that (2.16) holds. The formula (2.17) is obtained by applying (2.16) to x^Δ. □

Theorem 2.7 (Initial Value and Final Value Theorem)**.** *Assume* (2.2). *We have:*

(a) *If $x \in \mathcal{F}_\delta$ for some $\delta > 0$, then*

$$x(t_0) = \lim_{z\to\infty} \{z\widetilde{x}(z)\}. \tag{2.18}$$

(b) *If $x \in \mathcal{F}_\delta$ for all $\delta > 0$, then*

$$\lim_{n\to\infty} x(t_n) = \lim_{z\to 0}\{z\widetilde{x}(z)\}. \tag{2.19}$$

Proof. Assume $x \in \mathcal{F}_\delta$ for some $\delta > 0$. It follows from (2.5) that

$$\widetilde{x}(z) = \frac{\omega_0 x(t_0)}{1+\omega_0 z} + \frac{\omega_1 x(t_1)}{(1+\omega_0 z)(1+\omega_1 z)} + \frac{\omega_2 x(t_2)}{(1+\omega_0 z)(1+\omega_1 z)(1+\omega_2 z)} + \dots$$

and

$$(1+\omega_0 z)\widetilde{x}(z) = \omega_0 x(t_0) + \frac{\omega_1 x(t_1)}{1+\omega_1 z} + \frac{\omega_2 x(t_2)}{(1+\omega_1 z)(1+\omega_2 z)} + \dots.$$

Hence

$$\lim_{z\to\infty} \widetilde{x}(z) = 0 \quad \text{and} \quad \lim_{z\to\infty} \{(1+\omega_0 z)\widetilde{x}(z)\} = \omega_0 x(t_0),$$

which yield (2.18). To show (2.19), assume $x \in \mathcal{F}_\delta$ for all $\delta > 0$. In the proof of Theorem 2.6 we have obtained the formula

$$\sum_{n=0}^{\infty} \frac{x(t_{n+1}) - x(t_n)}{P_n(z)} = z\widetilde{x}(z) - x(t_0).$$

Hence, using Lemma 2.2 and taking into account that

$$\lim_{z\to 0} P_n(z) = 1 \quad \text{for any} \quad n \in \mathbb{N}_0,$$

it is not difficult to arrive at (2.19). □

3. THE CONVOLUTION

In this section we only assume (2.1). For a given function $f:\{t_n: n\in\mathbb{N}_0\}\to\mathbb{C}$, we consider the *shifting problem* (see (1.3))

$$\begin{gathered}\omega_m\left[\widehat{f}(t_{n+1},t_{m+1})-\widehat{f}(t_n,t_{m+1})\right]+\omega_n\left[\widehat{f}(t_n,t_{m+1})-\widehat{f}(t_n,t_m)\right]=0,\\ m,n\in\mathbb{N}_0,\quad n\ge m,\\ \widehat{f}(t_n,t_0)=f(t_n),\quad n\in\mathbb{N}_0.\end{gathered}\tag{3.1}$$

Theorem 3.1. *Assume* (2.1)*. For an arbitrary function* $f:\{t_n: n\in\mathbb{N}_0\}\to\mathbb{C}$*, the shifting problem* (3.1) *has a unique solution.*

Proof. Setting $\widehat{f}(t_n,t_m)=\widehat{f}_{n,m}$ for brevity, we rewrite the shifting problem (3.1) in the form

$$\omega_m\left(\widehat{f}_{n+1,m+1}-\widehat{f}_{n,m+1}\right)+\omega_n\left(\widehat{f}_{n,m+1}-\widehat{f}_{n,m}\right)=0,\quad m,n\in\mathbb{N}_0,\quad n\ge m,\tag{3.2}$$

$$\widehat{f}_{n,0}=f(t_n),\quad n\in\mathbb{N}_0,\tag{3.3}$$

where $\widehat{f}_{n,m}$ defined for $m,n\in\mathbb{N}_0$ with $m\le n$ is a desired solution. Note that for $m=n$ in (3.2) there arises the term $\widehat{f}_{n,n+1}$ in which the second index is greater than the first one, but this term arises in (3.2) in two places with the same coefficient and opposite signs and therefore this term cancels. Assume that $\widehat{f}_{n,m}$ is a solution of problem (3.2), (3.3). Putting in (3.2) $m=n$, we get

$$\widehat{f}_{n+1,n+1}=\widehat{f}_{n,n}\quad\text{for all}\quad n\in\mathbb{N}_0.$$

Therefore $\widehat{f}_{n,n}$ is constant for $n\in\mathbb{N}_0$, and since $\widehat{f}_{0,0}=f(t_0)$ by (3.3), we obtain

$$\widehat{f}_{n,n}=f(t_0)\quad\text{for all}\quad n\in\mathbb{N}_0.\tag{3.4}$$

Consequently, it is enough to show that equation (3.2) has a unique solution satisfying conditions (3.3) and (3.4). We will do this by showing that equation (3.2) can be solved recurrently under the conditions (3.3) and (3.4). For any $i\in\mathbb{N}_0$, let us set $\mathbb{N}_i=[i,\infty)\cap\mathbb{N}_0$. Putting $m=n-1$ with $n\in\mathbb{N}_1$ in (3.2), we get

$$\omega_{n-1}\left(\widehat{f}_{n+1,n}-\widehat{f}_{n,n}\right)+\omega_n\left(\widehat{f}_{n,n}-\widehat{f}_{n,n-1}\right)=0,\quad n\in\mathbb{N}_1.$$

Hence, taking into account (3.4), we get

$$\widehat{f}_{n+1,n}=\left(1-\frac{\omega_n}{\omega_{n-1}}\right)f(t_0)+\frac{\omega_n}{\omega_{n-1}}\widehat{f}_{n,n-1},\quad n\in\mathbb{N}_1,\tag{3.5}$$

and besides, by (3.3),

$$\widehat{f}_{1,0}=f(t_1).\tag{3.6}$$

Using the initial condition (3.6), we find $\widehat{f}_{n+1,n}$ from (3.5) recursively in a unique way for all $n\in\mathbb{N}_0$. Next, we put $m=n-2$ with $n\in\mathbb{N}_2$ in (3.2) to get

$$\omega_{n-2}\left(\widehat{f}_{n+1,n-1}-\widehat{f}_{n,n-1}\right)+\omega_n\left(\widehat{f}_{n,n-1}-\widehat{f}_{n,n-2}\right)=0,\quad n\in\mathbb{N}_2.$$

Hence

$$\widehat{f}_{n+1,n-1} = \left(1 - \frac{\omega_n}{\omega_{n-2}}\right) \widehat{f}_{n,n-1} + \frac{\omega_n}{\omega_{n-2}} \widehat{f}_{n,n-2}, \quad n \in \mathbb{N}_2, \tag{3.7}$$

and besides, by (3.3),

$$\widehat{f}_{2,0} = f(t_2). \tag{3.8}$$

In equation (3.7) the term $\widehat{f}_{n,n-1}$ is known for all $n \in \mathbb{N}_1$ from the first step. Therefore, using the initial value (3.8), we can find $\widehat{f}_{n+1,n-1}$ from (3.7) recursively in a unique way for all $n \in \mathbb{N}_1$. Repeating this procedure, we put $m = n - i$ for $n \in \mathbb{N}_i$ in (3.2) to get

$$\omega_{n-i} \left(\widehat{f}_{n+1,n-i+1} - \widehat{f}_{n,n-i+1}\right) + \omega_n \left(\widehat{f}_{n,n-i+1} - \widehat{f}_{n,n-i}\right) = 0, \quad n \in \mathbb{N}_i.$$

Hence

$$\widehat{f}_{n+1,n-i+1} = \left(1 - \frac{\omega_n}{\omega_{n-i}}\right) \widehat{f}_{n,n-i+1} + \frac{\omega_n}{\omega_{n-i}} \widehat{f}_{n,n-i}, \quad n \in \mathbb{N}_i, \tag{3.9}$$

and besides, by (3.3),

$$\widehat{f}_{i,0} = f(t_i). \tag{3.10}$$

In equation (3.9) the term $\widehat{f}_{n,n-i+1}$ is known for all $n \in \mathbb{N}_{i-1}$ from the previous step. Therefore, using the initial value (3.10), we can find $\widehat{f}_{n+1,n-i+1}$ from (3.9) recursively in a unique way for all $n \in \mathbb{N}_{i-1}$. Since $i \in \mathbb{N}$ can be taken arbitrarily, we see that $\widehat{f}_{n,m}$ is constructed in this way uniquely for all $m, n \in \mathbb{N}_0$ with $m \leq n$. □

We now introduce the following definition (see (1.4)) of the convolution of two functions.

Definition 3.2. Assume (2.1), let $f, g : \{t_n : n \in \mathbb{N}_0\} \to \mathbb{C}$ be two functions, and let $\widehat{f}$ be the solution of the shifting problem (3.1). Then the *convolution* $f * g$ of f and g is defined by $(f * g)(t_0) = 0$ and

$$(f * g)(t_n) = \sum_{k=0}^{n-1} \omega_k \widehat{f}(t_n, t_{k+1}) g(t_k), \quad n \in \mathbb{N}_0.$$

The following theorem is a discrete analogue of the classical theorem of Titchmarsh [9, 10] for the usual continuous convolution.

Theorem 3.3. *Assume* (2.1) *and let* $f, g : \{t_n : n \in \mathbb{N}_0\} \to \mathbb{C}$ *be two functions. If* $f * g$ *is identically equal to zero on* $\{t_n : n \in \mathbb{N}_0\}$*, then at least one of the functions* f *and* g *is identically equal to zero on* $\{t_n : n \in \mathbb{N}_0\}$.

Proof. Assume that $f * g$ is identically zero on $\{t_n : n \in \mathbb{N}_0\}$. Then we have for any $n \in \mathbb{N}_0$, by Definition 3.2 of the convolution and the notation $\widehat{f}(t_n, t_m) = \widehat{f}_{n,m}$,

$$\begin{cases} \omega_0 \widehat{f}_{1,1} g(t_0) = 0, \\ \omega_0 \widehat{f}_{2,1} g(t_0) + \omega_1 \widehat{f}_{2,2} g(t_1) = 0, \\ \omega_0 \widehat{f}_{3,1} g(t_0) + \omega_1 \widehat{f}_{3,2} g(t_1) + \omega_2 \widehat{f}_{3,3} g(t_2) = 0, \\ \qquad \vdots \\ \omega_0 \widehat{f}_{n,1} g(t_0) + \omega_1 \widehat{f}_{n,2} g(t_1) + \ldots + \omega_{n-1} \widehat{f}_{n,n} g(t_{n-1}) = 0, \end{cases} \tag{3.11}$$

where we can take $n \in \mathbb{N}$ as large as we wish. It is sufficient to show that if f is not identically zero on $\{t_n : n \in \mathbb{N}_0\}$, then g is identically zero on $\{t_n : n \in \mathbb{N}_0\}$. Thus assume that f is not identically zero on $\{t_n : n \in \mathbb{N}_0\}$. Let $f(t_m)$ with an $m \in \mathbb{N}_0$ be the first of the values of $f(t_0), f(t_1), \ldots$ that is different from zero. Hence

$$f(t_0) = \ldots = f(t_{m-1}) = 0 \quad \text{and} \quad f(t_m) \neq 0. \tag{3.12}$$

We have to show that then $g(t_n) = 0$ for all $n \in \mathbb{N}_0$. Let us consider the possible values of $m \in \mathbb{N}_0$ in (3.12) separately.

If $m = 0$ in (3.12), then we have $f(t_0) \neq 0$. Consider (3.11) as a homogeneous system of linear algebraic equations $A_0 x_0 = 0$ with $x_0 = (g(t_0), g(t_1), \ldots, g(t_{n-1}))^T$. The determinant of the matrix A_0 (being a triangular matrix) is equal to

$$\prod_{k=0}^{n-1} \omega_k \widehat{f}_{k+1,k+1} = [f(t_0)]^n \prod_{k=0}^{n-1} \omega_k,$$

where we have used (3.4). Hence the determinant of A_0 is different from zero by the assumption $f(t_0) \neq 0$. Therefore A_0 is invertible and the equation $A_0 x_0 = 0$ implies $x_0 = 0$, i.e., $g(t_0) = g(t_1) = \ldots = g(t_{n-1}) = 0$. Since $n \in \mathbb{N}$ is arbitrary, we get that g is identically zero on $\{t_n : n \in \mathbb{N}_0\}$.

If $m = 1$ in (3.12), then we have

$$f(t_0) = 0 \quad \text{and} \quad f(t_1) \neq 0. \tag{3.13}$$

In this case, (3.4) implies that

$$\widehat{f}_{n,n} = 0 \quad \text{for all} \quad n \in \mathbb{N}_0, \tag{3.14}$$

and the system (3.11) becomes

$$\begin{cases} \omega_0 \widehat{f}_{2,1} g(t_0) = 0, \\ \omega_0 \widehat{f}_{3,1} g(t_0) + \omega_1 \widehat{f}_{3,2} g(t_1) = 0, \\ \omega_0 \widehat{f}_{4,1} g(t_0) + \omega_1 \widehat{f}_{4,2} g(t_1) + \omega_2 \widehat{f}_{4,3} g(t_2) = 0, \\ \qquad \vdots \\ \omega_0 \widehat{f}_{n,1} g(t_0) + \omega_1 \widehat{f}_{n,2} g(t_1) + \ldots + \omega_{n-2} \widehat{f}_{n,n-1} g(t_{n-2}) = 0. \end{cases} \tag{3.15}$$

Next, since $f(t_0)=0$, we have from (3.5) that

$$\widehat{f}_{n+1,n}=\frac{\omega_n}{\omega_{n-1}}\widehat{f}_{n,n-1},\quad n\in\mathbb{N}_1.$$

Iterating this equation and taking into account (3.3), we find

$$\widehat{f}_{n+1,n}=\frac{\omega_n}{\omega_0}\widehat{f}_{1,0}=\frac{\omega_n}{\omega_0}f(t_1),\quad n\in\mathbb{N}_0. \tag{3.16}$$

Considering the system (3.15) as before as a system $A_1x_1=0$, the determinant of A_1 is found to be equal to

$$\prod_{k=0}^{n-2}\omega_k\widehat{f}_{k+2,k+1}=[f(t_1)]^{n-1}\prod_{k=0}^{n-2}\frac{\omega_{k+1}}{\omega_0}$$

and hence is different from zero by (3.13). Therefore $x_1=0$, i.e., $g(t_0)=g(t_1)=\ldots=g(t_{n-2})=0$, and since $n\in\mathbb{N}$ is arbitrary, we get that g is identically zero on $\{t_n:n\in\mathbb{N}_0\}$.

If $m=2$ in (3.12), then we have

$$f(t_0)=f(t_1)=0\quad\text{and}\quad f(t_2)\neq 0. \tag{3.17}$$

In this case, (3.14) and (3.16) still hold. Besides, by $f(t_1)=0$, equation (3.16) yields

$$\widehat{f}_{n+1,n}=0\quad\text{for all}\quad n\in\mathbb{N}_0. \tag{3.18}$$

Therefore the system (3.11) becomes

$$\begin{cases}\omega_0\widehat{f}_{3,1}g(t_0)=0,\\ \omega_0\widehat{f}_{4,1}g(t_0)+\omega_1\widehat{f}_{4,2}g(t_1)=0,\\ \omega_0\widehat{f}_{5,1}g(t_0)+\omega_1\widehat{f}_{5,2}g(t_1)+\omega_2\widehat{f}_{5,3}g(t_2)=0,\\ \quad\vdots\\ \omega_0\widehat{f}_{n,1}g(t_0)+\omega_1\widehat{f}_{n,2}g(t_1)+\ldots+\omega_{n-3}\widehat{f}_{n,n-2}g(t_{n-3})=0.\end{cases} \tag{3.19}$$

Next, since $\widehat{f}_{n,n-1}=0$ for $n\in\mathbb{N}_1$ by (3.18), we have from (3.7) that

$$\widehat{f}_{n+1,n-1}=\frac{\omega_n}{\omega_{n-2}}\widehat{f}_{n,n-2},\quad n\in\mathbb{N}_2.$$

Iterating the last equation, we find

$$\widehat{f}_{n+1,n-1}=\frac{\omega_n\omega_{n-1}}{\omega_1\omega_0}\widehat{f}_{2,0}=\frac{\omega_n\omega_{n-1}}{\omega_1\omega_0}f(t_2),\quad n\in\mathbb{N}_1.$$

Writing the system (3.19) again as $A_2x_2=0$, the determinant of A_2 is equal to

$$\prod_{k=0}^{n-3}\omega_k\widehat{f}_{k+3,k+1}=[f(t_2)]^{n-2}\prod_{k=0}^{n-3}\frac{\omega_{k+2}\omega_{k+1}}{\omega_1\omega_0}$$

and hence is different from zero by (3.17). Then $x_2=0$, i.e., $g(t_0)=g(t_1)=\ldots=g(t_{n-3})=0$, and since $n\in\mathbb{N}$ is arbitrary, we get that g is identically zero on $\{t_n:n\in\mathbb{N}_0\}$.

We see that one can discuss the system $A_m x_m = 0$ and argue in this way for any value of $m \in \mathbb{N}_0$ in (3.12) in order to obtain that $g(t_n) = 0$ for all $n \in \mathbb{N}_0$. □

Remark 3.4. Note that in the special case $t_n = hn$, $n \in \mathbb{N}_0$, where $h > 0$ is a fixed real number, for any two functions $f, g : \{t_n : n \in \mathbb{N}_0\} \to \mathbb{C}$ their convolution $f * g$ has the form (see [2])

$$(f * g)(nh) = h \sum_{k=0}^{n-1} f(nh - kh - h) g(kh) \quad \text{for} \quad n \in \mathbb{N}_0,$$

and therefore equations (3.11) in the proof of Theorem 3.3 take in this case the form

$$\begin{cases} f(0)g(0) = 0, \\ f(h)g(0) + f(0)g(h) = 0, \\ f(2h)g(0) + f(h)g(h) + f(0)g(2h) = 0, \\ \quad \vdots \\ f((n-1)h)g(0) + f((n-2)h)g(h) + \ldots + f(0)g((n-1)h) = 0, \end{cases}$$

which is much easier for illustrating the reasoning made in the proof of Theorem 3.3.

Theorem 3.5 (Convolution Theorem)**.** *Assume* (2.1) *and let* $f, g : \{t_n : n \in \mathbb{N}_0\} \to \mathbb{C}$ *be two functions such that* $\mathcal{L}\{f\}(z)$, $\mathcal{L}\{g\}(z)$, *and* $\mathcal{L}\{f * g\}(z)$ *exist for a given* $z \in \mathbb{C}$ *satisfying* (2.3)*. Then, at the point* z,

$$\mathcal{L}\{f * g\}(z) = \mathcal{L}\{f\}(z) \cdot \mathcal{L}\{g\}(z). \tag{3.20}$$

Proof. For brevity let us set

$$e_{n,m}(z) := e_z(t_n, t_m) \quad \text{and} \quad \widehat{f}_{n,m} := \widehat{f}(t_n, t_m).$$

Then (2.4) gives

$$e_{n,n}(z) = 1 \quad \text{for all} \quad n \in \mathbb{N}_0 \tag{3.21}$$

$$e_{n+1,m}(z) = (1 + \omega_n z) e_{n,m}(z) \quad \text{for all} \quad n, m \in \mathbb{N}_0 \quad \text{with} \quad n \geq m \tag{3.22}$$

$$e_{n,m+1}(z) = \frac{e_{n,m}(z)}{1 + \omega_m z} \quad \text{for all} \quad n, m \in \mathbb{N}_0 \quad \text{with} \quad n \geq m + 1, \tag{3.23}$$

and the shifting problem (3.1) can be rewritten as (3.2), (3.3). Using definition (2.5) of the Laplace transform and Definition 3.2 for the convolution, we have

$$\begin{aligned} \mathcal{L}\{f * g\}(z) &= \sum_{n=1}^{\infty} \frac{\omega_n (f * g)(t_n)}{e_{n+1,0}(z)} = \sum_{n=1}^{\infty} \frac{\omega_n}{e_{n+1,0}(z)} \sum_{k=0}^{n-1} \omega_k \widehat{f}_{n,k+1} g(t_k) \\ &= \sum_{k=0}^{\infty} \omega_k g(t_k) \sum_{n=k+1}^{\infty} \frac{\omega_n \widehat{f}_{n,k+1}}{e_{n+1,0}(z)}. \end{aligned}$$

Substituting here

$$e_{n+1,0}(z) = e_{n+1,k+1}(z) e_{k+1,0}(z),$$

we get

$$\begin{aligned}\mathcal{L}\{f*g\}(z) &= \sum_{k=0}^{\infty}\frac{\omega_k g(t_k)}{e_{k+1,0}(z)}\sum_{n=k+1}^{\infty}\frac{\omega_n\widehat{f}_{n,k+1}}{e_{n+1,k+1}(z)}\\ &= \mathcal{L}\{g\}(z)\sum_{n=k+1}^{\infty}\frac{\omega_n\widehat{f}_{n,k+1}}{e_{n+1,k+1}(z)}.\end{aligned}\tag{3.24}$$

Let us set

$$\Psi_m = \sum_{n=m}^{\infty}\frac{\omega_n\widehat{f}_{n,m}}{e_{n+1,m}(z)},\quad m\in\mathbb{N}_0.\tag{3.25}$$

We will show that Ψ_m is independent of $m\in\mathbb{N}_0$, which then implies

$$\sum_{n=k+1}^{\infty}\frac{\omega_n\widehat{f}_{n,k+1}}{e_{n+1,k+1}(z)} = \sum_{n=0}^{\infty}\frac{\omega_n\widehat{f}_{n,0}}{e_{n+1,0}(z)} = \sum_{n=0}^{\infty}\frac{\omega_n f(t_n)}{e_{n+1,0}(z)} = \mathcal{L}\{f\}(z),$$

and thus yields (3.20) by using (3.24). So, it remains to show that the quantity Ψ_m defined by (3.25) does not depend on $m\in\mathbb{N}_0$. We have, putting $e_{n,m} := e_{n,m}(z)$ and using (3.2) and (3.21), (3.22), and (3.23),

$$\begin{aligned}\Psi_{m+1} &= \sum_{n=m+1}^{\infty}\frac{\omega_n\widehat{f}_{n,m+1}}{e_{n+1,m+1}} = \sum_{n=m+1}^{\infty}\frac{\omega_n\widehat{f}_{n,m}+\omega_m\widehat{f}_{n,m+1}-\omega_m\widehat{f}_{n+1,m+1}}{e_{n+1,m+1}}\\ &= \sum_{n=m+1}^{\infty}\frac{\omega_n\widehat{f}_{n,m}}{e_{n+1,m+1}} - \omega_m\sum_{n=m+1}^{\infty}\left[\frac{\widehat{f}_{n+1,m+1}}{e_{n+1,m+1}}-\frac{\widehat{f}_{n,m+1}}{e_{n,m+1}}+\frac{\widehat{f}_{n,m+1}}{e_{n,m+1}}-\frac{\widehat{f}_{n,m+1}}{e_{n+1,m+1}}\right]\\ &= \sum_{n=m+1}^{\infty}\frac{\omega_n\widehat{f}_{n,m}}{e_{n+1,m}}(1+\omega_m z)+\omega_m\frac{\widehat{f}_{m+1,m+1}}{e_{m+1,m+1}}-\omega_m\sum_{n=m+1}^{\infty}\frac{\widehat{f}_{n,m+1}}{e_{n+1,m+1}}\omega_n z\\ &= (1+\omega_m z)\Psi_m-\omega_m\frac{\widehat{f}_{m,m}}{e_{m+1,m}}(1+\omega_m z)+\omega_m\widehat{f}_{m+1,m+1}-\omega_m z\Psi_{m+1}\\ &= (1+\omega_m z)\Psi_m-\omega_m\widehat{f}_{m,m}+\omega_m\widehat{f}_{m+1,m+1}-\omega_m z\Psi_{m+1}\\ &= (1+\omega_m z)\Psi_m-\omega_m z\Psi_{m+1},\end{aligned}$$

where we have used the fact that $\widehat{f}_{n,n} = f(t_0)$ for all $n\in\mathbb{N}_0$. Consequently

$$(1+\omega_m z)\Psi_{m+1} = (1+\omega_m z)\Psi_m,$$

and hence $\Psi_{m+1} = \Psi_m$ as $1+\omega_m z\neq 0$ under condition (2.3). □

4. THE INVERSE LAPLACE TRANSFORM

In this section we establish an inversion formula for the Laplace transform.

Theorem 4.1 (Uniqueness Theorem)**.** *Assume* (2.2) *and let* $x:\{t_n : n\in\mathbb{N}_0\}\to\mathbb{C}$ *be a function in the space* $\mathcal{F}_\delta$*, i.e.,* x *satisfies* (2.15)*. Further, let* $\widetilde{x}(z)$ *be the Laplace transform of* x *defined by* (2.5) *for* $z\in D_\delta$*. If* $\widetilde{x}(z)\equiv 0$ *for* $z\in D_\delta$*, then* $x(t_n)=0$ *for all* $n\in\mathbb{N}_0$*.*

Proof. By the assumption, we have

$$\frac{\omega_0 x(t_0)}{1+\omega_0 z}+\frac{\omega_1 x(t_1)}{(1+\omega_0 z)(1+\omega_1 z)}+\frac{\omega_2 x(t_2)}{(1+\omega_0 z)(1+\omega_1 z)(1+\omega_2 z)}+\ldots\equiv 0 \tag{4.1}$$

for $z\in D_\delta$. Multiplying (4.1) by $1+\omega_0 z$ and then passing to the limit as $|z|\to\infty$ (we can take a term-by-term limit due to the uniform convergence proved in Theorem 2.5), we get $x(t_0)=0$. Now we multiply the remaining equation (use $x(t_0)=0$ in (4.1))

$$\frac{\omega_1 x(t_1)}{(1+\omega_0 z)(1+\omega_1 z)}+\frac{\omega_2 x(t_2)}{(1+\omega_0 z)(1+\omega_1 z)(1+\omega_2 z)}+\ldots\equiv 0$$

by $(1+\omega_0 z)(1+\omega_1 z)$ and pass then to the limit as $|z|\to\infty$ to obtain $x(t_1)=0$. Repeating this procedure, we find that $x(t_0)=x(t_1)=x(t_2)=\ldots=0$. □

Theorem 4.1 implies that the inverse Laplace transform exists. The following theorem gives an integral formula for the inverse Laplace transform.

Theorem 4.2 (Inverse Laplace Transform)**.** *Assume* (2.2), *let* $x\in\mathcal{F}_\delta$, *and let* $\widetilde{x}(z)$ *be its Laplace transform defined by* (2.5). *Then*

$$x(t_n)=\frac{1}{2\pi\mathrm{i}}\int_\Gamma \widetilde{x}(z)\prod_{k=0}^{n-1}(1+\omega_k z)\mathrm{d}z \quad \text{for} \quad n\in\mathbb{N}_0, \tag{4.2}$$

where Γ *is any positively oriented closed curve in the region* D_δ *that encloses all the points* $\alpha_k=-\omega_k^{-1}$ *for* $k\in\mathbb{N}_0$.

Proof. Integrating the equality

$$\widetilde{x}(z)=\frac{\omega_0 x(t_0)}{1+\omega_0 z}+\frac{\omega_1 x(t_1)}{(1+\omega_0 z)(1+\omega_1 z)}+\frac{\omega_2 x(t_2)}{(1+\omega_0 z)(1+\omega_1 z)(1+\omega_2 z)}+\ldots \tag{4.3}$$

over the curve Γ with respect to z and noting that we can integrate term-by-term by the uniform convergence of the series proved in Theorem 2.5, we get

$$\int_\Gamma \widetilde{x}(z)\mathrm{d}z=\omega_0 x(t_0)\int_\Gamma\frac{\mathrm{d}z}{1+\omega_0 z}+\omega_1 x(t_1)\int_\Gamma\frac{\mathrm{d}z}{(1+\omega_0 z)(1+\omega_1 z)}+\ldots.$$

Next,

$$\int_\Gamma\frac{\mathrm{d}z}{1+\omega_0 z}=\frac{1}{\omega_0}\int_\Gamma\frac{\mathrm{d}z}{z-\alpha_0}=\frac{2\pi\mathrm{i}}{\omega_0}$$

and

$$\int_\Gamma\frac{\mathrm{d}z}{\prod_{k=0}^{n-1}(1+\omega_k z)}=0 \quad \text{for all} \quad n\in\mathbb{N}\setminus\{1\}$$

because if $P(z)$ is any polynomial of degree greater than or equal to two and if Γ is any closed contour that encloses all the roots of the polynomial $P(z)$, then

$$\int_\Gamma\frac{\mathrm{d}z}{P(z)}=0.$$

Therefore we find

$$x(t_0) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \widetilde{x}(z) \mathrm{d}z.$$

Now multiplying (4.3) by $1 + \omega_0 z$ and then integrating over Γ with respect to z, we obtain

$$\begin{aligned} \int_\Gamma (1+\omega_0 z)\widetilde{x}(z)\mathrm{d}z = \omega_0 x(t_0) \int_\Gamma \mathrm{d}z + \omega_1 x(t_1) \int_\Gamma \frac{\mathrm{d}z}{1+\omega_1 z} \\ + \omega_2 x(t_2) \int_\Gamma \frac{\mathrm{d}z}{(1+\omega_1 z)(1+\omega_2 z)} + \ldots. \end{aligned}$$

Next,

$$\int_\Gamma \mathrm{d}z = 0, \qquad \int_\Gamma \frac{\mathrm{d}z}{1+\omega_1 z} = \frac{1}{\omega_1} \int_\Gamma \frac{\mathrm{d}z}{z-\alpha_1} = \frac{2\pi \mathrm{i}}{\omega_1}$$

and

$$\int_\Gamma \frac{\mathrm{d}z}{\prod_{k=1}^{n}(1+\omega_k z)} = 0 \quad \text{for all} \quad n \in \mathbb{N} \setminus \{1\}.$$

Therefore we find

$$x(t_1) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \widetilde{x}(z)(1+\omega_0 z) \mathrm{d}z.$$

Repeating this procedure, we can obtain formula (4.2) for an arbitrary $n \in \mathbb{N}_0$. $\square$

5. EXAMPLES

Example 5.1. Let $t_n = hn$, $n \in \mathbb{N}_0$, where $h > 0$ is a fixed real number. In this case

$$\omega_n = t_{n+1} - t_n = (n+1)h - nh = h \quad \text{for all} \quad n \in \mathbb{N}_0.$$

Note that (2.2) holds with $\omega = h$. For a function $x : \{hn : n \in \mathbb{N}_0\} \to \mathbb{R}$, its Laplace transform (2.5) becomes

$$\widetilde{x}(z) = \mathcal{L}\{x\}(z) = h \sum_{n=0}^{\infty} \frac{x(nh)}{(1+hz)^{n+1}}.$$

The inversion formula (4.2) takes the form

$$x(nh) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \widetilde{x}(z)(1+hz)^n \mathrm{d}z, \quad n \in \mathbb{N}_0,$$

where Γ is a positively oriented curve that encloses the point $-1/h$.

Example 5.2. Let $t_n = q^n$, $n \in \mathbb{N}_0$, where $q > 1$ is a fixed real number. Then we have

$$\omega_n = t_{n+1} - t_n = q^{n+1} - q^n = (q-1)q^n = q' q^n \quad \text{for all} \quad n \in \mathbb{N}_0, \quad \text{where} \quad q' = q-1.$$

Note that (2.2) holds with $\omega = q'$. For a function $x : \{q^n : n \in \mathbb{N}_0\} \to \mathbb{R}$, its Laplace transform (2.5) becomes

$$\widetilde{x}(z) = \mathcal{L}\{x\}(z) = q' \sum_{n=0}^{\infty} \frac{q^n x(q^n)}{\prod_{k=0}^{n}(1 + q'q^k z)}.$$

The inversion formula (4.2) takes the form

$$x(q^n) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \widetilde{x}(z) \prod_{k=0}^{n-1}(1 + q'q^k z)\mathrm{d}z, \quad n \in \mathbb{N}_0,$$

where Γ is a positively oriented curve that encloses all the points $-(q'q^k)^{-1}$ with $k \in \mathbb{N}_0$.

Example 5.3. Let $t_n = n^p$, $n \in \mathbb{N}_0$, where p is a positive real number. Then we have

$$\omega_n = t_{n+1} - t_n = (n+1)^p - n^p \quad \text{for all} \quad n \in \mathbb{N}_0.$$

Next, applying the mean value theorem to the function $f(x) = x^p$ on the interval $[n, n+1]$, we obtain

$$(n+1)^p - n^p = f(n+1) - f(n) = f'(c) = pc^{p-1}, \quad \text{where} \quad n < c < n+1.$$

Therefore, taking into account that the function x^{p-1} is nondecreasing on $[0, \infty)$ if $p \geq 1$ and decreasing on $(0, \infty)$ if $p < 1$, we get

$$pn^{p-1} \leq \omega_n \leq p(n+1)^{p-1} \quad \text{if} \quad p \geq 1$$

and

$$p(n+1)^{p-1} < \omega_n < pn^{p-1} \quad \text{if} \quad p < 1.$$

Hence, we see that condition (2.1) holds for all $p > 0$, while condition (2.2) holds if and only if $p \geq 1$.

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